Holomorphic Mapping into Canonical Algebraic Varieties

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TABLE OF CONTENTS

1. Introduction and statement of main results

We will study holomorphic mappings

$$f: M \longrightarrow V$$

from a connected complex manifold M to a projective algebraic manifold V.[1] Since we are primarily interested in the *equi-dimensional case*, we will assume throughout that $\dim_{\mathbf{C}} M = n = \dim_{\mathbf{C}} V$. Using local holomorphic coordinates $z_1, \cdots, z_n$ on M and $w_1, \cdots, w_n$ on V, we may write f locally as

$$w_j = f_j(z_1, \cdots, z_n) \qquad (j = 1, \cdots, n) .$$

We will say that f is *non-degenerate* if the Jacobian determinant $\det\left(\partial f_j(z)/\partial z_k\right)$ is not identically zero. In this case the image $f(M)$ contains an open set in V.

Our results will generalize the following three classical theorems on algebraic curves:

THEOREM A′ (*Picard*). *Suppose that V is a complete curve of genus larger*

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[1] By definition, an *algebraic manifold* is a smooth and irreducible algebraic variety defined over $\mathbf{C}$. The adjective *projective* implies that V is complete (i.e. compact) and admits a projective embedding.

than one and that M is an arbitrary algebraic curve.[2] *Then a holomorphic mapping $f: M \to V$ is rational.*

THEOREM B′ (*Landau-Schottky*). *Let V be a complete curve of genus larger than one and $\Delta(\rho)$ the disc $\{z \in \mathbf{C}: |z| < \rho\}$ of radius ρ. Then for any holomorphic mapping $f: \Delta(\rho) \to V$ with $f(0) = x_0$ and $|f'(0)| \geqq 1$, we have the upper bound $\rho \leqq R$ where R is an absolute constant which is independent of f.*[3]

To state the last result, we let V be a complete non-singular curve and ds_V^2 an Hermitian metric on V. Given $x_0 \in V$, the symbol $\Delta(x_0, \rho)$ will denote the disc of radius ρ around x_0 on V, where distance is measured with respect to ds_V^2. For a holomorphic map $f: \Delta \to V$ of the unit disc $\{z \in \mathbf{C}: |z| < 1\}$ into V, let $|f'(0)|$ denote the length of the tangent vector $f_*(\partial/\partial z)$ in $\mathbf{T}_{f(0)}(V)$. A *univalent disc* $\Delta(x_0, \rho)$ for $f: \Delta \to V$ is by definition a disc $\Delta(x_0, \rho)$ on V such that f maps some open set U in Δ biholomorphically onto $\Delta(x_0, \rho)$.

THEOREM C′ (*Bloch*). *Let $f: \Delta \to V$ be an arbitrary holomorphic mapping with $|f'(0)| \geqq 1$. Then there exists an absolute constant $r > 0$ such that there is a univalent disc $\Delta(x_0, r)$ for $f: \Delta \to V$.*

Theorems A′ and B′ are discussed in Picard [10, pp. 59–68 and pp. 75–82]. These results together with Theorem C′ are also treated very nicely in Hille [7, chs. 14, 15, and 17].

To state the generalizations of Theorems A′, B′, and C′ which we shall prove, we consider an n-dimensional complete algebraic manifold V. Let $\omega_0, \cdots, \omega_N$ be a basis for the vector space of holomorphic n-forms on V. Then, letting $[\xi_0, \cdots, \xi_N]$ denote homogeneous coordinates in $\mathbf{P}_N$, the *canonical mapping*

$$x \longrightarrow [\omega_0(x), \cdots, \omega_N(x)] \qquad (x \in V)$$

gives a *rational map* from V to $\mathbf{P}_N$. We will say that V is a *canonical algebraic variety* if the above canonical mapping is a holomorphic immersion.[4]

[2] In other words, V is a compact Riemann surface with negative Euler-Poincaré characteristic and M is a compact Riemann surface with finitely many points deleted.

[3] Here x_0 is a fixed point of V, and the expression $f'(z)$ means that we have written $f: \Delta(\rho) \to V$ as $w = f(z)$ where w is a *fixed* local coordinate around x_0 on V. The adjective *absolute constant* will always mean a constant which is independent of all holomorphic mappings under consideration.

[4] In the language of holomorphic line bundles, the ω_α are holomorphic sections of the canonical line bundle $\mathbf{K} \to V$. To say that V is a canonical algebraic variety means first that the global sections of $\mathbf{K}$ generate each fibre, so that we have an exact sequence

$$0 \longrightarrow \mathbf{F}_x \longrightarrow H^0(V, \sigma(\mathbf{K})) \longrightarrow \mathbf{K}_x \longrightarrow 0 \qquad (x \in V),$$

and secondly that the natural mapping

$$\mathbf{F}_x \longrightarrow \mathbf{K}_x \otimes \mathbf{T}_x^*(V)$$

which sends a section σ of $\mathbf{K}$ with $\sigma(x) = 0$ into $d\sigma(x)$, should be surjective.

Our theorems are (see footnote 22 at and of paper)

THEOREM A. *Suppose that V is a complete canonical algebraic manifold and that M is an arbitrary smooth algebraic variety. Then any non-degenerate holomorphic mapping $f: M \to V$ is necessarily a rational mapping.*

COROLLARY. *If V is a complete canonical algebraic manifold, then any holomorphic mapping $f: \mathbf{C}^n \to V$ is degenerate.*

THEOREM B. *Let V be a complete canonical algebraic manifold and $B(\rho)$ the ball $\{z = (z_1, \cdots, z_n): \sum_{j=1}^n |z_j|^2 < \rho^2\}$ of radius ρ in $\mathbf{C}^n$. Then for any holomorphic mapping $f: B(\rho) \to V$ with $f(0) = x_0$, $|\det(\partial f_j/\partial z_k(0))| \geqq 1$, we have that the radius $\rho \leqq R$ where R is a constant which is independent of f.*[5]

Observe that the corollary of Theorem A is also a corollary of Theorem B.

To state our last result, we let ds_V^2 be an Hermitian metric on V. For a holomorphic mapping $f: B \to V$ of the unit ball

$$\{z = (z_1, \cdots, z_n) \in \mathbf{C}^n: \textstyle\sum_{j=1}^n |z_j|^2 < 1\}$$

into V, we consider the differential $f_*: \mathbf{T}_0(B) \to \mathbf{T}_{f(0)}(V)$. If $e_1, \cdots, e_n$ is a unitary basis for the tangent space $\mathbf{T}_{f(0)}(V)$, we write

$$f_*\left(\frac{\partial}{\partial z_1}\right) \wedge \cdots \wedge f_*\left(\frac{\partial}{\partial z_n}\right) = Jf(0)\ (e_1 \wedge \cdots \wedge e_n)$$

so that $|Jf(0)|$ is the length of $f_*(\partial/\partial z_1) \wedge \cdots \wedge f_*(\partial/\partial z_n)$.

THEOREM C. *Let V be a complete canonical algebraic variety and $f: B \to V$ an arbitrary holomorphic mapping with $|Jf(0)| \geqq 1$. Then there exists an absolute constant $r > 0$ such that there is a univalent ball $B(x_0, r)$ for $f: B \to V$.*[6]

The proofs of Theorems A, B, and C are rather simple and are somewhat similar. The essential ingredients are (1) the fact that a holomorphic mapping $f: M \to V$ into a canonical algebraic manifold is given by using the pull-back differential n-forms $f^*(\omega_a)$ as homogeneous coordinates of a mapping of M

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[5] As explained in footnote (3), the notation means that x_0 is a fixed point of V and $w_1, \cdots, w_n$ is a local coordinate system around x_0 such that f is given by $w_j = f_j(z)$ for $|z| < \varepsilon$.

