# Hyperbolic Exterior Differential Systems and their Conservation Laws*, Part I** 

R. Bryant, P. Griffiths and L. Hsu

## Introduction

This paper falls under the general subject of the geometric theory of differential equations. The theory, founded in the last century by Lie and Darboux and extensively developed by Goursat, Cartan, and others, seeks to understand differential equations through the study of their invariants under suitable groups of coordinate transformations (such as contact, point or gauge transformations). The goals of the theory are, first, to understand interesting special equations through explicit solutions or algorithms for solutions, or other properties such as conservation laws or estimates; and, second, to study the geometry of differential equations as a subject of interest in its own right (like Riemannian or CR geometry).

Much of the classical theory is centered on the attempt to describe the general solution of a given system of partial differential equations in some reasonably explicit way. The method of Darboux (a generalization of the method of characteristics which is so familiar in the study of hyperbolic systems) is perhaps the most successful of these techniques developed in the classical theory. However, the geometric methods developed during this period have significance even when no explicit general solution can be found. The application of these methods is in its infancy as regards modern issues in differential equations, such as inference of properties of solutions (especially global ones) rather than the explicit construction of solutions.

The variety of phenomena studied in the theory of partial differential equations is, of course, very great. To get to the deeper aspects of the subject, it is necessary to specialize to some extent. Thus, for example, while hyperbolic and elliptic systems share some very basic features, the sorts of interesting problems that one poses for these two classes of equations are very different. Naturally, this happens in geometry as well, with Euclidean and Lorentzian geometries bearing

[^0]superficial resemblances, but having quite different deeper behaviors. It would be unrealistic to expect that the geometric method applied to the study of partial differential equations would not exhibit the same sort of divisions. For this reason, we have opted in this paper to specialize the class of equations to be studied.

In fact, we shall introduce the concept of a hyperbolic exterior differential system, a proper generalization of the classical case of second order hyperbolic PDE in the plane, and shall begin the study of the conservation laws and other geometric properties of such systems. The conservation laws provide an interesting intrinsic invariant of the hyperbolic system and are of use in understanding its solutions. We expect the class of hyperbolic exterior differential systems to provide a good case study within the general program.

Hyperbolic exterior differential systems are divided into the sets $\mathcal{H}_{s}$ of systems of class $s$ where $s$ may be any non-negative integer. The classes $s=1$ and $s=3$ include hyperbolic Monge-Ampere systems and second order hyperbolic equations for one unknown function $z(x, y)$, respectively. The class $s=2$ includes the case of a hyperbolic pair of first order equations for unknowns $u(x, y)$ and $v(x, y)$. The various classes are inter-related by the constructions of prolongation and (more subtly) integrable extensions. ${ }^{1}$

A hyperbolic system of class $s=0$ is given by a transverse pair of decomposable 2 -forms $\Omega_{1}, \Omega_{2}$ on a 4 -manifold $M$ - thus we have

$$
\left\{\begin{array}{c}
\Omega_{1} \wedge \Omega_{1}=0=\Omega_{2} \wedge \Omega_{2} \\
\Omega_{1} \wedge \Omega_{2} \neq 0
\end{array}\right.
$$

This simple structure turns out to have a very rich geometry and appears in several guises in the course of this paper. Moreover, the application of the geometric method leads to some very special and interesting PDE systems, such as (1), (2), and (3) below.

In general, a hyperbolic exterior differential system $(M, \mathcal{I})$ of class $s$ is given by a differential ideal $\mathcal{I}$ on a manifold $M$ of dimension $s+4$ where $\mathcal{I}$ is generated algebraically by a rank $s$ Pfaffian system $I$ and a transverse pair of 2 -forms that are decomposable modulo $I$. The $k^{\text {th }}$ prolongation of $(M, \mathcal{I}) \in \mathcal{H}_{s}$ is a hyperbolic system $\left(M^{(k)}, \mathcal{I}^{(k)}\right) \in \mathcal{H}_{s+2 k}$. In order to understand the most interesting systems, which are those of classes $s=0,1,2$ and 3 , it seems to be advantageous to consider the whole set of hyperbolic exterior differential systems and its inter-relationships.

The most important objects associated to a hyperbolic system are the characteristic systems $\Xi_{1}, \Xi_{2}$ and their prolongations $\Xi_{1}^{(k)}, \Xi_{2}^{(k)}$. Each of $\Xi_{1}^{(k)}$ and $\Xi_{2}^{(k)}$ is

1) A system of partial differential equations canonically gives rise to an exterior differential system. Differentiation of the PDE system then corresponds to prolongation of the exterior differential system. An integrable extension roughly corresponds to adjoining the primitive of a conservation law as a new variable - in the setting of exterior differential systems this may be done in a canonical way.
a Pfaffian system of rank $s+2+k$ on the $(s+4+2 k)$-dimensional manifold $M^{(k)}$ and their geometry - especially their integrable subsystems - is the dominant geometric feature of the hyperbolic system. Classically the characteristic systems arose in attempting to find explicit solutions of hyperbolic PDEs of Monge-Ampere type. Later they were used by Riemann to produce explicit integral formulas for certain special hyperbolic systems. The synthesizing concept of Darboux integrability was introduced by Darboux in 1870. A hyperbolic system is Darboux integrable at level $k$ if there are rank 2 integrable systems

$$
\Delta_{1} \subset \Xi_{1}^{(k)}, \quad \Delta_{2} \subset \Xi_{2}^{(k)}
$$

which are transverse to the ideal $\mathcal{I}^{(k)}$. The solution to the Cauchy problem for such systems may be reduced to ODEs. Many interesting equations are Darboux integrable and first integrals may be found explicitly, leading to explicit forms for the general solution of the equation (well-known examples are the Liouville equation and the Weierstrass formulas for minimal surfaces).

An integral surface of a hyperbolic system $(M, \mathcal{I})$ is an immersed surface $S \subset$ $M$ such that all the forms in $\mathcal{I}$ pull-back to be zero on $S$. The characteristic systems induce on any integral surface a pair of foliations by curves. The initial value problem for such a system is the problem of how to extend a given integral curve of $\mathcal{I}$ to an integral surface of $\mathcal{I}$. We show that if the integral curve is 'non-characteristic' in the appropriate sense, then local solutions to the initial value problem always exist. More interestingly, the geometry of the characteristic curves allows us to intrinsically define a condition (which we call characteristic completeness) that is equivalent to the existence of global solutions to the initial value problem. The exterior differential system associated to many non-linear PDEs, such as the famous Fermi-Pasta-Ulam equation

$$
z_{y y}-\left(k\left(z_{x}\right)\right)^{2} z_{x x}=0
$$

where $k: \mathbb{R} \rightarrow \mathbb{R}$ is a diffeomorphism, are shown to admit unique global integral surfaces even though singularities necessarily develop for solutions to the PDE.

In this paper we are especially interested in the space $\mathcal{C}$ of conservation laws for a hyperbolic exterior differential system. From the general theory $\left[\mathrm{BG}_{1}\right]$ we know that (i) conservation laws have a normal form derived from the symbol of the exterior differential system, (ii) $\mathcal{C}$ is naturally given as the kernel of a linear differential operator (thus, we eliminate the trivial conservation laws), and (iii) $\mathcal{C}$ is filtered by subspaces $\mathcal{C}_{k}$ where $\Phi \in \mathcal{C}_{k}$ means that $\Phi$, assumed in normal form, is defined on $M^{(k)} .^{2}$ One of the principal objectives of this paper is the study of conservation laws of hyperbolic systems of class $s=0$.
2) Traditionally, a conservation law is given by a 1 -form $\varphi$ involving the unknown function $z$ together with its derivatives $z_{x}, z_{y}, z_{x x}, \ldots$ up to some finite order such that $d \varphi=0$ whenever we substitute in a solution to the equation. For us conservation laws are given by closed 2 -form $\Phi \in \mathcal{I}$; writing locally $\Phi=d \varphi$ gives the usual conservation law. In Section 2.1 we have included an introduction to the theory of conservation laws so that this paper can be read independently of $\left[B G_{1}\right]$.

Our main general results on conservation laws are:
(i) $\mathcal{C}_{2 k}=\mathcal{C}_{2 k-1}$, thus new conservation laws can be added only at odd levels;
(ii) $\Phi \in \mathcal{C}_{2 k+1}$ has a highest order part $B=\left(B_{1}, B_{2}\right)$ with the properties that $B$ uniquely determines $\Phi$ modulo $\mathcal{C}_{2 k-1}$ and that for a pair $\Phi, \bar{\Phi} \in \mathcal{C}_{2 k+1}$

$$
\left\{\begin{array}{l}
d\left(B_{1} / \bar{B}_{1}\right) \in \Xi_{1}^{(k)} \\
d\left(B_{2} / \bar{B}_{2}\right) \in \Xi_{2}^{(k)}
\end{array}\right.
$$

Thus, there is a direct relationship between conservation laws and integrable subsystems of the characteristic systems. This relationship has several consequences.

At one extreme, we can add (at least) two functions of two variables worth of new conservation laws when we pass from $\mathcal{C}_{2 k-1}$ to $\mathcal{C}_{2 k+1}$, and this many are added if the hyperbolic system is Darboux integrable at this (or possibly a lower) level. The converse to this statement is quite plausible but we have not attempted to formulate and prove it in this paper. At the other extreme, if there are no integrable subsystems of the characteristic systems at level $2 k+1$, then

$$
\operatorname{dim} \mathcal{C}_{2 k+1} / \mathcal{C}_{2 k-1} \leqq 2
$$

On the other hand, the $s=0$ sine-Gordon system

$$
\begin{align*}
u_{x} & =\sin v \\
v_{y} & =\sin u \tag{1}
\end{align*}
$$

has rank one integrable subsystems of the characteristic systems and is shown to have the property that $\operatorname{dim} \mathcal{C}_{0}=1$ while $\operatorname{dim} \mathcal{C}_{1} / \mathcal{C}_{0}=3$. Perhaps most interesting will be those hyperbolic systems which satisfy $\operatorname{dim} \mathcal{C}=\infty$ but $\operatorname{dim} \mathcal{C}_{k}<\infty$ for all $k$. The study of such systems will be the objective of a future paper in this series.