[6] Here $B(x_0, r)$ is the ball of radius r with respect to ds_V^2 around $x_0 \in V$, and to say that $B(x_0, r)$ is a *univalent ball* for $f: B \to V$ means that f maps an open subset $U \subset B$ biholomorphically onto $B(x_0, r)$. I should also like to point out Wu's comment that the Bloch Theorem is false for $n > 1$ unless we have some restrictive assumption on V (cf. [12]). In fact, the mappings $f_n: \mathbf{C}^2 \to \mathbf{C}^2$ given by $f_n(z_1, z_2) = (nz_1, (1/n)z_2)$ satisfy $|\det(\partial f_{n,j}(0)/\partial z_k)| = 1$ but there is obviously no univalent ball for all of the f_n. In particular, Theorem C is not true for $V = \mathbf{P}_n$ (the canonical bundle of $\mathbf{P}_n$ is *negative*) or for V an abelian variety (the canonical bundle is *trivial*), so it would seem that some sort of positivity of the canonical bundle $\mathbf{K}_V$ is indeed essential.

into $\mathbf{P}_N$; (2) the fact that the *canonical bundle* $\mathbf{K} \to V$ carries a natural metric with positive curvature; and (3) the principle of *hyperbolic complex analysis*, which in the case at hand will give that $f: M \to V$ is *volume decreasing* for suitable domain spaces M.

In addition to our main results listed above, we should like to call attention to a related result of Mrs. Kwack [9], which we have given below as Theorem 6.2 together with a different proof from that presented in [9]. Also, another result (Theorem 6.8) has been given in § 6 as an illustration of how one may look up standard theorems on conformal mapping and then prove the analogous result for several variables so long as we are mapping into canonical algebraic manifolds.

It is my pleasure to acknowledge very helpful conversations with H. Wu. In particular, I learned about the Bloch-type theorems from him, and essential use is made of his paper [12] for the proof of Theorem C. Also, the papers of Kobayashi [8] and Chern [8] have been very useful and between them contain the essential idea necessary to prove that our holomorphic mappings are volume decreasing in the situations we will need for applications.

2. Volume decreasing holomorphic mappings

Let M be a complex manifold of dimension n. A *volume form* μ on M is given by a real and positive differential form of type (n, n) defined everywhere on M. In local coordinates $z_1, \cdots, z_n$ on M we will then have an expression

$$\mu = i^n h(z) dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n$$

where $h(z)$ is a positive C^∞ function. Associated to the volume form μ we define the *Ricci form* $R(\mu)$ by

$$R(\mu) = id'd'' \log h(z) = i\{\textstyle\sum_{j,k} R_{jk}(\mu) dz_j \wedge d\bar{z}_k\} .$$ [7]

Example 1. Suppose that we have an Hermitian metric ds_M^2 on M. Locally we may choose an orthonormal co-frame $\omega_1, \cdots, \omega_m$ for M such that $ds_M^2 = \sum_j \omega_j \bar{\omega}_j$.[8] The *associated volume form* is then

$$\mu = i^{n^2+n} \omega_1 \wedge \cdots \wedge \omega_n \wedge \bar{\omega}_1 \wedge \cdots \wedge \bar{\omega}_n = i^n \omega_1 \wedge \bar{\omega}_1 \wedge \cdots \wedge \omega_n \wedge \bar{\omega}_n .$$

It is well known that ds_M^2 induces an intrinsic connection (cf. Chern [3]) on M and we let $\Omega_j^i = \sum_{k,l} R_{jkl}^i \omega_k \wedge \bar{\omega}_l$ be the curvature. Then the Ricci form $R(\mu) = i\{\sum_{j,k} R_{jk}(\mu) \omega_j \wedge \bar{\omega}_k\}$ where $R_{jk}(\mu) = -\sum_l R_{ljk}^l$ is minus the trace of the

[7] Note that our definition of the Ricci form differs by a minus sign from that of Kobayashi and Chern. The reason for this is that it is notationally more convenient if the Ricci forms used in proving Theorems A, B, and C are positive. Observe that a volume form is the same as a metric in the canonical bundle of M.

[8] Recall that locally $\omega_j = \sum_k A_{jk} dz_k$, $\det(A_{jk}) \neq 0$, and that the ω_j are defined up to a unitary transformation.

curvature.

We are especially interested in the case when the Hermitian metric ds^2_M is an *Einstein-Kähler metric*; by definition this means that, in the above notation,

$$R(\mu) = i\lambda\{\textstyle\sum_{j=1}^n \omega_j \wedge \bar{\omega}_j\} , \qquad \lambda > 0 . \tag{2.1}$$

In words, the Ricci form should be a *constant positive* multiple of the 2-form $\sum_j \omega_j \wedge \bar{\omega}_j$ associated to the Hermitian metric ds^2_M.

Example 2. Let V be a complete canonical algebraic manifold. We consider a basis $\omega_0, \cdots, \omega_N$ for the vector space of holomorphic n-forms on V with the property that

$$i^{n^2+n}\int_V \omega_\alpha \wedge \bar{\omega}_\beta = \delta^\alpha_\beta \qquad (\alpha, \beta = 0, \cdots, N) .$$

Such a basis will be said to be *orthonormal*; it is determined up to a unitary transformation. Now the (n, n) form

$$\mu = i^{n^2+n}\{\textstyle\sum_\alpha \omega_\alpha \wedge \bar{\omega}_\alpha\} \tag{2.2}$$

is an intrinsically defined volume form on V. (Using the language of line bundles as in footnote (4), the μ defined by (2.2) is a volume form if, and only if, the restriction mappings $H^0(V, \sigma(\mathbf{K})) \rightarrow \mathbf{K}_x$ are onto for all $x \in V$.)

We now claim that the Ricci form $R(\mu)$ of μ defined by (2.2) is positive definite. To see this we write locally that $\omega_\alpha = g_\alpha(z)dz_1 \wedge \cdots \wedge dz_n$. Then

$$\mu = i^n\{\textstyle\sum_\alpha |g_\alpha(z)|^2\}dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n .$$

The fact that the canonical map $V \rightarrow \mathbf{P}_N$ is an immersion means that we may choose the orthonormal basis ω_α of $H^0(V, \sigma(\mathbf{K}))$ and the local coordinates $z_1, \cdots, z_n$ so that the equations

$$\begin{cases} g_0(z) = 1 + [2] & \\ g_j(z) = z_j + [2] & j = 1, \cdots, n , \\ g_a(z) = [2] & a > n , \end{cases}$$

where [2] denotes terms of at least 2nd order, will hold. Then the Ricci form is given by

$$R(\mu) = i\left\{\sum_{j,k} \frac{\partial^2 \log\{|g_0(z)|^2 + \cdots + |g_N(z)|^2\}}{\partial z_j \partial \bar{z}_k} dz_j \wedge d\bar{z}_k\right\}$$

so that, at $z = 0$, we have

$$R(\mu)_{z=0} = i\{\textstyle\sum_{j=1}^n dz_j \wedge d\bar{z}_j\} .$$[9]

[9] The fact that this Ricci form $R(\mu)$ is positive definite is perhaps best explained by saying that $R(\mu)$ is the 1st *Chern class* of the hyperplane line bundle $\mathbf{K} \rightarrow V$ relative to the canonical immersion $V \subset \mathbf{P}_N$ where this Chern class has been expressed by using the curvature of the metric connection in $\mathbf{K} \rightarrow V$.

As a consequence of this discussion we have

(2.3) LEMMA. *Let V be a complete canonical algebraic manifold, μ the intrinsic volume form* (2.2) *and $R(\mu)$ the Ricci form of μ. Then there is a positive constant $c > 0$ such that we have everywhere on V the estimate*

$$\mu \leqq cR(\mu)^n . \tag{2.4}$$

Example 3. This is a continuation of Example 1 where we take M to be a polycylinder $P(\rho)$ of radius ρ defined by $\{z = (z_1, \cdots, z_n): |z_j| < \rho\}$. When $\rho = 1$ we will speak of the unit polycylinder and write P for $P(1)$. There is defined on $P(\rho)$ the standard *Poincaré metric*

$$ds^2_{P(\rho)} = a\left\{\sum_{j=1}^n \frac{\rho^2 dz_j d\bar{z}_j}{(\rho^2 - |z_j|^2)^2}\right\} \qquad (a > 0) . \tag{2.5}$$

It is well known that this is an Einstein-Kähler metric such that the constant factor λ in (2.1) is just $4/a$.

Example 4. This is again a continuation of Example 1 where we take M to be the ball $B(\rho)$ of radius ρ defined by $\{z = (z_1, \cdots, z_n) \in \mathbf{C}^n: \sum_{j=1}^n |z_j|^2 < \rho^2\}$. As before, when $\rho = 1$ we will write B for $B(\rho)$. On $B(\rho)$ there is defined the standard *hyperbolic metric*

$$ds^2_{B(\rho)} = \left\{\frac{b}{\rho^2 - r^2}\right\}\sum_j dz_j d\bar{z}_j + \left\{\frac{4r^2}{(\rho^2 - r^2)^2}\right\}d'rd''r , \quad r^2 = \sum_{j=1}^n |z_j|^2 . \tag{2.6}$$

This metric is again an Einstein-Kähler metric such that the constant factor λ in (2.1) is $2n(n + 1)/b$.

The main tool in our proofs of Theorems A, B, and C is the following

(2.7) PROPOSITION. *Let M be either the unit polycylinder or unit ball in $\mathbf{C}^n$ as discussed in Examples* 3 *and* 4 *above. Let μ_M be the volume element deduced from the standard metric ds^2_M given by* (2.5) *and* (2.6). *Suppose that V is a complete canonical algebraic manifold with intrinsic volume form μ_V as discussed in Example* 2 *above. Then, by suitable choice of the constants a and b in* (2.5) *and* (2.6), *we have that any holomorphic mapping $f: M \rightarrow V$ is volume decreasing in the sense that $f^*\mu_V \leqq \mu_M$.*

Proof. We first remark that similar volume-decreasing theorems have been proved by Chern [3] and Kobayashi [8], among others. All proofs seem to closely follow the lines of the original argument for $n = 1$ given by Ahlfors. We shall prove (2.7) when $M = P$ is the polycylinder, as the other case of the ball follows by the same argument.

Write $f^*\mu_V = \varphi\mu_P$ where φ is a non-negative function on P. We want to prove that, with a suitable choice of constants, we have everywhere $\varphi \leqq 1$.

The idea is to use the *maximum principle*, and so we first show that it will suffice to consider the case when φ assumes its maximum at some interior point of the polycylinder P.

To see this we restrict f to the smaller polycylinder $P(\rho)$ where $\rho < 1$. Then we write $f^*\mu_V = \varphi(\rho)\mu_{P(\rho)}$. From (2.5) it follows that $\lim_{\rho\to 1}\varphi(\rho)(x) = \varphi(x)$ for fixed $x \in P(\rho)$. On the other hand, for fixed ρ, we have $\lim_{x\to\partial P(\rho)}\varphi(\rho)(x) = 0$ since again from (2.5) it follows that the volume form $\mu_{P(\rho)}$ goes to infinity everywhere at the boundary of the polycylinder $P(\rho)$. Thus the inequality $\varphi(\rho) \leqq 1$ for all $\rho < 1$ will give $\varphi \leqq 1$, while it is certainly the case that $\varphi(\rho)$ has an interior maximum point for $\rho < 1$.

Let $x_0 \in P$ be a maximum point for φ. We may assume that $\varphi(x_0) > 0$ so that f is biholomorphic in a neighborhood of x_0. Choose local holomorphic coordinates $u_1, \cdots, u_n$ around x_0 on M and local holomorphic coordinates $w_1, \cdots, w_n$ around $f(x_0)$ such that $f_j(u) = w_j$. Writing $\mu_V = i^n h_V dw_1 \wedge d\bar{w}_1 \wedge \cdots \wedge dw_n \wedge d\bar{w}_n$ and $\mu_P = i^n h_P du_1 \wedge d\bar{u}_1 \wedge \cdots \wedge du_n \wedge d\bar{u}_n$, we have that $\varphi = h_V/h_P$. It follows that

$$id'd'' \log \varphi = f^*R(\mu_V) - R(\mu_P) \ . \tag{2.8}$$

Since at a maximum point we have $id'd'' \log \varphi \leqq 0$, we find the inequality

$$f^*R(\mu_V)(x_0) \leqq R(\mu_P)(x_0) \ .$$ [10]

Passing to n^{th} exterior powers and using (2.4), we have

$$\frac{1}{c} f^*\mu_V(x_0) \leqq \left(\frac{4}{a}\right)^n \mu_P(x_0) \ .$$

Choosing $a = 4\sqrt[n]{c}$ we arrive at $\varphi(x_0) \leqq 1$, from which it follows that $\varphi(x) \leqq 1$ for all $x \in P$. Q.E.D.

3. Proof of Theorem A

We want to formulate a local result which will imply Theorem A. For this we define a *punctured polycylinder* P^* to be a product $\underbrace{\Delta^* \times \cdots \times \Delta^*}_{k} \times \underbrace{\Delta \times \cdots \times \Delta}_{n-k}$ of k punctured unit discs $\Delta^* = \{z \in \mathbf{C}: 0 < |z| < 1\}$ with $n - k$ ordinary unit discs $\Delta = \{z \in \mathbf{C}: |z| < 1\}$. We will think of P^* as being the complement of the divisor D given by $z_1 \cdots z_k = 0$ in the usual polycylinder P. Thus we may write that $P^* = P - D$.

Now let M be an arbitrary smooth and irreducible algebraic variety. A *smooth compactification* of M is given by a smooth and complete algebraic

[10] If $\psi = i\{\sum_{j,k} g_{jk} dz_j \wedge d\bar{z}_k\}$ is a (1, 1) form with $g_{jk} = \bar{g}_{kj}$, the inequality $\psi \leqq 0$ means that the Hermitian matrix (g_{jk}) is negative semi-definite. For two such forms ψ_1 and ψ_2, inequality $\psi_1 \leqq \psi_2$ means by definition that $\psi_2 - \psi_1 \geqq 0$.

variety $\bar{M}$ which contains M as a Zariski open set defined locally by $z_1 \cdots z_k \neq 0$ for suitable local holomorphic coordinates $z_1, \cdots, z_n$ on $\bar{M}$. In other words, a smooth compactification of M is a smooth completion of M such that locally around infinity M looks like a punctured polycylinder. By Hironaka's *resolution of singularities* such smooth compactifications of M exist.

From this discussion it is clear that Theorem A will follow from

THEOREM D. *Let P^* be a punctured polycylinder and V a complete canonical algebraic manifold. Then any non-degenerate holomorphic mapping $f: P^* \rightarrow V$ extends to a meromorphic mapping of the whole polycylinder into V.*[11]

The proof of this theorem will be given in a series of lemmas. To state the first one, we coordinatize our punctured polycylinder by letting

$$P^* = \{(z, w): z = (z_1, \cdots, z_k) \in \mathbf{C}^k;\ \ w = (w_1, \cdots, w_{n-k}) \in \mathbf{C}^{n-k};\\ 0 < |z_j| < 1 \quad \text{and} \quad |w_\alpha| < 1\}\,.$$

Then if $s = (s_1, \cdots, s_k)$ is a k-tuple of real numbers with $0 < s_j < 1/2$ for all j, we define the *annulus* $A(s)$ *of inner radius* s by

$$A(s) = \left\{(z, w) \in P^*: s_j \leqq |z_j| \leqq \frac{1}{2} \quad \text{and} \quad |w_\alpha| \leqq \frac{1}{2}\right\}.$$

(3.1) LEMMA. *Let φ be a holomorphic n-form on the punctured polycylinder P^*. Then φ extends to a holomorphic n-form on the whole polycylinder P if, and only if, we have an estimate*

$$\left|\int_{A(s)} \varphi \wedge \bar{\varphi}\right| < \text{(constant)} \tag{3.2}$$

for all annuli $A(s)$ of inner radius s with $0 < s_j < 1/2$.