In this paper we shall work primarily with systems that are symmetric in that they exhibit symmetric behavior in the characteristic systems (the precise definition is given in Section 1.5). For these systems we shall determine those hyperbolic systems of class $s=0$ for which the space $\mathcal{C}_{0}$ of level zero conservation laws or as we shall say classical conservation laws - has infinite dimension. ${ }^{3}$ The word "determine" here has two meanings: First, we shall find the conditions imposed on the invariants of the system in order that $\operatorname{dim} \mathcal{C}_{0}=\infty$. In practice, this gives an algorithm for checking whether or not a given PDE system has an infinite number of classical conservation laws. Secondly, we shall derive a normal form for hyperbolic systems having $\operatorname{dim} \mathcal{C}_{0}=\infty$. Among the corollaries of our analysis is the result:
If $\operatorname{dim} \mathcal{C}_{0} \geqq 7$, then $\operatorname{dim} \mathcal{C}_{0}=\infty$. There is exactly one non-linear symmetric hyperbolic exterior differential system having an infinite number of classical conservation

[^1]laws; namely, the exterior differential system associated to the $s=0$ Liouville system
\[

$$
\begin{align*}
u_{y} & =e^{v} \\
v_{x} & =e^{u} . \tag{2}
\end{align*}
$$
\]

Among hyperbolic exterior differential systems of class $s=0$ especially interesting are those of Euler-Lagrange type. Given a symplectic 4-manifold ( $M, \Phi$ ) and a. 2 -form $\Lambda$, the critical points of the functional

$$
S \rightarrow \int_{S} \Lambda
$$

defined on $\Phi$-Lagrangian surfaces are solutions to an exterior differential system

$$
\Phi=\Psi=0
$$

where $\Psi$ is a closed 2 -form constructed from $\Lambda$. It turns out that this theory has a striking internal symmetry which exchanges the roles of $\Phi$ and $\Psi$. The $s=0$ Goursat systems

$$
\begin{align*}
& u_{y}=\frac{c \sqrt{u v}}{(x+y)}  \tag{3}\\
& v_{x}=\frac{c \sqrt{u v}}{(x+y)}, \quad c \in \mathbb{R}^{*}
\end{align*}
$$

are 'half' of a certain one-parameter family of systems which may be uniquely characterized as being Euler-Lagrange in a two parameter family of geometrically distinct ways - the maximum possible for hyperbolic systems of class $s=0$. A variant of Noether's theorem gives an isomorphism between the symmetries and conservation laws modulo $\Phi, \Psi$ of Euler-Lagrange systems, and we show that in this case $\mathcal{C}_{0}$ is infinite dimensional if, and only if, the system is linearizable. ${ }^{4}$

The study of Euler-Lagrange systems and other considerations suggests the importance of symplectic hyperbolic systems ( $M, \Phi, \mathcal{I}$ ), this being a hyperbolic system $(M, \mathcal{I})$ together with a symplectic form $\Phi$ such that $\Phi \in \mathcal{I}$. Automorphisms of such systems must preserve both $\mathcal{I}$ and $\Phi$. We show that there are three classes of (symmetric) symplectic hyperbolic systems having an infinite number of classical conservation laws. These are:
4) One of the unexpected discoveries of our study is the seeming ubiquity of linear hyperbolic systems. Of course, a highly non-linear PDE may be linearizable as an exterior differential system, as is the case for the system (3); moreover, an integrable extension of a non-linear PDE may linearize it, as is the case for (2). The fact that many interesting PDEs may be linearized in a variety of ways (hodographic transformation, Legendre transformations) is of course classical [CH]. The setting of exterior differential systems appears to synthesize and extend these classical methods.

Class A: consisting of systems that may be symplectically linearized;
Class B: consisting of symplectic hyperbolic systems that become Darboux integrable at level one (such as the system modeled on (2) above); and
Class C: consisting of symplectic hyperbolic systems modeled on the non-linear PDE system

$$
\begin{aligned}
u_{y} & =F(x, y) \sqrt{u v} \\
v_{x} & =F(x, y) \sqrt{u v}
\end{aligned}
$$

Note that when $F(x, y)=c /(x+y)$ we recover the $s=0$ Goursat systems above.
Our technique of proof is to use the equivalence method of É. Cartan to introduce a canonical $G$-structure and class of pseudo-connections intrinsically associated to the hyperbolic system. This describes the intrinsic "geometry" associated to a PDE system up to contact equivalence, and associated to this geometry are the invariants or "curvatures" of the system. The condition that $\mathcal{C}_{0}$ have a certain structure - e.g., that there be infinitely many classical conservation laws - then imposes constraints on these invariants. Some of these constraints have direct geometric meaning - for example, if the characteristic systems do not each have an integrable subsystem then $\operatorname{dim} \mathcal{C}_{0} \leq 5$. Others are expressed by algebraic conditions imposed on the various "curvatures". As an application, using the general theory we may easily write down the explicit form of the conservation laws for explicitly given systems such as (1), (2), or (3). For example, for (2) the conservation laws are

$$
\begin{equation*}
\varphi=f(x)\left(d u-e^{v} d y-e^{u} d x\right)+g(y)\left(d v-e^{u} d x-e^{v} d y\right) \tag{4}
\end{equation*}
$$

where $f$ and $g$ are arbitrary functions, each of one variable.
A very interesting issue is the extent to which these intrinsic invariants relate to the more traditional estimates in PDE theory. The ultimate objective of each is, of course, to provide a means to understand the "solutions" of the system. As an illustration, we use the conservation laws (4) to infer that singularities of solutions to (2) are essentially of the form

$$
\begin{aligned}
& u(\xi, t)=-c \log \xi+(\text { regular terms }) \\
& v(\xi, t)=c \log \xi+\text { (regular terms) }
\end{aligned}
$$

for some constant $c>0$, in space-time coordinates $(\xi, t)$ as $\xi \downarrow 0$.
We also use the method of equivalence to answer a number of natural geometric questions concerning hyperbolic systems of class $s=0$. Thus, in Section 1.5, we give necessary and sufficient conditions, expressed in terms of the torsion and curvature of the system, that it be linearizable. For example, this general result tells us that (3) is linearizable. This should be compared with the usual Goursat equation

$$
z_{x y}=\frac{c \sqrt{z_{x} z_{y}}}{x+y}
$$

which is obtained from the linearized form of (3) by a non-linear integrable extension; it is well-known that the usual Goursat equation is not linearizable within the class $\mathcal{H}_{1}$. A linearizable system has a pair of intrinsic curvatures $K$ and $F$ and those for which $K$ and $F$ are constant, say $K=c^{2}>0$ and $F=K \beta$ are locally equivalent to the exterior differential system associated to the PDE system

$$
\begin{align*}
u_{x} & =\frac{v}{\cos ^{1+\beta}(c(x+y))} \\
v_{y} & =\frac{u}{\cos ^{1-\beta}(c(x+y))} \tag{5}
\end{align*}
$$

Another natural question is: When is a hyperbolic exterior differential system Darboux integrable at level one? Both the systems (2) and (3) (with $c=2$ ) satisfy these conditions and once the general result is known, the explicit integration may be carried out. (In fact, guided by the general theory, the explicit linearization and/or integration of examples such as the above may seem more straightforward than some of the calculations in the classical theory (cf. [Go]).) We show that there are, in fact, only two equivalence classes of hyperbolic systems of class $s=0$ that are Darboux integrable at level one, namely that corresponding to (2) and linear equations with constant curvature.

We also want to remark that this may be related to the interesting recent paper of Anderson and Kamran [AK] which studies the conditions that a hyperbolic system of class $s=3$ (e.g., a scalar second order hyperbolic equation for one function of two independent variables) be Darboux integrable at any level.

This paper is part of the general subject of the "geometry" associated to a differential equation. By geometry, we mean a $G$-structure together with an intrinsic class of pseudo-connections. The structure given by a 4 -manifold together with a pair of transverse, non-integrable 2-plane fields is sufficiently simple that we are able to study it in an essentially self-contained manner using the general theory as a guide. It is, on the other hand, a very rich structure and the study in this paper is incomplete in two major aspects. The first is the analysis of systems that admit an infinite number of conservation laws of all levels. The second is the relation of the geometry of the unique global smooth integral surfaces "upstairs" with given noncharacteristic initial data to the existence and uniqueness of global shock solutions to hyperbolic PDE systems of conservation laws with that same initial data. Both of these topics have been introduced and illustrated in the present work, but their satisfactory understanding is far from complete at this time.

## §1 Hyperbolic Exterior Differential Systems

### 1.1 Basic definition and examples.

1.1.1 Some terminology. First, for convenience of the reader, we will recall some of the basic ideas of the theory of exterior differential systems. A much more extensive treatment of these concepts can be found in $\left[\mathrm{BCG}^{3}\right]$.

An exterior differential system is given by a manifold $M$ and a differential ideal $\mathcal{I} \subset \Omega^{*}(M)$ in the algebra of smooth differential forms on $M$. We denote the exterior differential system by $(M, \mathcal{I})$. Recall that by definition the differential ideal $\mathcal{I}$ is homogeneous in the sense that

$$
\mathcal{I}=\oplus_{q \geq 0} \mathcal{I}^{q},
$$

where $\mathcal{I}^{q}=\mathcal{I} \cap \Omega^{q}(M)$ is the space of $q$-forms in $\mathcal{I}$, and that $\mathcal{I}$ is closed under exterior differentiation $d: \mathcal{I} \rightarrow \mathcal{I}$.

A symmetry of an exterior differential system $(M, \mathcal{I})$ is a diffeomorphism $f: M \rightarrow M$ which satisfies $f^{*} \mathcal{I}=\mathcal{I}$. An exterior differential system $(M, \mathcal{I})$ is said to be equivalent to an exterior differential system $(\tilde{M}, \tilde{\mathcal{I}})$ in case there is a diffeomorphism of $M$ with $\tilde{M}$ which takes $\tilde{\mathcal{I}}$ to $\mathcal{I}$.

Given a set $\theta_{1}, \theta_{2}, \ldots$ of forms of degrees $q_{1}, q_{2}, \ldots$, we denote by $\left\{\theta_{1}, \theta_{2}, \ldots\right\}$ the algebraic ideal they generate in $\Omega^{*}(M)$. The differential ideal generated by $\theta_{1}, \theta_{2}, \ldots$ is then denoted

$$
\mathcal{I}=\left\{\theta_{1}, \theta_{2}, \ldots ; d \theta_{1}, d \theta_{2}, \ldots\right\}
$$

In practice, differential ideals are locally generated by a finite set of forms in this way. In case all the forms $\theta_{1}, \theta_{2}, \ldots$ have degree one, $\mathcal{I}$ is called a Pfaffian system.

An integral manifold of an exterior differential system $(M, \mathcal{I})$ is an immersed submanifold $f: N \rightarrow M$ which satisfies $f^{*} \theta=0$ for all $\theta \in \mathcal{I}$. When written out in local coordinates, this condition is a system of partial differential equations for the mapping $f$. Conversely, a sufficiently regular PDE system gives rise, in a canonical way, to an exterior differential system such that the solutions to the PDE system and the integral manifolds of the exterior differential system which satisfy a transversality condition are locally in one-to-one correspondence.

However, the notion of equivalence of exterior differential systems is different from that for partial differential equations. In particular, the symmetry group of an exterior differential system can be strictly larger than that of the PDE system from which it arises. This important point is explained and illustrated in Section 1.2 of [ $\mathrm{BG}_{1}$ ]; a consequence is that an exterior differential system has fewer invariants than the generating PDE system. Conforming to classical terminology, we shall call the symmetries of an exterior differential system contact transformations.

As always in the theory of exterior differential systems, we shall, without further mention, make suitable constant rank assumptions. For example, if $\theta_{1}, \theta_{2}, \ldots$
are 1 -forms generating a Pfaffian system, then for each point $x \in M$ the values $\theta_{1}(x), \theta_{2}(x), \ldots$ span a linear subspace $I_{x} \subset T_{x}^{*} M$. We shall assume that $\operatorname{dim} I_{x}$ is constant (i.e., independent of $x$ ) and shall denote by $I$ either the corresponding sub-bundle of $T^{*} M$ or the sub-module of $\Omega^{1}(M)$ generated by $\theta_{1}, \theta_{2}, \ldots$; the context should make clear which of these interpretations to use. We shall generally use the notation

$$
I=\left[\theta_{1}, \theta_{2}, \ldots\right]
$$

in either of the above two senses. (In general, the square bracket $\left[\theta_{1}, \theta_{2}, \ldots\right]$ will denote the linear span over the functions of a set $\theta_{1}, \theta_{2}, \ldots$ of differential forms.)