Proof. We will use the following standard multi-index notation: $I = (i_1, \cdots, i_k)$ is a k-tuple of integers; $z^I = (z_1)^{i_1} \cdots (z_k)^{i_k}$; $s^I = (s_1)^{i_1} \cdots (s_k)^{i_k}$. Writing the n-form φ as

$$\varphi = h(z, w) dz_1 \wedge \cdots \wedge dz_k \wedge dw_1 \wedge \cdots \wedge dw_{n-k}\,,$$

we will have a *Laurent series expansion* in the z_j's of the holomorphic function $h(z, w)$

$$h(z, w) = \textstyle\sum_I A_I(w) z^I$$

[11] For our purposes a *meromorphic mapping* $f: M \rightarrow V$ of an arbitrary complex manifold M into V may be defined as being given by a holomorphic mapping of a Zariski open subset of M into V such that any rational function on V pulls back to a meromorphic function on M. (Recall that a Zariski open subset of M is by definition the complement of a proper analytic subvariety of M.)

which is absolutely and uniformly convergent on each annulus $A(s)$ and where the coefficients $A_I(w)$ are holomorphic functions of w. From the formula

$$i^{n^2+n}\int_{A(s)}\varphi\wedge\bar{\varphi} = i^n\int_{\substack{s_j\leqq|z_j|\leqq 1/2\\ |w_\alpha|\leqq 1/2}}|h(z,w)|^2 dz_1\wedge d\bar{z}_1\wedge\cdots\wedge dw_{n-k}\wedge d\bar{w}_{n-k}$$

and the elementary fact that

$$i\int_{s\leqq|t|\leqq 1/2} t^m\bar{t}^l dt d\bar{t} = \delta_l^m\gamma_m(s)$$

where $\gamma_m(s)$ is strictly positive for $0 < s < 1/2$ and where $\lim_{s\to 0}\gamma_m(s) = +\infty$ if $m \leqq -1$,[12] we find that

$$\left|\int_{A(s)}\varphi\wedge\bar{\varphi}\right| = \sum\nolimits_I \|A_I\|^2\gamma_I(s) \tag{3.3}$$

where

$$\|A_I\|^2 = i^{n-k}\int_{|w_\alpha|\leqq 1/2}|A_I(w)|^2 dw_1\wedge d\bar{w}_1\wedge\cdots\wedge dw_{n-k}\wedge d\bar{w}_{n-k}$$

and where $\gamma_I(s) = \gamma_{i_1}(s_1)\cdots\gamma_{i_k}(s_k)$. Using (3.3) we see that the estimate (3.2) implies $A_I(w)\equiv 0$ if some component index $i_j \leqq -1$, so that φ is in fact holomorphic. Conversely, if φ is holomorphic, then it is obvious that (3.2) holds.

Q.E.D.

To state the second lemma, we shall use the *Poincaré metric* $ds^2_{P^*}$ on the punctured polycylinder. To define this metric, we identify the universal covering of P^* with the usual polycylinder P by the sequence of maps

$$\underbrace{\Delta\times\cdots\times\Delta}_{n}\xrightarrow{\psi_1}\underbrace{H\times\cdots\times H}_{k}\times\underbrace{\Delta\times\cdots\times\Delta}_{n-k}$$
$$\xrightarrow{\psi_2}\underbrace{\Delta^*\times\cdots\times\Delta^*}_{k}\times\underbrace{\Delta\times\cdots\times\Delta}_{n-k}$$

where ψ_1 transforms the first k-factors into upper-half-planes H by the usual linear fraction transformations on $\mathbf{P}_1$ while leaving the last $(n-k)$-factors alone, and where ψ_2 is the exponential function in the first k-factors while leaving fixed the last $(n-k)$-factors. Using the just-defined universal covering mapping $\psi\colon P\to P^*$, the Poincaré metric $ds^2_{P^*}$ on the punctured polycylinder is the unique metric on P^* such that the pull-back $\psi^*(ds^2_{P^*})$ is the Poincaré metric ds^2_P given by (2.5) on P. Denote by μ_{P^*} the volume form associated to $ds^2_{P^*}$.

(3.4) LEMMA. *Let $A(s)$ be an annulus of inner radius s as defined just above Lemma* 3.1. *Then we have an estimate*

[12] To be precise, we have the formulae $\gamma_m(s) = 2\pi/(m+1)[(1/2)^{2m+2} - s^{2m+2}]$ for $m \neq -1$ while $\gamma_{-1}(s) = 2\pi[\log(1/2) - \log s]$.

$$\int_{A(s)} \mu_{P^*} \leqq (\text{constant}) < \infty \tag{3.5}$$

for all s with $0 < s_j < 1/2$.

Proof. We introduce coordinates (z, w) in P^* as was done just above Lemma 3.1. Using polar coordinates $z_j = \rho_j e^{i\theta_j}$ and $w_a = \sigma_a e^{i\varphi_a}$, we find by an easy computation that

$$\mu_{P^*} = \prod_{j=1}^{k} \frac{d\rho_j d\theta_j}{\rho_j(\log \rho_j)^2} \prod_{a=1}^{n-k} \frac{d\sigma_a d\varphi_a}{(1 - \sigma_a^2)^2} .$$

The desired result follows by inspection of this expression for μ_{P^*}.[13]

(3.6) LEMMA. *Let* $f: P^* \to V$ *be as in the statement of Theorem D and let* ω *be a non-zero holomorphic n-form on V. Then the pull-back* $f^*\omega$ *extends to a non-zero holomorphic n-form on the whole polycylinder P.*

Proof. The pull-back $f^*\omega$ is not identically zero because f is non-degenerate. To see that $f^*\omega$ is holomorphic, we may restrict ω to be one of the orthonormal basis $\omega_0, \cdots, \omega_N$ in Example 2 of § 2. Then

$$\left|\int_{A(s)} f^*\omega \wedge \overline{f^*\omega}\right| = \left|\int_{A(s)} f^*(\omega \wedge \bar{\omega})\right| = \left|\int_{f(A(s))} \omega \wedge \bar{\omega}\right| \leqq \mu_V(f(A(s))) .$$

Here $\mu_V(f(A(s)))$ is the volume of the image $f(A(s))$ computed (with multiplicities) using the canonical volume form μ_V on V.

Now by Proposition 2.7 we have

$$\mu_V(f(A(s))) \leqq \mu_{P^*}(A(s)) \leqq (\text{constant})$$

where the last step follows Lemma 3.4. Combining we have the estimate

$$\left|\int_{A(s)} f^*\omega \wedge \overline{f^*\omega}\right| < c$$

where c is a constant independent of ω, f, and s. Our result follows from this together with Lemma 3.1.