As another illustration of implicit constant rank assumptions, suppose we are given a Pfaffian system generated by $I \subset \Omega^{1}(M)$. Then the exterior derivative induces a $C^{\infty}(M)$-linear mapping

$$
\delta: I \rightarrow \Omega^{2}(M) /\{I\}
$$

where we recall our convention that $\{I\}$ is the algebraic ideal generated by $I$ (since the meaning is clear, we do not use the more correct but clumsy notation $\{I\} \cap$ $\Omega^{2}(M)$ in the denominator above). We shall assume without further mention that $\delta$ has pointwise constant rank. Then $\operatorname{ker} \delta=I^{\{1\rangle}$ generates another Pfaffian system called the first derived system of $I$. Setting

$$
I^{\langle 2\rangle}=\left(I^{\langle 1\rangle}\right)^{\langle 1\rangle}
$$

and so forth leads to the derived flag

$$
I \supset I^{\langle 1\rangle} \supset I^{\langle 2\rangle} \supset \cdots
$$

of $I$. We shall think of the $I^{\langle k\rangle}$ as either sub-bundles of $T^{*} M$ or sub-modules of $\Omega^{1}(M)$. This construction will play a central role below.
1.1.2 Hyperbolic systems. We will now define the main object of study of this paper, this being a hyperbolic exterior differential system - or briefly a hyperbolic system. This is a special type of exterior differential system which is meant to capture the essential features of the classical theory of hyperbolic (systems of) PDE in the plane. We shall give a more intrinsic definition below, but, informally stated, a hyperbolic system is an exterior differential system $(M, \mathcal{I})$ where $M$ is a manifold of dimension $s+4$ and $\mathcal{I}$ is a differential ideal with the property that every point of $M$ lies in a neighborhood $U$ on which there exists a coframing (i.e., a basis of 1-forms)

$$
(\theta ; \omega)=\left(\theta^{1}, \ldots, \theta^{s} ; \omega^{1}, \omega^{2}, \omega^{3}, \omega^{4}\right)
$$

so that, on $U$, the ideal $\mathcal{I}$ is generated algebraically in the form

$$
\mathcal{I}=\left\{\theta^{1}, \ldots, \theta^{s} ; \omega^{1} \wedge \omega^{2}, \omega^{3} \wedge \omega^{4}\right\}
$$

Such a coframing $(\theta ; \omega)$ will be referred to as an admissable local coframing for $\mathcal{I}$. The condition that $\mathcal{I}$ be locally generated in the stated fashion implies that, for any admissable local coframing, there are relations of the form

$$
d \theta^{\alpha} \equiv A^{\alpha} \omega^{1} \wedge \omega^{2}+B^{\alpha} \omega^{3} \wedge \omega^{4} \bmod \left\{\theta^{\alpha}\right\}
$$

about which we will have more to say in Section 1.5.
It is not immediately apparent what a hyperbolic system in this sense has to do with hyperbolic PDE. However, as we will show by examples later in this section, many classical or well-known hyperbolic PDE systems for functions of two independent variables can be reformulated as hyperbolic systems in our sense.

From now on, $\mathcal{I}$ will denote a hyperbolic system on $M$. We will call $s$ the class of the hyperbolic system under discussion. Note that $s$ may be any non-negative integer. It is easy to see that our assumptions imply that $\mathcal{I}^{q}=\Omega^{q}(M)$ for all $q \geq 3$. In particular, it follows that the dimension of an integral manifold of $\mathcal{I}$ is at most 2 . Integral surfaces of $\mathcal{I}$ will generally be referred to as solution surfaces of $\mathcal{I}$.

We can give a more "intrinsic" description of what a hyperbolic system is, but first we want to make two remarks.

The first remark concerns the ambiguity in the choice of an admissable local coframing for $\mathcal{I}$. If $(\theta ; \omega)$ and $(\bar{\theta} ; \bar{\omega})$ are two admissable local coframings on the same domain in $M$, then, by inspection, we see that the span of the 1 -forms $\theta^{\alpha}$ and that of the 1 -forms $\bar{\theta}^{\alpha}$ must be equal. Thus there must exist a globally defined Pfaffian system $I \subset T^{*} M$ of rank $s$ of which the $\theta^{\alpha}$ in any local admissable coframing are local sections. Moreover, as is easy to see, there must exist non-zero functions $\lambda$ and $\mu$ so that either

$$
\left.\left.\begin{array}{l}
\bar{\omega}^{1} \wedge \bar{\omega}^{2} \equiv \lambda \omega^{1} \wedge \omega^{2} \\
\bar{\omega}^{3} \wedge \bar{\omega}^{4} \equiv \mu \omega^{3} \wedge \omega^{4}
\end{array}\right\} \bmod \theta^{\alpha} \quad \text { or } \quad \begin{array}{l}
\bar{\omega}^{1} \wedge \bar{\omega}^{2} \equiv \lambda \omega^{3} \wedge \omega^{4} \\
\bar{\omega}^{3} \wedge \bar{\omega}^{4} \equiv \mu \omega^{1} \wedge \omega^{2}
\end{array}\right\} \bmod \theta^{\alpha}
$$

Our second remark is about linear algebra. Let $V$ be any vector space of dimension four. A non-zero 2 -form $\Omega \in \Lambda^{2} V^{*}$ is by definition decomposable if it can be written in the form $\Omega=\omega^{1} \wedge \omega^{2}$ for some $\omega^{1}, \omega^{2} \in V^{*}$. For a non-zero $\Omega$, decomposability is equivalent to the condition $\Omega \wedge \Omega=0$. Because we are in dimension 4, a non-zero decomposable 2-form $\Omega=\omega^{1} \wedge \omega^{2}$ determines a 2-plane

$$
\Omega^{\perp}=\left\{v \in V: \omega^{1}(v)=\omega^{2}(v)=0\right\}
$$

We will say that a pair of non-zero decomposable 2 -forms $\left(\Omega_{1}, \Omega_{2}\right)$ is transverse if the corresponding 2-planes are transverse: i.e., $\Omega_{1}^{\frac{1}{1}} \cap \Omega \frac{1}{2}$ is a point. It is easy to see that this is equivalent to the condition $\Omega_{1} \wedge \Omega_{2} \neq 0$. In this case, there always exists a basis ( $\omega^{1}, \omega^{2}, \omega^{3}, \omega^{4}$ ) of $V^{*}$ so that $\Omega_{1}=\omega^{1} \wedge \omega^{2}$ and $\Omega_{2}=\omega^{3} \wedge \omega^{4}$.

A pencil of 2 -forms is, by definition, given by a line $L \subset \mathbb{P}\left(\Lambda^{2} V^{*}\right)$. If $\Phi_{1}, \Phi_{2} \in$ $\Lambda^{2} V^{*}$ are independent 2 -forms that generate the pencil, then elements of $L$ may be represented as 2 -forms $\Phi(\mu)=\mu_{1} \Phi_{1}+\mu_{2} \Phi_{2}$ where $\mu=\left[\mu_{1}, \mu_{2}\right]$ are homogeneous coordinates in $L \cong \mathbb{P}^{1}$. The decomposable elements of the pencil are given by solutions to the exterior equation

$$
\Phi(\mu) \wedge \Phi(\mu)=0
$$

Assuming that this equation is not identically satisfied for all $\mu$, it is a homogeneous quadratic equation in $\mu_{1}, \mu_{2}$. We will say that the pencil $L$ is hyperbolic in case this equation has distinct, real roots. In this case, it is not hard to see that there exists a transverse decomposable pair of generators for the pencil.

We now give the promised more intrinsic definition of a hyperbolic system.
Definition: A hyperbolic system of class $s$ is given by an exterior differential system $(M, \mathcal{I})$ where $M$ is a manifold of dimension $s+4$ and $\mathcal{I}=\oplus_{q>0} \mathcal{I}^{q}$ is a differential ideal satisfying
(i) $\mathcal{I}^{1}=I$ is a Pfaffian system of rank $s$;
(ii) $\mathcal{I}^{2} /\{I\}$ is a hyperbolic pencil at each point.

More explicitly, for each $x \in M$, the subspace $I_{x}^{\perp} \subset T_{x} M$ is a 4-dimensional vector space $V$ with dual $V^{*} \simeq T_{x}^{*} / I_{x}$. The values

$$
\Phi(x) \in \Lambda^{2}\left(T_{x}^{*} / I_{x}\right), \quad \Phi \in \mathcal{I}^{2} /\{I\}
$$

are well-defined. Condition (ii) then means that the 2-forms $\Phi(x)$ should then give a hyperbolic pencil.

Note that this condition has the effect of defining a $\mathbb{P}^{1}$-bundle $L \subset \mathbb{P}\left(\Lambda^{2}\left(T^{*} / I\right)\right)$, each fiber of which contains two special points, the decomposable elements of the pencil. These points fit together to form a smooth double cover of $M$. We are going to assume that this double cover is trivial, mainly for ease of exposition, though, in fact, this condition is satisfied in all of the interesting examples anyway.

In fact, by passing to a finite cover of $M$ (with index at most 8 ), we can and shall assume that there exists a pair of non-vanishing, decomposable 2 -forms $\Omega_{10}$ and $\Omega_{01}$ on $M$ so that $\mathcal{I}$ is generated algebraically by the sections of $I$ together with $\Omega_{10}$ and $\Omega_{01}$. This necessarily implies that $\Omega_{10} \wedge \Omega_{01} \neq 0$, since these 2 -forms must restrict to each $I_{x}^{\perp}$ to be a transverse pair. (The reason for the peculiar indexing of the $\Omega$ 's will be explained below.)
1.1.3 Some examples. We will now give some examples of hyperbolic systems, starting with those of class $s=0$. Such a system $\mathcal{I}$ is defined on a 4 -manifold $M$ by a pair $\Omega_{10}, \Omega_{01}$ of everywhere transverse decomposable 2 -forms.

Example 1: Over a domain $U \subset \mathbb{R}^{2}$ with coordinates $(x, y)$ we consider a quasilinear first order hyperbolic PDE system for two unknown functions $u$, $v$. Such a system is usually written in the form

$$
\begin{equation*}
\mathbf{A}(x, y, u, v)\binom{u_{x}}{v_{x}}+\mathbf{B}(x, y, u, v)\binom{u_{y}}{v_{y}}+\mathbf{C}(x, y, u, v)=\mathbf{0} \tag{1}
\end{equation*}
$$

where $\mathbf{A}$ and $\mathbf{B}$ are 2-by-2 matrices and $\mathbf{C}$ is a 2 -by- 1 column, with entries which are functions of $(x, y, u, v)$ defined on some open set $M \subset \mathbb{R}^{4}$.

The usual condition that a solution $(u, v)=(f(x, y), g(x, y))$ be hyperbolic (or, more properly speaking, that (1) be hyperbolic at the solution $(u, v)=$ $(f(x, y), g(x, y)))$ is that the quadratic form

$$
\begin{aligned}
Q_{(f, g)}\left(\xi_{1}, \xi_{2}\right) & =\operatorname{det}\left(\xi_{1} \mathbf{A}(x, y, f, g)+\xi_{2} \mathbf{B}(x, y, f, g)\right) \\
& =Q_{(f, g)}^{11} \xi_{1}^{2}+2 Q_{(f, g)}^{12} \xi_{1} \xi_{2}+Q_{(f, g)}^{22} \xi_{2}^{2}
\end{aligned}
$$

have real, distinct linear factors, i.e., its discriminant $\Delta_{(f, g)}=Q_{(f, g)}^{11} Q_{(f, g)}^{22}-$ $\left(Q_{(f, g)}^{12}\right)^{2}$ should be negative.

To write (1) as an exterior differential system in the open set $M \subset \mathbb{R}^{4}$, we consider the pair of 2 -forms $\Phi^{1}$ and $\Phi^{2}$ defined by the matrix equation

$$
\binom{\Phi^{1}}{\Phi^{2}}=\mathbf{A}(x, y, u, v)\binom{d u \wedge d y}{d v \wedge d y}-\mathbf{B}(x, y, u, v)\binom{d u \wedge d x}{d v \wedge d x}+\mathbf{C}(x, y, u, v) d x \wedge d y
$$

The graph in $M$ of a solution to (1) is easily seen to be an integral surface of the exterior differential system $\mathcal{I}$ generated by $\Phi^{1}$ and $\Phi^{2}$. Conversely, integral surfaces of $\mathcal{I}$ on which $d x \wedge d y$ is non-zero are locally graphs of solutions to (1). Thus, locally the solutions to the former are in one-to-one correspondence with those integral surfaces of the latter which satisfy a transversality condition.

Now,

$$
\left(\lambda_{1} \Phi^{1}+\lambda_{2} \Phi^{2}\right)^{2}=P\left(\lambda_{1}, \lambda_{2}\right) d u \wedge d v \wedge d x \wedge d y
$$

where $P=P^{11} \lambda_{1}^{2}+2 P^{12} \lambda_{1} \lambda_{2}+P^{22} \lambda_{2}^{2}$ is a quadratic form whose coefficients are functions of $x, y, u$, and $v$. Moreover, computation shows that its discriminant $D=P^{11} P^{22}-\left(P^{12}\right)^{2}$ has the property that, on the graph of a solution to (1), we have

$$
D(x, y, f(x, y), g(x, y))=\Delta_{(f, g)}
$$

It follows that the pair of 2 -forms $\Phi^{1}, \Phi^{2}$ span a hyperbolic pencil (and thus define a hyperbolic exterior differential system of class $s=0$ ) on the open subset of $M$ which has the property that it contains the graphs of the hyperbolic solutions of (1).

Example 2: Consider the general hyperbolic exterior differential system $I$ with $s=0$ generated by a hyperbolic pencil of 2 -forms $\left\{\Phi^{1}, \Phi^{2}\right\}$. In any local coordinate system $(x, y, u, v)$, there are expressions

$$
\begin{aligned}
& \Phi^{1}=A^{1} d u \wedge d y+B^{1} d x \wedge d u+C^{1} d v \wedge d y+D^{1} d x \wedge d v+E^{1} d u \wedge d v+F^{1} d x \wedge d y \\
& \Phi^{2}=A^{2} d u \wedge d y+B^{2} d x \wedge d u+C^{2} d v \wedge d y+D^{2} d x \wedge d v+E^{2} d u \wedge d v+F^{2} d x \wedge d y
\end{aligned}
$$

where the coefficient functions $A^{1}$, etc., can be essentially arbitrary functions of $x$, $y, u$, and $v$ (subject only to the conditions that the forms $\Phi^{1}$ and $\Phi^{2}$ generate a hyperbolic pencil). Then the integral surfaces of $\mathcal{I}$ to which, say, $x$ and $y$ restrict to be independent functions are locally graphs of the form $(x, y, u(x, y), v(x, y))$ where $u$ and $v$ satisfy the pair of first order equations

$$
\begin{aligned}
& 0=A^{1} u_{x}+B^{1} u_{y}+C^{1} v_{x}+D^{1} v_{y}+E^{1}\left(u_{x} v_{y}-v_{x} u_{y}\right)+F^{1} \\
& 0=A^{2} u_{x}+B^{2} u_{y}+C^{2} v_{x}+D^{2} v_{y}+E^{2}\left(u_{x} v_{y}-v_{x} u_{y}\right)+F^{2}
\end{aligned}
$$

Note that if the functions $E^{1}$ and $E^{2}$ vanish identically in this coordinate system, this reduces to a quasi-linear first order hyperbolic PDE system as studied in Example 1. In fact, in Section 1.1.4 below we shall show that if the system $\mathcal{I}$ is real analytic in some local coordinate system, then each point of $M$ lies in a neighborhood on which there exists a local coordinate system in which the generating 2 -forms $\Phi^{1}, \Phi^{2}$ have no $d u \wedge d v$ terms. Thus, at least in the real-analytic category (and, presumably in the smooth category as well, though we do not know this) the general hyperbolic system with $s=0$ is locally equivalent to a hyperbolic pair of first-order quasi-linear PDE.

Example 3: Not all hyperbolic systems with $s=0$ come directly from a first order system. Other natural constructions also yield these systems. Consider the second order Monge-Ampere equation

$$
E\left(z_{x x} z_{y y}-z_{x y}^{2}\right)+A z_{x x}+2 B z_{x y}+C z_{y y}+D=0
$$

where $A, B, C, D$, and $E$ are functions of $x, y, p=z_{x}$, and $q=z_{y}$ only, i.e, they have no explicit dependence on $z$. In this case, the exterior differential system $\tilde{\mathcal{I}}$ generated in $x y p q$-space $\tilde{M}$ by

$$
\begin{aligned}
& \Phi=d p \wedge d x+d q \wedge d y \\
& \Psi=E d p \wedge d q+A d p \wedge d y+B(d q \wedge d y+d x \wedge d p)+C d x \wedge d q+D d x \wedge d y
\end{aligned}
$$

has the following property: Integral surfaces on which $d x \wedge d y$ is non-zero are locally graphs

$$
(x, y) \rightarrow(x, y, p(x, y), q(x, y))
$$

and $\Phi=0$ implies that there is (at least locally) a function $z$ so that

$$
p=z_{x} \quad \text { and } \quad q=z_{y}
$$

Then $\Psi$ vanishes on this graph exactly when $z(x, y)$ is a solution to the MongeAmpere equation above. The condition that this solution be hyperbolic in the classical sense (see [CH]) is easily found to be equivalent to the condition that the graph lie in the open set where $\Phi$ and $\Psi$ generate a hyperbolic pencil.

We shall study several explicit examples in later sections. However, this might be a good place to illustrate the notion of equivalence that we alluded to earlier. According to the general procedure that we have just outlined, the classical wave equation $z_{y y}-z_{x x}=0$ corresponds to the ideal $\mathcal{I}_{1}$ on $\mathbb{R}^{4}$ defined by

$$
\mathcal{I}_{1}=\{d p \wedge d x+d q \wedge d y, d x \wedge d q-d p \wedge d y\}
$$

while the Monge-Ampere equation $z_{x x} z_{y y}-z_{x y}^{2}=-1$ corresponds to the ideal $\mathcal{I}_{2}$ on $\mathbb{R}^{4}$ defined by

$$
\mathcal{I}_{2}=\{d p \wedge d x+d q \wedge d y, d p \wedge d q+d x \wedge d y\}
$$

Even though there is no change of variables in $x y z$-space which will convert one of these equations into the other, the ideals $\mathcal{I}_{i}$ are diffeomorphic: the diffeomorphism $f: \mathbb{R}^{4} \rightarrow \mathbb{R}^{4}$ defined by the formula

$$
f(x, y, p, q)=(-p, y, x, q)
$$

clearly has the property that $f^{*}\left(\mathcal{I}_{1}\right)=\mathcal{I}_{2}$ and $f^{*}\left(\mathcal{I}_{2}\right)=\mathcal{I}_{1}$. Thus, the two systems are (globally) equivalent.

This example points out the importance of understanding when two systems are equivalent. Later sections of this paper will be directed at understanding how one can develop invariants which distinguish hyperbolic systems. The study of these invariants will then point out that several classically studied systems can be characterized by invariant properties. For example, in later sections our study will uncover, among others, the system

$$
\begin{aligned}
& u_{x}-v_{x}=\sin (u+v) \\
& u_{y}+v_{y}=\sin (u-v)
\end{aligned}
$$

which generates the famous Bäcklund transformation for the sine-Gordon equation $z_{x y}=\sin z$, the system

$$
\begin{aligned}
u_{x} & =e^{v} \\
v_{y} & =e^{u}
\end{aligned}
$$

which is important in the study of the classical Liouville equation $z_{x y}=e^{z}$, and the system

$$
\begin{aligned}
u_{x} & =\frac{v}{x+y} \\
v_{y} & =\frac{u}{x+y}
\end{aligned}
$$

which turns out to be a 'linearization' of Goursat's equation $z_{x y}=2 \sqrt{z_{x} z_{y}} /(x+y)$.
The invariants to be discussed also influence the space of conservation laws of the systems in question, and other special systems will turn up in that context.

Example 4: Among the most interesting and important exterior differential systems are those that arise from critical points of functionals. In the present setting we consider a symplectic manifold $(M, \Phi)$ where $\operatorname{dim} M=4$ and $\Phi$ is a symplectic form on $M$. Recall that an immersed surface

$$
f: S \rightarrow M
$$

is said to be Lagrangian if $f^{*} \Phi=0$. Let $\Lambda$ be another 2-form. Consider the functional on Lagrangian surfaces

$$
\begin{equation*}
L(f)=\int_{S} f^{*}(\Lambda) \tag{2}
\end{equation*}
$$

We will now construct an exterior differential system, called the Euler-Lagrange system $\mathcal{E}(\Lambda)$ associated to $\Lambda$, whose integral surfaces are the critical points of the functional (2).

To do this, we first write $d \Lambda=\Phi \wedge \psi$ for some (unique) 1-form $\psi$. (This form exists and is unique since $\Phi_{\wedge}: T^{*} M \rightarrow \Lambda^{3} T^{*} M$ is an isomorphism.) We will say that (2) is non-degenerate in case the 2 -form $\Psi=d \psi$ is a symplectic form on $M$, i.e., $\Psi \wedge \Psi$ is nowhere vanishing.

We now define $\mathcal{E}(\Lambda)$ to be the ideal generated by $\Phi$ and $\Psi$. We are now going to show that a Lagrangian immersion $f: S \rightarrow M$ is critical for the functional (2) with respect to (compactly supported) variations of $f$ through $\Phi$-Lagrangian immersions if and only if it is an integral manifold of $\mathcal{E}(\Lambda)$.

To see this let $f_{t}: S \rightarrow M$ be a compactly supported variation through $\Phi$ Lagrangian surfaces. Assuming for simplicity of notation that the $f_{t}$ are imbeddings and setting $S_{t}=f_{t}(S)$, a standard calculation gives

$$
\left.\frac{d}{d t}\left(\int_{S_{t}} \Lambda\right)_{t=0}=\int_{S} v\right\lrcorner d \Lambda
$$

where $v$ is the variational vector field along $S=S_{0}$. Now, restricted to $S$ we have

$$
\begin{aligned}
v\lrcorner(\Phi \wedge \psi) & =(v\lrcorner \Phi) \wedge \psi \\
& =d h \wedge \psi
\end{aligned}
$$

for a suitable function $h$ that depends linearly on $v$. The first equation follows from $\left.\Phi\right|_{S}=0$, and the relation

$$
v-\left.\Phi\right|_{S}=d h
$$

follows from the fact that the $S_{t}$ are $\Phi$-Lagrangian surfaces. By Stokes' theorem

$$
\frac{d}{d t}\left(\int_{S_{t}} \Lambda\right)_{t=0}=-\int_{S} h \Psi
$$

If this vanishes for all $\Phi$-Lagrangian variations, then $\left.\Psi\right|_{S}=0$, as we wanted to show.