4. Proof of Theorem B

Let $f: B(\rho) \to V$ be as in the theorem. From the proof of Proposition 2.7, we have

$$f^*\mu_V(0) \leqq c\left(\frac{2n(n+1)}{b}\right)^n \mu_{B(\rho)}(0) \tag{4.1}$$

[13] This lemma is perhaps best understood as follows: Let $H = \{z + x + iy: y > 0\}$ be the upper-half-plane with Poincaré metric $(dx^2 + dy^2)/y^2$ and associated volume form $dxdy/y^2$. Then the volume of the vertical strip defined by $\{-1/2 \leqq x \leqq 1/2,\ y \geqq 1\}$ is finite; this is essentially the same as saying that the volume of a fundamental domain for the modular group is finite. The fact that this vertical strip has finite non-Euclidean volume is in turnes sentially equivalent to Lemma 3.4.

where c is as in Lemma 2.3, b is as in Example 4, both being independent of f. Writing B for $B(1)$, it is clear from Example 4 that

$$\mu_{B(\rho)}(0) = \frac{b^n}{\rho^{2n}} \mu_B(0) , \tag{4.2}$$

where in fact $\mu_B(0) = i^n dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n$. Using this last expression we see that $|\det(\partial f_j/\partial z_k(0))| \geqq 1$ implies that

$$e\mu_B(0) \leqq f\mu_V(0) \tag{4.3}$$

for some positive constant e independent of f. Combining (4.1)–(4.3), we get

$$e\mu_B(0) \leqq c\left(\frac{2n(n+1)}{b}\right)^n \frac{b^n}{\rho^{2n}} \mu_B(0) ,$$

or

$$\rho = \left(\frac{c}{e}\right)^{1/2n} (2n(n+1))^{1/n} .$$

Letting R be this last constant, we are done.[14]

5. Proof of Theorem C and an alternate proof of Theorem B

Let B and V be as in the statement of Theorem C, and let $\mathcal{F}$ be the family of all holomorphic mappings $f: B \to V$ which satisfy the condition $|Jf(0)| \geqq 1$ as explained above Theorem C. Our proof of Theorem C is based on

(5.1) PROPOSITION. *Given an arbitrary sequence of maps* $\{f_k^\sharp\} \in \mathcal{F}$, *there is a subsequence* $\{f_k\}$ *of* $\{f_k^\sharp\}$, *a meromorphic mapping* $g: B \to V$, *and a number* ρ_0 *with* $0 < \rho_0 \leqq 1$ *such that the restriction* $g: B(\rho_0) \to V$ *is holomorphic and such that the mappings* $f_k: B \to V$ *converge uniformly on compact sets to* g *on the smaller ball* $B(\rho_0)$.

Proof of Proposition 5.1. Let ω be one of the basis $\omega_0, \cdots, \omega_N$ in Example 2 of § 2 and let $f: B \to V$ be a mapping in the class $\mathcal{F}$. Given $z \in B$ we assert that there is a ball $B(z, \varepsilon)$ of (Euclidean) radius ε with $0 < \varepsilon < 1 - |z|$ around z and an absolute constant $c = c(\omega, z, \varepsilon)$ such that we have the estimate

$$\left| \int_{B(z,\varepsilon)} f^*\omega \wedge \overline{f^*\omega} \right| \leqq c . \tag{5.2}$$

Indeed, using Proposition 2.7 we have as in the proof of Lemma 3.6 that

$$\left| \int_{B(z,\varepsilon)} f^*\omega \wedge \overline{f^*\omega} \right| = \left| \int_{f[B(z,\varepsilon)]} \omega \wedge \bar{\omega} \right| \leqq \mu_V\{f[B(z, \varepsilon)]\} \leqq \mu_B(B(z, \varepsilon)) ,$$

[14] The original proof of Theorem B was somewhat vague, and the above argument was suggested by the referee. The referee also kindly pointed out to me that Proposition 2.7 above follows from Theorem 3 of [8], where in fact the image manifold V need only have a positive canonical bundle. Thus Theorem B is also true in this more general situation.

so that we may in fact take the constant c to be $\mu_B(B(z, \varepsilon))$ in (5.2).

Keeping now the notation ω for an arbitrary holomorphic n-form on V, we set $\omega_k^\# = (f_k^\#)^*\omega$. Writing

$$\omega_k^\# = h_k^\#(z)dz_1 \wedge \cdots \wedge dz_n$$

where $h_k^\#(z)$ is a holomorphic function on B, it follows from (5.2) that given $z \in B$ and $\varepsilon < 1 - |z|$, we have the estimate

$$\int_{B(z,\varepsilon)} |h_k^\#(z)|^2 d\mu \leqq (\text{constant}) \tag{5.3}$$

where $d\mu$ denotes Euclidean measure in $\mathbf{C}^n$ and where the constant is independent of the index k. In other words, the sequence of holomorphic functions $\{h_k^\#(z)\}$ on B is *locally uniformly bounded in the* L_2*-sense.* It is then a standard result in complex function theory that there is a subsequence $\{h_k(z)\}$ such that the holomorphic functions h_k converge uniformly on compact sets to a holomorphic limit function $h(z)$. Then the corresponding holomorphic n-forms $\omega_k = h_k(z)dz_1 \wedge \cdots \wedge dz_n$ converge uniformly on compact sets to the holomorphic n-form $\varphi = h(z)dz_1 \wedge \cdots \wedge dz_n$.

Now let $\omega_0, \cdots, \omega_N$ be a basis for the vector space of holomorphic n-forms on V. By what has just been said we may assume that a subsequence $\{f_k\}$ of $\{f_k^\#\} \subset \mathcal{F}$ has been selected such that the pull-backs $f_k^*\omega_\alpha$ converge uniformly on compact sets to a holomorphic n-form φ_α ($\alpha = 0, \cdots, N$). Furthermore, the condition $|Jf(0)| \geqq 1$ implies that the φ_α are linearly independent and that there is a ρ_0 with $0 < \rho_0 \leqq 1$ such that, for each point $z \in B(\rho_0)$, at least one $\varphi_\alpha(z) \neq 0$. The meromorphic mapping $f: B \rightarrow V$ given by thinking of V as being immersed in $\mathbf{P}_N$ and setting $f(z) = [\varphi_0(z), \cdots, \varphi_N(z)]$ will then be holomorphic on $B(\rho_0)$ and we will have that $\lim_{k\to\infty} f_k = f$ uniformly on compact subsets of $B(\rho_0)$. Q.E.D.

Proof of Theorem C. Given $f \in \mathcal{F}$ we let $\rho(f)$ denote the radius of the maximal univalent ball $B(x, \rho(f)) \subset V$. If the theorem is false, there is a sequence $\{f_k^\#\} \subset \mathcal{F}$ such that $\lim_{k\to\infty} \rho(f_k^\#) = 0$. Let $\{f_k\}$ be a subsequence as given by Proposition 5.1. We now replace B by the smaller ball $B(\rho_0)$, $\mathcal{F}$ by the corresponding family $\mathcal{G}$ of all holomorphic mappings $g: B(\rho_0) \rightarrow V$ satisfying $|Jg(0)| \geqq 1$, and we let g_k be the restriction of f_k to $B(\rho_0)$. Then we find that g_k tends uniformly on compact sets to a limit mapping $g \in \mathcal{G}$. On the other hand, the radii of the maximal univalent balls $\rho(g_k)$ of the mappings $g_k: B(\rho_0) \rightarrow V$ will tend to zero because it is clear that $\rho(g_k) \leqq \rho(f_k)$. Now the statements

$$\begin{cases} \lim_{k\to\infty} g_k = g \in \mathcal{G} \\ \lim_{k\to\infty} \rho(g_k) = 0 \end{cases} \tag{5.4}$$

contradict Theorem A in Wu [12, p. 200]. This contradiction arose from the assumption $\lim_{k\to\infty}\rho(f_k^\sharp) = 0$, so that we will have an estimate $\rho(f) \geqq r > 0$ for all $f \in \mathcal{F}$. Q.E.D.