In case $\Phi$ and $\Psi$ span a hyperbolic pencil, it follows that the Euler-Lagrange system $\mathcal{E}(\Lambda)$ is a hyperbolic exterior differential system of class $s=0$.

We shall now give some examples of hyperbolic systems with $s>0$.

Example 5: A general Monge-Ampere equation

$$
E\left(z_{x x} z_{y y}-z_{x y}^{2}\right)+A z_{x x}+2 B z_{x y}+C z_{y y}+D=0
$$

as mentioned above, except where, now, the coefficients $A, B, C, D$, and $E$ are functions of all five variables $x, y, z, z_{x}$, and $z_{y}$, may be written as an exterior differential system $(M, \mathcal{I})$ on a 5 -manifold. In fact, $M$ is a suitable open set in the jet space $J^{1}\left(\mathbb{R}^{2}, \mathbb{R}\right)$ with coordinates $(x, y, z, p, q)$ and $\mathcal{I}$ is generated as a differential ideal by the contact 1 -form

$$
\theta=d z-p d x-q d y
$$

and the 2 -form

$$
\Psi=E d p \wedge d q+A d p \wedge d y+B(d q \wedge d y+d x \wedge d p)+C d x \wedge d q+D d x \wedge d y
$$

Algebraically $\mathcal{I}$ is generated by the 1 -form $\theta$ and the 2 -forms $\Theta=d \theta$ and $\Psi$. As expected, the Monge-Ampere equation turns out to be hyperbolic in the usual sense if and only if $\Theta$ and $\Psi$ generate a hyperbolic pencil modulo $\theta$.

The relation between this example and the hyperbolic system $(\tilde{M}, \tilde{\mathcal{I}})$ of class $s=0$ constructed above is that, in the case the coefficients $A, B, \ldots, E$ do not depend on $z$, the system $(M, \mathcal{I})$ is an integrable extension of $(\tilde{M}, \tilde{\mathcal{I}})$ in the sense of Section 6 in [ $\mathrm{BG}_{2}$ ].

Explicitly, following the notation in Example 3, we note that $\varphi=p d x+q d y$ satisfies $d \varphi=\Phi \in \tilde{\mathcal{I}}$. Thus, $\varphi$ is closed modulo $\tilde{\mathcal{I}}$. It follows that, on $M^{5}=\tilde{M} \times \mathbb{R}$ with coordinate function $z$ on the $\mathbb{R}$-factor, we may introduce a 1 -form

$$
\theta=d z-p d x-q d y
$$

on $M$. The exterior differential system generated by $\theta$ and $\Psi$ (same notation in both examples) is then $\mathcal{I}$ as introduced above. We may think of the fibration $M \rightarrow \tilde{M}$ as being given by the primitive of the 1 -form giving a conservation law for $\tilde{\mathcal{I}}$.

Example 6: Suppose that we consider the general first-order system for two functions of two unknowns:

$$
\begin{aligned}
& F^{1}\left(x, y, u, v, u_{x}, u_{y}, v_{x}, v_{y}\right)=0 \\
& F^{2}\left(x, y, u, v, u_{x}, u_{y}, v_{x}, v_{y}\right)=0
\end{aligned}
$$

We shall naturally assume that these equations are non-degenerate in the sense that it is always possible to solve them locally for two of the partials. On $\mathbb{R}^{8}$ with coordinates $x, y, u, v, p, q, r$, and $s$, consider the differential ideal generated by the two 1 -forms

$$
\begin{aligned}
& \theta^{1}=d u-p d x-q d y \\
& \theta^{2}=d v-r d x-s d y
\end{aligned}
$$

Pull these forms back to the submanifold $M^{6} \subset \mathbb{R}^{8}$ defined by the equations

$$
F^{1}(x, y, u, v, p, q, r, s)=F^{2}(x, y, u, v, p, q, r, s)=0
$$

Then they generate a rank 2 Pfaffian system $I$. Let $\mathcal{I}$ denote the ideal generated by $\left\{\theta^{1}, \theta^{2}, d \theta^{1}, d \theta^{2}\right\}$.

It is not difficult to show that a solution $(u, v)=(f(x, y), g(x, y))$ of the PDE system is hyperbolic in the usual PDE sense if and only if its 1-graph

$$
(x, y) \mapsto\left(x, y, f(x, y), g(x, y), f_{x}(x, y), f_{y}(x, y), g_{x}(x, y), g_{y}(x, y)\right)
$$

(which is clearly an integral manifold of $\mathcal{I}$ ) lies in the open subset of $M$ on which $\mathcal{I}$ is a hyperbolic exterior differential system of class $s=2$.

Example 7: Now consider a single second-order hyperbolic equation

$$
F\left(x, y, z, z_{x}, z_{y}, z_{x x}, z_{x y}, z_{y y}\right)=0
$$

which we assume to be non-degenerate, i.e., one can, at least locally, always solve this equation for one of the second order partials.

On $\mathbb{R}^{8}$ with coordinates $x, y, z, p\left(=z_{x}\right), q\left(=z_{y}\right), r\left(=z_{x x}\right), s\left(=z_{x y}\right)$, and $t\left(=z_{y y}\right)$, we consider the Pfaffian system generated by the three 1-forms

$$
\begin{aligned}
& \theta^{0}=d z-p d x-q d y \\
& \theta^{1}=d p-r d x-s d y \\
& \theta^{2}=d q-s d x-t d y
\end{aligned}
$$

We pull these 1 -forms back to the hypersurface $M^{7}$ in $\mathbb{R}^{8}$ defined by the equation

$$
F(x, y, z, p, q, r, s, t)=0
$$

and let $\mathcal{I}$ denote the differential ideal generated by these forms. Now, $d \theta^{0} \equiv$ $0 \bmod \theta^{0}, \theta^{1}, \theta^{2}$ as is easily verified, so it follows that $\mathcal{I}$ is generated algebraically by the forms $\left\{\theta^{0}, \theta^{1}, \theta^{2}, d \theta^{1}, d \theta^{2}\right\}$. It is not difficult to show that the above equation is hyperbolic at a solution $z=f(x, y)$ if and only if its 2-graph

$$
(x, y) \mapsto\left(x, y, f(x, y), f_{x}(x, y), f_{y}(x, y), f_{x x}(x, y), f_{x y}(x, y), f_{y y}(x, y)\right)
$$

(which is clearly an integral manifold of the system $\mathcal{I}$ ) lies in the open subset of $M$ on which $\mathcal{I}$ is a hyperbolic exterior differential system. Thus, $(M, \mathcal{I})$ is hyperbolic of class $s=3$.

Example 8: In the previous examples, we have shown how several classical hyperbolic PDEs can be recast as hyperbolic exterior differential systems. In fact, in geometry, this is frequently the reverse of the natural order. One generally encounters problems cast naturally in the form of a differential system and the "reduction" to an equation or system of equations can only be done after a somewhat arbitrary choice of coordinates. (One of the motives of studying exterior differential systems is to avoid having to make these choices of coordinates.) In this example, we want to show how a hyperbolic system arises naturally in differential geometry. We will assume some familiarity with the language of moving frames.

Let $\left(N^{3}, g\right)$ be an oriented Riemannian 3 -manifold. Let $\pi: F \rightarrow N$ be its oriented orthonormal frame bundle. Thus, an element $f \in F$ is a quadruple $f=\left(x ; e_{1}, e_{2}, e_{3}\right)$ where $x=\pi(f)$ is a point of $N$ and $\left(e_{1}, e_{2}, e_{3}\right)$ is an oriented orthonormal basis of $T_{x} N$. There are canonical 1-forms $\omega_{i}$ defined on $F$ by the rule

$$
\omega_{i}(v)=e_{i} \cdot \pi^{\prime}(v), \quad \text { for all } v \in T_{f} F
$$

It is easy to show that $\pi^{*}(g)=\omega_{1}^{2}+\omega_{2}^{2}+\omega_{3}^{2}$. By the Fundamental Lemma of Riemannian geometry, there exist unique 1 -forms $\omega_{i j}=-\omega_{j i}$ (the connection forms) so that the following so-called structure equations hold:

$$
d \omega_{i}=-\omega_{i j} \wedge \omega_{j}
$$

The 1 -forms $\left(\omega_{1}, \omega_{2}, \omega_{3}, \omega_{23}, \omega_{31}, \omega_{12}\right)$ then form a global coframing of $F$ (which has dimension 6). One proves that the connection forms satisfy structure equations of the form

$$
d \omega_{i j}=-\omega_{i k} \wedge \omega_{k j}+\frac{1}{2} R_{i j k l} \omega_{k} \wedge \omega_{l}
$$

where $R_{i j k l}=-R_{j i k l}=-R_{i j l k}$, represent the components of the Riemann curvature tensor.

Now let $M^{5} \subset T N$ be the unit sphere bundle in the tangent bundle of $N$. There is a canonical map $\sigma: F \rightarrow M$ given by $\sigma\left(x ; e_{i}\right)=\left(x ; e_{3}\right)$ and this map is a submersion. Looking at the definition of $\omega_{3}$ on $F$, it is clear that there exists a unique 1 -form $\theta$ on $M$ so that $\sigma^{*}(\theta)=\omega_{3}$. Since

$$
\sigma^{*}\left(\theta \wedge d \theta^{2}\right)=\omega_{3} \wedge\left(d \omega_{3}\right)^{2}=-2 \omega_{1} \wedge \omega_{2} \wedge \omega_{3} \wedge \omega_{31} \wedge \omega_{32} \neq 0
$$

it follows that $\theta$ defines a contact structure on $M$. Moreover, because the fibers of $\sigma$ are the leaves of the Pfaffian system generated by $\left\{\omega_{1}, \omega_{2}, \omega_{3}, \omega_{31}, \omega_{32}\right\}$, the structure equations imply that there exist well-defined 2 -forms $\Omega_{0}, \Omega_{1}, \Omega_{2}$ on $M$ so that

$$
\begin{aligned}
& \sigma^{*}\left(\Omega_{0}\right)=\omega_{1} \wedge \omega_{2} \\
& \sigma^{*}\left(\Omega_{1}\right)=\omega_{31} \wedge \omega_{2}+\omega_{1} \wedge \omega_{32} \\
& \sigma^{*}\left(\Omega_{2}\right)=\omega_{31} \wedge \omega_{32}
\end{aligned}
$$

In order to understand the geometric meaning of these forms, we consider the geometry of the integral surfaces of $\{\theta, d \theta\}$. For any immersed oriented surface $f: S \rightarrow N$, there is a canonical lifting $\tilde{f}: S \rightarrow M$ defined by

$$
\tilde{f}(s)=(f(s), \nu(s))
$$

where $\nu(s) \in T_{f(s)} N$ is the oriented normal to the oriented 2-plane $f^{\prime}(s)\left(T_{s} S\right) \subset$ $T_{f(s)} N$. It is easy to see that $\tilde{f}^{*}(\theta)=0$, so that $\tilde{f}: S \rightarrow M$ is an integral surface of $\{\theta, d \theta\}$. Conversely, any integral surface $g: S \rightarrow M$ of $\theta$ which satisfies the condition that $g^{*}\left(\Omega_{0}\right) \neq 0$ is easily seen to be of the form $g=\tilde{f}$ for some immersion $f: S \rightarrow N$.

Moreover, a simple calculation shows that, first, $\tilde{f}^{*}\left(\Omega_{0}\right)=d A$ is the area form induced on $S$ by its immersion into $N$, second, $\tilde{f}^{*}\left(\Omega_{1}\right)=2 H d A$, where $H$ is the mean curvature of the oriented immersion, and, third, $\tilde{f}^{*}\left(\Omega_{2}\right)=K d A$ where $K$ is the product of the principal curvatures of the immersion. ${ }^{5}$

Now let $\lambda_{0}, \lambda_{1}$, and $\lambda_{2}$ be constants (not all zero) and let

$$
\Omega=\lambda_{0} \Omega_{0}+\lambda_{1} \Omega_{1}+\lambda_{2} \Omega_{2}
$$

5) Note that, unless the metric $g$ is flat, this function $K$ will not generally be the Gauss curvature of the induced metric $f^{*}(g)$ on $S$.

Consider the ideal $\mathcal{I}$ on $M$ generated by $\{\theta, d \theta, \Omega\}$. It follows from the above discussion that the integral surfaces of $\mathcal{I}$ on which $\Omega_{0}$ is non-zero are exactly the canonical lifts of immersions into $N$ which satisfy the relation

$$
\lambda_{2} K+2 \lambda_{1} H+\lambda_{0}=0
$$

This will be a hyperbolic system of class $s=1$ exactly when the pair $\{d \theta, \Omega\}$ generates a hyperbolic pencil modulo $\theta$. A straightforward computation gives

$$
\left(\xi_{1} \Omega+\xi_{2} d \theta\right)^{2}=\left(\left(\lambda_{1}^{2}-\lambda_{0} \lambda_{2}\right) \xi_{1}^{2}+\xi_{2}^{2}\right)(d \theta)^{2}
$$

It follows that $\mathcal{I}$ will be hyperbolic if and only if $\lambda_{1}^{2}-\lambda_{0} \lambda_{2}<0$.
Thus, for example, when $(N, g)$ is flat Euclidean 3 -space, setting $\left(\lambda_{0}, \lambda_{1}, \lambda_{2}\right)=$ $(1,0,1)$ gives the system whose integral manifolds are the surfaces in 3 -space of constant Gauss curvature $K=-1$. Even in flat 3-space, however, there is no natural coordinate system for reducing this hyperbolic system to a partial differential equation. The introduction of the Tschebycheff coordinates usually associated with this problem (which are crucial in the proof of Hilbert's theorem that there are no complete surfaces of constant negative curvature in Euclidean 3-space) depend, as we shall see, on understanding the conservation laws of this system in a coordinatefree manner. The relation of this system with the Bäcklund transformation for the sine-Gordon equation is well-known and will appear later in the paper.

In later sections, we will see further examples of hyperbolic systems with class $s>0$ and will explore some of the relationships among systems of different classes. Although we are primarily interested in equations of class $s=0$ or $s=1$, it turns out that considering the more general case simplifies our study considerably.
1.1.4 A local normal form for hyperbolic systems of class $s=0$. Let $(M, \mathcal{I})$ be a hyperbolic system of class $s=0$. We will show that in the real analytic case such a system is locally equivalent to the system associated to a quasi-linear first order hyperbolic PDE system exhibited in Example 1 in Section 1.1.3 above. We begin by proving the following

Proposition: Let $(M, \mathcal{I})$ be a real analytic hyperbolic system of class $s=0$. Then given any point $p \in M$ there is a neighborhood $U$ of $p$ and a non-zero 2-form $\Omega$ defined on $U$ satisfying
(i) $\Omega \wedge \Omega=0$;
(ii) $\Omega \wedge \Phi=0$ for all 2 -forms $\Phi \in \mathcal{I}$;
(iii) $\Omega$ is integrable, i.e., $d \Omega \equiv 0 \bmod \Omega$.

Proof: We begin with a brief linear algebra discussion. Let $V$ be a vector space of dimension 4 and $\mathbb{P}=\mathbb{P}\left(\Lambda^{2} V^{*}\right)$ the projective space associated to the vector space of 2 -forms on $V$. Then $\operatorname{dim} \mathbb{P}=5$ and in $\mathbb{P}$ the equation

$$
\Omega_{\wedge} \Omega=0, \quad \Omega \in \Lambda^{2} V^{*}
$$

defines the Grassmannian $\mathbb{G}$ of decomposable 2-planes. (More precisely, a point in $\mathbb{P}$ is the line $[\Omega]$ spanned by a non-zero $\Omega \in \Lambda^{2} V^{*}$, and the validity of the above equation is independent of which point on the line we choose. We shall follow similar conventions for the remainder of this argument.) Let $L \subset \mathbb{P}$ be a hyperbolic pencil with intersection

$$
L \cap \mathbb{G}=\left\{\left[\Omega_{1}\right],\left[\Omega_{2}\right]\right\}
$$

where $\Omega_{1}$ and $\Omega_{2}$ are the two decomposable elements of the pencil, each of which is defined up to non-zero scalars. For a point $[\Omega] \in \mathbb{G}$ the equation

$$
\Omega_{\wedge} \Omega_{1}=0
$$

is equivalent to the condition that $\Omega$ and $\Omega_{1}$ have a common non-zero linear factor. Since the projectivized tangent space

$$
\mathbb{P}\left(T_{\left[\Omega_{1}\right]} \mathbb{G}\right)=\left\{[\Phi] \in \mathbb{P}: \Phi \wedge \Omega_{1}=0\right\}
$$

we see that the above equation exactly defines

$$
\mathbb{P}\left(T_{\left[\Omega_{1}\right]} \mathbb{G}\right) \cap \mathbb{G}
$$

Thus the double intersection

$$
S=\mathbb{P}\left(T_{\left[\Omega_{1}\right]} \mathbb{G}\right) \cap \mathbb{P}\left(T_{\left[\Omega_{2}\right]} \mathbb{G}\right) \cap \mathbb{G}
$$

is a smooth surface that defines the set of points $[\Omega] \in \mathbb{G}$ such that $\Omega$ has a common linear factor with each of $\Omega_{1}$ and $\Omega_{2}$. For any such $\Omega$ we may choose a basis $\omega^{1}, \omega^{2}, \omega^{3}, \omega^{4}$ for $V^{*}$ such that

$$
\begin{align*}
\Omega_{1} & =\omega^{1} \wedge \omega^{2} \\
\Omega_{2} & =\omega^{3} \wedge \omega^{4}  \tag{3}\\
\Omega & =\omega^{1} \wedge \omega^{3} .
\end{align*}
$$

For each point $p \in M$ we may apply this construction to $V=T_{p} M$ and obtain a surface $S_{p} \subset \mathbb{P}\left(\Lambda^{2} T_{p}^{*} M\right)$. In fact, there is an obvious 6-manifold $\Sigma \subset \mathbb{P}\left(\Lambda^{2} T^{*} M\right)$ given by $\cup_{p \in M} S_{p}$. To give local coordinates on $\Sigma$ we choose a point $\left(p_{0},\left[\Omega_{0}\right]\right)$ of $\Sigma$ and a basis $\omega^{1}, \omega^{2}, \omega^{3}, \omega^{4}$ in $T_{p_{0}}^{*} M$ for which (3) holds. We may extend the $\omega^{i}$ to a coframe near $p_{0}$, and then for $(p,[\Omega])$ a point of $\Sigma$ close to ( $\left.p_{0},\left[\Omega_{0}\right]\right)$ we will have

$$
\begin{equation*}
\Omega=\left(\omega^{1}+u \omega^{2}\right) \wedge\left(\omega^{3}+v \omega^{4}\right) \tag{4}
\end{equation*}
$$

for unique $(u, v) \in \mathbb{R}^{2}$. These $u, v$ together with local coordinates near $p_{0}$ on $M$ will give a local coordinate system on $\Sigma$.

On $\Sigma$ we have a tautological 2 -form $\Omega$, defined up to non-zero factors, and the exterior differential system that we want to consider is

$$
\begin{equation*}
d \Omega \equiv 0 \bmod \Omega \tag{5}
\end{equation*}
$$

subject to the condition

$$
\omega^{1} \wedge \omega^{2} \wedge \omega^{3} \wedge \omega^{4} \neq 0
$$

An integral manifold of this system is then locally given by a section $f: M \rightarrow \Sigma$ such that $f^{*} \Omega$ satisfies the conditions of the proposition. Locally we write

$$
\Omega=\pi_{1} \wedge \pi_{3}
$$

where $\pi_{1}=\omega^{1}+u \omega^{2}$ and $\pi_{3}=\omega^{3}+v \omega^{4}$ are the linear factors appearing in (4). Equation (5) above then implies

$$
\begin{aligned}
& d \pi_{1 \wedge} \pi_{1 \wedge} \wedge \pi_{3}=0 \\
& d \pi_{3} \wedge \pi_{3} \wedge \pi_{1}=0 .
\end{aligned}
$$

If we write

$$
\begin{aligned}
d u & =u_{i} \omega^{i} \\
d v & =v_{i} \omega^{i}
\end{aligned}
$$

then solutions of the exterior differential system are given by solutions to the PDE system

$$
\begin{aligned}
& u_{4}=f\left(u, v, u_{1}, u_{2}, u_{3}, v_{1}, v_{3}, v_{4}\right) \\
& v_{2}=g\left(u, v, u_{1}, u_{2}, u_{3}, v_{1}, v_{3}, v_{4}\right)
\end{aligned}
$$

where $f=g=0$ whenever $u=v=0$. This is a determined PDE system for 2 unknown functions on an open set in $\mathbb{R}^{4}$, and by the Cauchy-Kowaleska Theorem it will have local solutions in the real analytic case.

Remark: The above PDE system is in fact hyperbolic with characteristic variety given by the union of the two hyperplanes $\omega^{2}=0$ and $\omega^{4}=0$ in the tangent space at ( $p_{0},\left[\Omega_{0}\right]$ ). Even though this characteristic variety is singular, it is possible that the methods of [Ya] might apply to give a $C^{\infty}$ result.

As an application of the proposition we have the
Corollary: Any real analytic hyperbolic system of class $s=0$ is locally equivalent to the exterior differential system associated to a quasi-linear hyperbolic PDE system.

Proof: We may find local functions $x, y$ such that $\Omega$ is a non-zero multiple of $d x \wedge d y$. If we then complete $x, y$ to a local coordinate system $x, y, u, v$ then each of the generators of the hyperbolic system will have no $d u \wedge d v$ term. The result then follows from the discussion below Example 2 in Section 1.1.3.