Alternate proof of Theorem B. If Theorem B is false then we will have a sequence of numbers $\rho_k^\sharp$ tending to $+\infty$ such that there is a holomorphic mapping $f_k^\sharp: B(\rho_k^\sharp) \to V$ with $f_k^\sharp(0) = x_0$ and $|\det(\partial f_{k,j}^\sharp(0)/\partial z_l)| \geqq 1$. Define $h_k^\sharp: B \to V$ by $h_k^\sharp(z) = f_k^\sharp(\rho_k^\sharp z)$. This gives a sequence of maps $h_k^\sharp: B \to V$ with $h_k^\sharp(0) = x_0$ and $|\det(\partial h_{k,j}^\sharp(0)/\partial z_l)| \geqq (\rho_k^\sharp)^n$. Using Proposition 5.1 we may find a subsequence $\{h_k\}$ of $\{h_k^\sharp\}$ and a ρ_0 with $0 < \rho_0 \leqq 1$ such that the restricted mappings $h_k: B(\rho_0) \to V$ converge uniformly on compact sets to a holomorphic mapping $h: B(\rho_0) \to V$. But this contradicts the estimate

$$\left|\det\left(\frac{\partial h_{k,j}(0)}{\partial z_l}\right)\right| \geqq (\rho_k)^n, \quad \lim_{k\to\infty}\rho_k = \infty .$$

6. Some miscellaneous comments

(**a**) *Remarks on the problem of extending holomorphic mappings across subvarieties.* Let M and V be complex manifolds and Z a proper analytic subvariety of M. An interesting problem is to find conditions under which an arbitrary holomorphic mapping $f: M - Z \to V$ extends to either a holomorphic or meromorphic mapping $f: M \to V$. (Since the problem is local in M, we may take M to be a polycylinder if this is convenient.) We may rephrase Theorem D by saying that f extends as a meromorphic mapping provided that $\dim M = \dim V$, f is non-degenerate, and that V is a complete canonical algebraic manifold.

Now this same result is probably still true if, instead of assuming that the canonical bundle $\mathbf{K} \to V$ is *very ample* as described in footnote (4), we only assume that $\mathbf{K} \to V$ is positive in the sense of Kodaira.[15] In other words, it should be the case that a curvature assumption in the canonical bundle $\mathbf{K} = \Lambda^n\mathbf{T}^*(V)$ leads to an extension theorem for equi-dimensional, non-degenerate holomorphic mappings.

On the other hand, a recent theorem of Mrs. Kwack [9] shows that a curvature assumption on the *full cotangent bundle* $\mathbf{T}^*(V)$ leads to an extension theorem for holomorphic mappings and arbitrary M and Z. Her proof of this result relies on an ingenious but non-transparent argument using the Cauchy

[15] By definition, $\mathbf{K} \to V$ is *positive in the sense of Kodaira* if the 1st Chern class $c_1(\mathbf{K})$ is represented using *de Rham cohomology* by a real differential form of type (1, 1) $\psi = i(\sum_{j,k} g_{j\bar{k}} dz_k \wedge d\bar{z}_k)$ where the Hermitian matrix $(g_{j\bar{k}})$ is positive definite. This is the same as saying that there is a volume form μ_V on V whose associated Ricci form $R(\mu_V)$ is positive definite. Proposition 2.7 remains true if we only assume that $\mathbf{K}$ is positive in this sense.

integral formula—this argument seems to go back to the paper of Grauert-Reckziegel [6]. We should like to present here a hopefully somewhat more conceptual proof of Mrs. Kwack's theorem.

Thus let V be a compact, complex manifold on which there is an Hermitian metric ds_V^2 such that the *holomorphic sectional curvatures* are all negative. For a discussion of holomorphic sectional curvatures and a proof of the following lemma, we refer to Wu [12]:

(6.1) LEMMA.[16] *Let P^* be the punctured polycylinder with Poincaré metric $ds_{P^*}^2$ as given in the proof of Lemma* 3.4. *Then we may choose the constant a in* (2.6) *such that any holomorphic mapping $f: P^* \to V$ decreases distance in the sense that $f^*(ds_V^2) \leqq ds_{P^*}^2$.*

Mrs. Kwack's theorem is

(6.2) THEOREM. *Let V be a compact Hermitian manifold with negative holomorphic sectional curvatures. Then any holomorphic mapping $f: M - Z \to V$ extends to a holomorphic mapping $f: M \to V$.*

Proof. Elementary reasoning given in [9] shows that the theorem is true if $\operatorname{codim}(Z) \geqq 2$. Using this it will suffice to prove the theorem when Z is a non-singular divisor on M. Localizing, we may assume then that $M = \Delta^* \times \underbrace{\Delta \times \cdots \times \Delta}_{m-1}$ is the product of a punctured disc with a polycylinder. To prove the result for such an M, the essential case is that of a holomorphic mapping $f: \Delta^* \to V$ of the punctured disc $\{z: 0 < |z| < 1\}$ into V. Thus we shall restrict ourselves to proving

(6.3) THEOREM. *Let V be a (not necessarily compact) Hermitian manifold whose holomorphic sectional curvatures are negative and bounded away from zero. Let $f: \Delta^* \to V$ be an arbitrary holomorphic mapping such that, for some sequence of points $\{z_k\}$ in Δ^* with $\lim_{k\to\infty} |z_k| = 0$, we have $\lim_{k\to\infty} f(z_k) = x_0$ in V.*[17] *Then f extends to a holomorphic mapping $f: \Delta \to V$ of the whole disc into V.*

Proof of Theorem 6.3. We consider the product manifolds $N^* = \Delta^* \times V$ and $N = \Delta \times V$, and we denote π_{Δ^*}, π_Δ, and π_V the projections of N^* and N onto the various coordinate axes. We may write $N^* = N - D$ where $D = \{0\} \times V$ is a divisor on N. Finally, we denote by $ds_{N^*}^2 = ds_{\Delta^*}^2 + ds_V^2$ and $ds_N^2 = ds_\Delta^2 + ds_V^2$ the natural metrics induced on N^* and N.

[16] This lemma, which is essentially due to Ahlfors, is proved in the same way as Proposition 2.7. The lemma is true for non-compact V if we assume that the holomorphic sectional curvatures are all bounded above by a negative constant.

[17] Observe that this condition is automatically satisfied if V is compact.

Let $G_f \subset N^*$ be the *graph* of the holomorphic mapping $f: \Delta^* \to V$ and denote by $\bar{G}_f$ the (topological) closure of G_f in N. Then it is clear that f extends if, and only if, the following conditions are fulfilled:

(6.4) $\bar{G}_f$ is an analytic subvariety of N;

(6.5) there is exactly one point y of $\bar{G}_f$ such that $\pi_\Delta(y) = 0$.

Now it is easy to see that, if (6.4) is satisfied, then there is *at most* one point $y \in \bar{G}_f$ such that $\pi_\Delta(y) = 0$. On the other hand the assumption $\lim_{k\to\infty} f(z_k) = x_0$ in V where $\lim_{k\to\infty} |z_k| = 0$ in Δ shows that there is *at least* one such point in $\bar{G}_f$. Thus it will suffice to prove (6.4).

For this we will use the results of Bishop as reported on in the notes of Stolzenberg [11]. In the situation at hand, Bishop's theorem may be stated as follows [11, Th. (F), p. 2]:

The necessary and sufficient condition that $\bar{G}_f$ be an analytic subvariety of N is that, given $y \in D$, there is a neighborhood U of y in N such that the volume $\mu_N(G_f \cap U)$ of the intersection of G_f with U is finite (here volume is computed with respect to the Hermitian metric ds_N^2).