An important special class of first order quasi-linear hyperbolic systems are the hyperbolic systems of conservation laws, which are PDE systems for functions $u$ and $v$ of $x$ and $y$ of the form

$$
\begin{equation*}
\partial_{x}(\mathbf{F}(x, y, u, v))+\partial_{y}(\mathbf{G}(x, y, u, v))=0 \tag{6}
\end{equation*}
$$

where $\mathbf{F}$ and $\mathbf{G}$ are $\mathbb{R}^{2}$-valued functions. We may equivalently express (6) by the statement that the $\mathbb{R}^{2}$-valued 2 -form

$$
\binom{\Phi_{1}}{\Phi_{2}}=d \mathbf{F} \wedge d y-d \mathbf{G} \wedge d x=d(\mathbf{F} d y-\mathbf{G} d x)
$$

vanishes on graphs $(x, y) \rightarrow(x, y, u(x, y), v(x, y))$. We clearly have

$$
d \Phi_{1}=d \Phi_{2}=0
$$

so that the exterior differential system associated to a hyperbolic system of conservation laws is generated by a pair of closed 2 -forms. Using the above proposition we may prove a converse.

Proposition: A real analytic hyperbolic system $(M, \mathcal{I})$ of class $s=0$ is locally equivalent to the exterior differential system associated to a hyperbolic system of conservation laws if, and only if, $\mathcal{I}$ is generated by a pair of closed 2-forms.

Proof: Let $\Phi_{1}$ and $\Phi_{2}$ be closed generators of $\mathcal{I}$. We want to show that on a neighborhood of each point of $M$ there are functions $x$ and $y$ together with $\mathbb{R}^{2}$-valued functions $\mathbf{F}, \mathbf{G}$ such that

$$
\binom{\Phi_{1}}{\Phi_{2}}=d(\mathbf{F} d y-\mathbf{G} d x)
$$

Let $x, y, u, v$ be coordinates as in the above corollary and $\Phi \in \mathcal{I}$ be any closed 2 -form. Since $\Phi$ is locally exact we have

$$
\Phi=\alpha \wedge d x+\beta \wedge d y+f_{1} d x \wedge d y=d \gamma
$$

for some 1 -forms $\alpha, \beta$ and $\gamma$, and a function $f_{1}$. Clearly, it suffices to show that $\gamma$ may be chosen so that $\gamma \equiv 0 \bmod d x, d y$.

In the above expression for $\Phi$ we may assume that

$$
\begin{aligned}
& \alpha=a d u+b d v \\
& \beta=c d u+e d v
\end{aligned}
$$

for suitable functions $a, b, c$ and $e$. Looking at the coefficient of $d u \wedge d v \wedge d x$ in $d \Phi$ gives

$$
a_{v}=b_{u}
$$

so that

$$
a=A_{u}, \quad b=A_{v}
$$

for a suitable function $A$. Then

$$
\Phi-d(A d x)=\beta \wedge d y+f_{2} d x \wedge d y
$$

for some possibly new function $f_{2}$. Repeating the argument gives the existence of functions $B$ and $f$ such that

$$
\Phi-d(A d x+B d y)=f d x \wedge d y
$$

Taking the exterior derivatives of both sides gives

$$
d f \wedge d x \wedge d y=0
$$

and so $f=f(x, y)$. Then by modifying $A$ and $B$ appropriately we obtain

$$
\Phi=d(A d x+B d y)
$$

as desired.
1.2 The characteristic variety and the initial value problem. The fundamental feature of hyperbolic (as opposed to, say, elliptic) PDE is the existence of the so-called "characteristics". In the classical theory of (determined) hyperbolic PDE in two independent variables, one studies the initial value problem, which is usually posed as follows: Along a curve in the domain of the independent variables, one specifies initial data (the values of the dependent variables and their partial derivatives) which satisfy the so-called "strip conditions". If this data is "non-characteristic" in an appropriate sense, then one hopes to prove that there is a unique solution to the PDE on an open neighborhood of the initial curve which agrees with the given initial data along that curve. (See $[\mathrm{CH}]$ for a discussion of these ideas.) Then the notion of a "characteristic" emerges as being a curve in the domain of a solution along which the initial value problem would not be well-posed.

In this section, we will explore this concept in the context of hyperbolic exterior differential systems.
1.2.1 Some linear algebra. Before turning to the details of hyperbolic systems, it is useful to briefly continue the linear algebra discussion on 4-dimensional vector spaces that we began in Section 1.1.2. We will keep the notation established there and let $\mathbb{P}=\mathbb{P} V$ denote the projective space of lines through the origin in $V$. An isomorphism $V \cong \mathbb{R}^{4}$ induces an identification $\mathbb{P} \cong \mathbb{P}^{3}$, and all the statements below may be easily verified by choosing coordinates and making the appropriate calculations.

Let $\Omega_{1}, \Omega_{2} \in \Lambda^{2} V^{*}$ be decomposable 2-forms satisfying $\Omega_{1} \wedge \Omega_{2} \neq 0$. The 2 planes $\Omega_{1}^{\perp}, \Omega_{2}^{\frac{1}{2}}$ in $V$ then give a pair of lines $L_{1}, L_{2}$ in the projective space $\mathbb{P}$, and the condition $\Omega_{1} \wedge \Omega_{2} \neq 0$ translates into the condition that $L_{1}$ and $L_{2}$ be skew lines. A 2-plane $E \subset V$ gives a line $L_{E} \subset \mathbb{P}$ and for each $\alpha=1,2$ the conditions
(i) $\left.\Omega_{\alpha}\right|_{E}=0$
(ii) $L_{\alpha}$ meets $L_{E}$
are equivalent. Thus, the 2-planes for which (i) holds for $\alpha=1,2$ are given by the lines $L_{E} \subset \mathbb{P}$ which meet each of $L_{1}$ and $L_{2}$.


Fig. 1
Moreover, given any point $P \in \mathbb{P}$ not on $L_{1}$ or $L_{2}$, there is a unique line $L_{P}$ passing through $P$ and meeting each of $L_{1}$ and $L_{2}$ in points $P_{1}$ and $P_{2}$. This gives a fibration

$$
\begin{gathered}
\mathbb{P}^{3} \backslash\left(L_{1} \cup L_{2}\right) \\
\downarrow \\
L_{1} \times L_{2}
\end{gathered}
$$

with fibre $\mathbb{P}^{\mathbb{1}} \backslash\{2$ points $\} \cong \mathbb{R}^{*}$. In particular, the set of lines meeting each of $L_{1}$ and $L_{2}$ is bijective to $L_{1} \times L_{2}$. Back on $V$ the statement is: The set of 2-planes $E \subset V$ meeting each of the 2-planes $\Omega_{1}^{\perp}, \Omega_{2}^{\frac{1}{2}}$ in a line is a $\mathbb{P}^{1} \times \mathbb{P}^{1}$.
1.2.2 The Characteristic Variety. Now, recall (see $\left[\mathrm{BCG}^{3}\right]$ ) that an integral element (of dimension $q$ ) of an exterior differential system $(M, \mathcal{I})$ is a $q$-plane $E \subset T_{x} M$ such that all the forms in $\mathcal{I}$ restrict to zero on $E$; i.e.,

$$
\left.\theta(x)\right|_{E}=0, \quad \theta \in \mathcal{I} .
$$

Intuitively, the $q$-dimensional elements are the candidates for tangent planes to $q$-dimensional integral manifolds.

Hyperbolic systems (which, as we have seen, satisfy $\mathcal{I}^{q}=\Omega^{q}(M)$ for all $q \geq 3$ ) have integral elements of dimensions $q=0,1,2$, which we now want to describe. Let $\mathcal{I}=\left\{\theta^{1}, \ldots, \theta^{s} ; \Omega_{10}, \Omega_{01}\right\}$ be a hyperbolic exterior differential system on $M^{s+4}$
and let $x \in M$. Then the subspace $I_{x}^{\perp} \subset T_{x} M$, defined as the set of $v \in T_{x} M$ so that

$$
\theta_{x}^{1}(v)=\cdots=\theta_{x}^{s}(v)=0
$$

is a vector space of dimension 4 and we denote by $\Omega_{1}, \Omega_{2}$ the restrictions of $\Omega_{10}, \Omega_{01}$ to $I_{x}$.

Integral elements of dimension one in $T_{x} M$ are simply lines in $T_{x} M$ which lie in $I_{x}^{\perp}$ and are thus given by points $P \in \mathbb{P} I_{x}^{\perp}$. By the linear algebra discussion above, integral elements of dimension two in $T_{x} M$ are 2-planes in $I_{x}^{\perp}$ which meet each of the 2 -planes $\Omega_{1}^{\perp}$ and $\Omega_{2}^{\perp}$ and thus are given by lines $L \subset \mathbb{P} I_{x}^{\perp}$ meeting each of the skew lines $L_{1}=\mathbb{P}\left(\Omega_{1}^{\perp}\right)$ and $L_{2}=\mathbb{P}\left(\Omega_{2}^{\perp}\right)$. The condition that a point $P \in \mathbb{P} I_{x}^{\perp}$ lies on a unique line meeting each of $L_{1}$ and $L_{2}$ is that $P$ not lie on either $L_{1}$ or $L_{2}$.

We recall (again, see $\left[\mathrm{BCG}^{3}\right]$ for a more general discussion) that, by definition, the characteristic variety $\Xi$ of the exterior differential system $\mathcal{I}$ consists of all integral elements of dimension one that do not lie in a unique integral element of dimension 2. It follows from the above discussion that the base point mapping $\Xi \rightarrow M$ (which sends each characteristic integral element to its base point) is a fibration with fibre $\mathbb{P}^{1} \cup \mathbb{P}^{1}$. Informally, we say that the characteristics of a hyperbolic exterior differential system define a $\mathbb{P}^{1} \cup \mathbb{P}^{1}$ over each point of $M$.

This leads us to a very important definition for the subsequent theory. Recall our standing assumption from Section 1.1.2 that $\mathcal{I}$ can be generated on $M$ by the sections of $I$ and two globally defined, non-vanishing, decomposable 2 -forms $\Omega_{10}$ and $\Omega_{01}$. Clearly, we can choose a local adapted coframing $(\theta ; \omega)$ so that $\Omega_{10}=$ $\omega^{1} \wedge \omega^{2}$ and $\Omega_{01}=\omega^{3} \wedge \omega^{4}$.

Definition: The ( $0^{\text {th }}$ ) characteristic system $\Xi_{10}$ is the Pfaffian system generated by $\left\{\theta^{1}, \ldots, \theta^{s}, \omega^{1}, \omega^{2}\right\}$ while the $\left(0^{t h}\right)$ characteristic system $\Xi_{01}$ is the Pfaffian system generated by $\left\{\theta^{1}, \ldots, \theta^{s}, \omega^{3}, \omega^{4}\right\}$.