Now comparing $ds^2_{\Delta^*}$ with ds^2_Δ around $z = 0$, we see that $\mu_N(G_f \cap U)$ will certainly be finite if $\mu_{N^*}(G_f \cap U)$ is finite. On the other hand, and this is the whole point, $f: \Delta^* \to V$ being distance decreasing (Lemma 6.1) means that we have the estimate

$$\mu_{N^*}(G_f \cap U) = \mu_{\Delta^*}\big(\pi_{\Delta^*}(U)\big) + \mu_V\big(f\big(\pi_{\Delta^*}(U)\big)\big) \leqq 2\mu_{\Delta^*}\big(\pi_{\Delta^*}(U)\big) .$$

On the other hand, we may assume that $\mu_{\Delta^*}\big(\pi_{\Delta^*}(U)\big)$ is finite by a suitable choice of U (cf. Lemma 3.4).

(b) *Remarks on Theorems* A′ *and* A *as related to the general problem of value distributions of holomorphic mappings in several complex variables.* Perhaps the most important case of the (global) study of holomorphic mappings in several complex variables is the situation of a holomorphic mapping $f: A \to V$ where A and V are smooth algebraic varieties and where V is assumed to be complete.[18] The case when $\dim A = \dim V = 1$ is the so-called *Nevanlinna theory of value distributions* of which excellent expositions are given by Chern [4] and Wu [13]. One of the consequences of this general theory is the following theorem (cf. Chern [4, Th. 3, p. 336]):

[18] The simplest case of such holomorphic mappings is when A is also complete. Then f is a rational mapping. It is also true that f is rational in case A is of the form $A = \bar{A} - D$ where $\bar{A}$ is smooth and complete and D is a divisor on V such that the *Levi form* of the *normal bundle* of D has at least one negative eigen-value. This is the so-called *pseudo-concave case* and arises, for example, if $D \subset \bar{A}$ is obtained by blowing up a subvariety of codimension at least two in some other completion $\bar{A}'$ of A. The opposite, and most interesting, extreme is when A is an affine variety (e.g. a finitely sheeted algebraic covering of $\mathbf{C}^n$).

THEOREM A″. *Let A and V be as just above and let $f: A \to V$ be an arbitrary holomorphic mapping. Given a finite set of points $x_1, \cdots, x_N$ on V there are associated deficiency numbers $\delta(x_\alpha)$ with $0 \leqq \delta(x_\alpha) \leqq 1$ such that, if f is a transcendental mapping, we have the Nevanlinna inequality*

$$\sum_{\alpha=1}^{N} \delta(x_\alpha) \leqq \chi(V) \tag{6.6}$$

where $\chi(V)$ is the Euler-Poincaré characteristic of V. In particular, there is no such transcendental mapping if $\chi(V) < 0$.[19]

Thus we see that Nevanlinna theory implies, among many other things, both the usual Picard theorem as well as Theorem A′ (also due to Picard). In my opinion then, the proper understanding of Theorem A′ comes about through the general study of the value distributions of holomorphic mappings. *The same should be true of Theorem* A; that is to say, eventually Theorem A should be a consequence of a general understanding of the value distribution of holomorphic mappings in several complex variables.

The beginnings of such a general theory have been given by Chern [5] and Wu [14] among others. The situation is greatly complicated by the *Fatou-Bieberbach example* (cf. Bochner-Martin [2, p. 45]) of a one-to-one holomorphic mapping $f: \mathbf{C}^2 \to \mathbf{C}^2$ such that the image $f(\mathbf{C}^2)$ omits an open set. Thus it would seem that a satisfactory study of the value distributions of $f: A \to V$ can take place only for those mappings f which have been *rigidified* in some sense. For example, Wu's theorem that a non-degenerate holomorphic mapping $f: \mathbf{C}^n \to \mathbf{P}_n$ does not omit an open set if f is *uniformly quasi-conformal* is an example of one such rigidification.

Now it is also possible to "rigidify the situation" $f: A \to V$ by placing restrictions on the domain variety A and / or the image variety V. For example, referring to footnote (18) we see that such rigidification occurs when we assume that A is pseudo-concave (cf. [14, Th. 4.1]). At the other end, we should perhaps explain Theorem 6.2 by saying that assuming V to be *negatively curved* rigidifies the situation $f: A \to V$ so that f is in fact rational. Similarly, we might explain Theorem A as saying the assumption that V is a canonical algebraic manifold plus the assumption that we are in the equi-dimensional case and f is non-degenerate again forces a rigidification of the situation $f: A \to V$ (cf. Proposition 5.1). In general, assuming A to be affine, we may perhaps expect a nice value distribution theory for $f: A \to V$ by placing as-

[19] Obviously f is defined to be *transcendental* if it is *not* a rational mapping. Since $\delta(x_\alpha) = 1$ if f does not assume the value x_α, we see that the Nevanlinna inequality (6.6) implies the usual Picard theorem by taking $V = \mathbf{P}_1$. (The "usual Picard theorem" is the statement that a non-constant entire meromorphic function cannot omit three values.)

sumptions both on V and f, and where the stronger assumptions on f come with weaker assumptions on V. From this point of view, the understanding of the situation $f: \mathbf{C}^n \to \mathbf{P}_n$ would be the most difficult and, for example, it might be profitable to look at holomorphic mappings $f: A \to V$ when V is an *abelian variety*.[20]

(c) *Remarks on Theorem* B *as related to the general problem of parametrizing an algebraic variaty.* Let V be a complete n-dimensional algebraic manifold. We define a *parametrization* of V to be given by a holomorphic mapping $f: M \to V$ where M is an n-dimensional Stein manifold which is topologically a cell, and where the image $f(M)$ is required to contain a Zariski open subset of V. Presumably the most important cases are when $M = \mathbf{C}^n$ or when M is a contractible bounded domain in $\mathbf{C}^n$.

Example 1. When V is a *unirational variety*, we may take $M = \mathbf{C}^n$ and f a rational map.

Example 2. When V is an *abelian variety* we may take $M = \mathbf{C}^n$ and f is given by *multiple-periodic functions* relative to the period lattice of V.

Example 3. We may also take $M = \mathbf{C}^2$ when V is a special type of K–3 surface having a certain addition theorem (cf. Andreotti [1]).

Example 4. When V is a curve of genus larger than one, we may take $M = \Delta$ to be the unit disc and $f: \Delta \to V$ to be given by automorphic functions (*uniformization theorem*).

The above examples, especially 2, 3, and 4, illustrate the point that parametrizations are especially interesting to a complex analyst because of the nice meromorphic functions which turn up in giving the mapping $f: M \to V \hookrightarrow \mathbf{P}_N$.

We may think of Theorem B as restricting the possible parametrizations of a canonical algebraic manifold; it was this point of view which led me to study the questions in this paper.

In the context of parametrizations, we should like to mention the following theorem which we shall discuss at a later time:

(6.7) THEOREM. *Let V be an arbitrary smooth algebraic variety. Then there exists a bounded and contractible pseudo-convex domain $D \subset \mathbf{C}^n$ and a nowhere degenerate holomorphic mapping $f: D \to V$ of D onto a Zariski open subset of V. Furthermore, if we take a projective embedding $V \subset \mathbf{P}_N$ and*

[20] To an algebraic-geometer this perhaps makes sense because the birational theory of rational varieties is enormously complicated whereas the birational theory of abelian varieties and canonical varieties is somewhat more simple.

write $f(z) = (f_1(z), \cdots, f_N(z))$ *where* $f_1, \cdots, f_N$ *are meromorphic functions on* D, *then none of the* f_α *can be analytically continued as a meromorphic function across any part of the boundary of* D.

(**d**) *A distortion theorem as an example of further possible results about holomorphic mappings into canonical algebraic manifolds.* It is fairly clear that the methods used to prove Theorems A-C may be used to prove further analogues of classical results on conformal mapping. As an illustration of this, let me give the following n-dimensional version of *Koebe's Distortion Theorem* (cf. [7, p. 353]).

(6.8) THEOREM. *Let* $f: B \rightarrow V$ *be a holomorphic mapping of the unit ball in* $\mathbf{C}^n$ *into an* n*-dimensional canonical algebraic manifold. Assume that* f *satisfies the normalization conditions* $f(0) = x_0$ *and* $|Jf(0)| = 1$, *and let* $\mathrm{dist}_V(x, x')$ *be the distance function on* V *coming from an Hermitian metric* ds_V^2. *Then there exists* ρ_0 *with* $0 < \rho_0 \leqq 1$ *and a strictly increasing upper-semi-continuous function* $\varphi(\rho)$ *defined for* $0 \leqq \rho \leqq \rho_0$ *such that we have the estimate*

$$\varphi(||z||) \leqq \mathrm{dist}_V(x_0, f(z))$$

for all $z \in B(\rho_0)$ *and all holomorphic mappings* f *as above.*[21]

Proof. Let $\mathcal{F}$ be the class of mappings $f: B \rightarrow V$ satisfying the normalization conditions given above. Our proof is based on the following two lemmas:

(6.9) LEMMA. *There exists* ρ_1 *with* $0 < \rho_1 \leqq 1$ *such that, if* $\{f_n^\#\}$ *is any sequence in* $\mathcal{F}$, *there is a subsequence* $\{f_n\}$ *of* $\{f_n^\#\}$ *and a holomorphic mapping* $g: B(\rho_1) \rightarrow V$ *such that* f_n *converges to* g *uniformly on compact subsets of* $B(\rho_1)$.

Remark. This lemma may be compared with Proposition 5.1, which is essentially the same statement except that it is not proved there that we may choose the constant ρ_1 to work for *all* sequences in $\mathcal{F}$.

(6.10) LEMMA. *There is a constant* ρ_2 *with* $0 < \rho_2 \leqq 1$ *such that any* $f \in \mathcal{F}$ *is one-to-one on* $B(\rho_2)$.

We will complete the proof of Theorem 6.8 and then return to discuss the lemmas. Let $f \in \mathcal{F}$ and define $\varphi_f(\rho)$ for $0 \leqq \rho \leqq 1$ by

$$\varphi_f(\rho) = \inf_{||z||=\rho} \mathrm{dist}_V(x_0, f(z)) .$$

Then by Lemma 6.10 we may find an absolute constant ρ_3 with $0 < \rho_3 \leqq 1$ such that $\varphi_f(\rho)$ is continuous and strictly increasing on the interval $0 \leqq \rho \leqq \rho_3$. We now define $\varphi(\rho)$ by

$$\varphi(\rho) = \inf_{f \in \mathcal{F}} \varphi_f(\rho) .$$

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[21] By definition, $||z|| = (\sum_{j=1}^n |z_j|^2)^{1/2}$.

This function is defined and upper-semi-continuous for $0 \leqq \rho \leqq 1$. It is also clear that we have

$$\varphi(||z||) \leqq \operatorname{dist}_V(x_0, f(z))$$

for $0 \leqq ||z|| < 1$. It remains to prove that there is ρ_0 with $0 < \rho_0 \leqq 1$ such that $\varphi(\rho)$ is strictly increasing for $0 \leqq \rho \leqq \rho_0$.

To say what ρ_0 is, we observe that combining Lemmas 6.9 and 6.10 we may find an absolute constant ρ_4 with $0 < \rho_4 \leqq 1$ such that $\rho_4 \leqq \rho_1$ in Lemma 6.9 and such that, in the notation used in that lemma, all $f \in \mathcal{F}$ and all limits $g\colon B(\rho_1) \to V$ of sequences from $\mathcal{F}$ are one-to-one on $B(\rho_4)$. We now define ρ_0 to be minimum of ρ_3 and ρ_4, and it remains to show that we have $\varphi(\rho) < \varphi(\rho')$ if $0 \leqq \rho < \rho' \leqq \rho_0$. Let $\{f_n^\#\}$ be a sequence in $\mathcal{F}$ and $z_n^\#$ a sequence of points with $||z_n^\#|| = \rho'$ such that $\lim_{n\to\infty} \operatorname{dist}_V(x_0, f_n^\#(z_n^\#)) = \varphi(\rho')$. By passing to subsequences, we may assume that $\{f_n\}$ converges uniformly on $B(\rho_0)$ to a one-to-one holomorphic mapping $g\colon B(\rho_0) \to V$, and we may furthermore assume that $\{z_n\}$ converges to some $z \in B(\rho_0)$ with $||z|| = \rho'$. Passing to the limit as $n \to \infty$ in the inequalities

$$\varphi(\rho) \leqq \operatorname{dist}_V(x_0, f_n(w)) < \operatorname{dist}_V(x_0, f_n(z_n)) ,$$

where $||w|| = \rho$, we arrive at

$$\varphi(\rho) \leqq \operatorname{dist}_V(x_0, g(w)) \leqq \operatorname{dist}_V(x_0, g(z)) = \varphi(\rho') .$$

Since g is one-to-one we must now have the desired inequality $\varphi(\rho) < \varphi(\rho')$.

Q.E.D.

We will now prove Lemma 6.9 only, as the proof of Lemma 6.10 is similar to, but easier than, the corresponding proof of Theorem C from Proposition 5.1 (cf. [12, proof of Th. B]). Referring to Proposition 5.1, we see that if Lemma 6.9 were false, then we would be able to find a double sequence $\{f_{k,l}\}$ of maps in $\mathcal{F}$ and a sequence of meromorphic mappings $g_k\colon B(\rho_k) \to V$ such that we have (i) $\lim_{l\to\infty} f_{k,l} = g_k$ uniformly on compact subsets of $B(\rho_k)$, (ii) g_k is not holomorphic in a ball of radius larger than ρ_k, and (iii) $\lim_{k\to\infty} \rho_k = 0$. Letting $\omega_0, \cdots, \omega_N$ be an orthonormal basis for $H^0(V, \mathcal{O}(\mathbf{K}_V))$, we may give $f_{k,l}$ and g_k explicitly by using the homogeneous coordinate mappings

$$z \longrightarrow [f_{k,l}^*\omega_0(z), \cdots, f_{k,l}^*\omega_N(z)]$$
$$z \longrightarrow [g_k^*\omega_0(z), \cdots, g_k^*\omega_N(z)] .$$

The fact that g_k cannot be extended beyond $B(\rho_k)$ means that we will have $g_k^*\omega_\alpha(z_k) = 0$ $(\alpha = 0, \cdots, N)$ for some z_k with $||z_k|| = \rho_k$.

Referring to the proof of Proposition 5.1, we see that the holomorphic n-forms $g_k^*\omega_\alpha$ are locally uniformly bounded in the L_2-sense on B. Thus we may assume that the sequence $\{g_k^*\omega_\alpha\}$ converges uniformly on compact subsets of B. Letting $\mu_V = i^{n^2+n}\{\sum_{\alpha=0}^{N} \omega_\alpha \wedge \bar{\omega}_\alpha\}$ be the canonical volume element on V, the normalization condition gives at $z = 0$ that $g_k^* \mu_V(0) = i^{n^2+n} dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n$. This contradicts the assumption that $g_k^* \mu_V(z_k) = 0$ and $\lim_{k\to\infty} z_k = 0$. Q.E.D.

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[22] *Added in proof*: It has been pointed out to me by Bombieri that a small modification in the proof gives Theorem A if we only assume that the canonical bundle of V is positive in the sense of Kodaira. This result is then fairly sharp, as there are easy examples of non-constant transcendental holomorphic mappings of C into algebraic surfaces with positive canonical bundle.