Note that each of the characteristic systems is a Pfaffian system of rank $s+2$. The importance of the characteristic systems is that a 1-dimensional integral element $E_{1}$ of $\mathcal{I}$ is characteristic if and only if it lies in either the 2 -plane $\Xi_{10}$ or the 2 -plane $\Xi_{01}^{\frac{1}{0}}$. In fact, we immediately see that the characteristic variety bundle $\Xi \rightarrow M$ decomposes into two disjoint $\mathbb{P}^{1}$-bundles as

$$
\Xi=\mathbb{P}\left(\Xi_{10}^{1}\right) \cup \mathbb{P}\left(\Xi_{01}^{1}\right)
$$

Now, since, by definition, each of $\Omega_{01}$ and $\Omega_{10}$ vanish on any integral surface $f: S \rightarrow M$ of $\mathcal{I}$, it follows that each of the characteristic systems $\Xi_{10}$ and $\Xi_{01}$ pulls back via such an $f$ to be a Pfaffian system of rank 1. In particular, there are two everywhere transverse foliations $\mathcal{F}_{10}$ and $\mathcal{F}_{01}$ of $S$ by curves so that the leaves of $\mathcal{F}_{10}$ map to integral curves of $\Xi_{10}$ and the leaves of $\mathcal{F}_{01}$ map to integral curves of
$\Xi_{01}$. As will be seen, in the classical cases, these foliations are merely the foliations of the domain of the solution by the so-called Monge characteristics.

Now a transverse pair of curve-foliations of a surface has no local geometry. However, the exterior differential system perspective shows that these are merely specializations to each solution surface of the "ambient" characteristic systems $\Xi_{10}$ and $\Xi_{01}$. These Pfaffian systems do have local geometry and it is this geometry which strongly influences conservation laws. We shall explore it more deeply in the following sections.

ExAmple 1: Consider the equation $z_{x y}=f\left(x, y, z, z_{x}, z_{y}\right)$ which, as we showed in the last section, can be associated to the hyperbolic system $\mathcal{I}$ of class $s=1$ on $\mathbb{R}^{5}$

$$
\mathcal{I}=\{d z-p d x-q d y,(d p-f(x, y, z, p, q) d y) \wedge d x,(d q-f(x, y, z, p, q) d x) \wedge d y\}
$$

Thus, the characteristic systems are given by

$$
\begin{aligned}
& \Xi_{10}=\{d z-p d x-q d y,(d p-f(x, y, z, p, q) d y), d x\} \\
& \Xi_{01}=\{d z-p d x-q d y,(d q-f(x, y, z, p, q) d x), d y\}
\end{aligned}
$$

Of course, when we restrict to any solution of the above equation, $\Xi_{10}$ (respectively, $\Xi_{01}$ ) pulls back to be the multiples of $d x$ (respectively, $d y$ ). Thus the characteristic foliations in the $x y$-plane are simply the foliations by the $x$-lines and $y$-lines, just as we expected.

Example 2: Consider the Monge-Ampere equation $z_{x x} z_{y y}-z_{x y}^{2}=-1$ which gives rise to the hyperbolic system $\mathcal{I}$ of class $s=1$ on $\mathbb{R}^{5}$

$$
\mathcal{I}=\{d z-p d x-q d y,(d p-d y) \wedge(d q+d x),(d p+d y) \wedge(d q-d x)\}
$$

The characteristic systems are given by

$$
\begin{aligned}
& \Xi_{10}=\{d z-p d x-q d y,(d p-d y),(d q+d x)\} \\
& \Xi_{01}=\{d z-p d x-q d y,(d p+d y),(d q-d x)\}
\end{aligned}
$$

Thus, for example, on the solution surface

$$
(x, y, z, p, q)=\left(x, y, f(x)+x y, f^{\prime}(x)+y, x\right)
$$

(where $f$ is an arbitrary smooth function of one variable), the foliation $\mathcal{F}_{10}$ is just the foliation induced by the slices $x=$ const while the foliation $\mathcal{F}_{01}$ is the foliation induced by the slices $2 y+f^{\prime}(x)=$ const .
1.2.3 The initial value problem for hyperbolic systems. As mentioned earlier, the notion of an integral element of an exterior differential system was meant to capture the idea of an "infinitesimal" integral manifold. Since a characteristic integral element is, by definition, one for which the extension to a higher dimensional integral element is not unique, it is the infinitesimal version of the usual PDE notion of a characteristic.

In the case of a hyperbolic exterior differential system $(M, \mathcal{I})$, the notion of "initial data satisfying a strip condition" corresponds exactly to an integral curve (i.e., one dimensional integral manifold) $\phi: C \rightarrow M$ of the Pfaffian system $I$. We say that $\phi$ is non-characteristic if the tangent line $\phi^{\prime}(t)\left(T_{t} C\right) \subset T_{\phi(t)} M$ is not a characteristic integral element of $\mathcal{I}$ for any $t \in C$, i.e., if $\phi^{\prime}(t)\left(T_{t} C\right) \nexists \Xi$ for all $t \in C$. As ought to be expected, this reduces in the classical cases to the notion of initial data being non-characteristic.

EXAMPLE 2 (continued): A curve $\phi:(0,1) \rightarrow \mathbb{R}^{5}$ of the form

$$
\phi(t)=\left(x_{0}(t), y_{0}(t), z_{0}(t), p_{0}(t), q_{0}(t)\right)
$$

is an integral curve of $I$ (i.e., satisfies the strip conditions) if and only if $z_{0}^{\prime}(t)=$ $p_{0}(t) x_{0}^{\prime}(t)+q_{0}(t) y_{0}^{\prime}(t)$. It is non-characteristic if and only if it is not tangent to either of the distributions $\Xi_{10}^{\frac{1}{10}}$ or $\Xi_{01}^{\perp}$, in other words, if and only if, for all $0<t<1$, we have

$$
\begin{aligned}
& \left(p_{0}^{\prime}(t)-y_{0}^{\prime}(t)\right)^{2}+\left(q_{0}^{\prime}(t)+x_{0}^{\prime}(t)\right)^{2} \neq 0 \\
& \left(p_{0}^{\prime}(t)+y_{0}^{\prime}(t)\right)^{2}+\left(q_{0}^{\prime}(t)-x_{0}^{\prime}(t)\right)^{2} \neq 0
\end{aligned}
$$

(Note that the condition $x_{0}^{\prime}(t)^{2}+y_{0}^{\prime}(t)^{2} \neq 0$ is not required, which it would be if we were to think of $\phi$ as defining initial data satisfying a strip condition along a smooth curve in the domain of the independent variables.) Assuming that these inequalities are satisfied, we can construct an integral surface $f:(0,1) \times(0,1) \rightarrow \mathbb{R}^{5}$ of $\mathcal{I}$ which satisfies $f(t, t)=\phi(t)$ for all $0<t<1$ by setting

$$
\begin{aligned}
& x(s, t)=\frac{1}{2}\left(x_{0}(s)+x_{0}(t)-q_{0}(s)+q_{0}(t)\right) \\
& y(s, t)=\frac{1}{2}\left(y_{0}(s)+y_{0}(t)+p_{0}(s)-p_{0}(t)\right) \\
& p(s, t)=\frac{1}{2}\left(p_{0}(s)+p_{0}(t)+y_{0}(s)-y_{0}(t)\right) \\
& q(s, t)=\frac{1}{2}\left(q_{0}(s)+q_{0}(t)-x_{0}(s)+x_{0}(t)\right)
\end{aligned}
$$

and then defining, for some chosen $t_{0} \in(0,1)$,

$$
z(s, t)=z_{0}\left(t_{0}\right)+\int_{\left(t_{0}, t_{0}\right)}^{(s, t)} p d x+q d y
$$

where the line integral is taken over any curve in $(0,1) \times(0,1)$ joining $\left(t_{0}, t_{0}\right)$ to $(s, t)$. The reader might check that the hypothesis that the curve $\phi$ be non-characteristic forces $f$ to be an immersion (on the whole open square) and that the characteristic foliations are simply the foliations by the $s$ - and $t$-slices in the square. We will comment further on the method we used to construct this solution in Section 1.4.

In fact, the general initial value problem for hyperbolic exterior differential systems has been studied quite extensively in the literature, see [Ka] and [Ya]. In particular, an elementary consequence of Theorem $5.1 \mathrm{in}[\mathrm{Ka}]$ is the following existence and uniqueness result:

ThEOREM: Let $\left(M^{s+4}, \mathcal{I}\right)$ be a smooth hyperbolic exterior differential system and let $\phi:(0,1) \rightarrow M$ be a non-characteristic smooth integral curve of $\mathcal{I}$. Then there exists an open neighborhood $U_{\phi} \subset(0,1) \times(0,1)$ of the diagonal and a smooth map $f_{\phi}: U_{\phi} \rightarrow M$ which is an integral surface of $\mathcal{I}$, satisfies $f_{\phi}(t, t)=\phi(t)$, and has the property that its characteristic foliation $\mathcal{F}_{10}$ is given by the slices $t=$ const while its characteristic foliation $\mathcal{F}_{01}$ is given by the slices $s=$ const. Moreover, if $\left(\tilde{U}_{\phi}, \tilde{f}_{\phi}\right)$ is any other pair with these properties, then $f_{\phi}$ and $\tilde{f}_{\phi}$ agree on any subsquare of the form $(a, b) \times(a, b)$ which lies in the intersection $U_{\phi} \cap \tilde{U}_{\phi}$.

Note that the condition of normalizing the characteristic foliations has the effect of removing the reparametrization ambiguity which normally affects integral manifolds of exterior differential systems. Also, note that while one would like to be able to say that the solution surface $f$ is defined on the entire open unit square, this will not generally be the case. A very interesting problem is to try to characterize in terms of some sort of completeness, those systems $(M, \mathcal{I})$ which have the property that every non-characteristic integral curve $\phi:(0,1) \rightarrow M$ has an extension as above with $U_{\phi}=(0,1) \times(0,1)$. Example 2 above clearly does have this property, so it should be thought of as "characteristically complete".

However, we caution the reader that the integral surface so constructed may not be representable as a graph of $z$ as a function of $x$ and $y$ even if it is so representable in a neighborhood of the initial curve $\phi$. Thus, from the PDE perspective, a given PDE may not have global existence for all initial data even when the corresponding hyperbolic exterior differential system is complete.

Example 2 (continued): Suppose that, for the Monge-Ampere equation above, we consider the initial curve given by the data

$$
\begin{aligned}
& x_{0}(t)=\cos t-\sin t \\
& y_{0}(t)=\cos t+\sin t \\
& z_{0}(t)=1 \\
& p_{0}(t)=\cos t-\sin t \\
& q_{0}(t)=\cos t+\sin t,
\end{aligned}
$$

$$
z_{0}(t)=1 \quad \text { with solution }
$$

$$
\begin{aligned}
& x(s, t)=\cos t-\sin s \\
& y(s, t)=\cos s+\sin t \\
& z(s, t)=t-s+\cos (t-s) \\
& p(s, t)=\cos s-\sin t \\
& q(s, t)=\cos t+\sin s
\end{aligned}
$$

Note that, as expected, this is an immersed surface in $\mathbb{R}^{5}$. However, it cannot be represented globally as a graph of $z$ as a function of $x$ and $y$. In fact, we have

$$
d x \wedge d y=\cos (t-s) d t \wedge d s
$$

Thus, for example, we cannot solve smoothly for $s$ and $t$ as functions of $x$ and $y$ in any region where $\cos (t-s)$ vanishes. In fact, restricting to the region where $|t-s|<\pi / 2$, we can solve for $z$ as a function of $x$ and $y$, getting

$$
z(x, y)=\sin ^{-1}\left(1-\frac{1}{2}\left(x^{2}+y^{2}\right)\right)+\cos \left(\sin ^{-1}\left(1-\frac{1}{2}\left(x^{2}+y^{2}\right)\right)\right) .
$$

Note that this solution cannot be extended smoothly beyond the punctured disk $0<x^{2}+y^{2}<2$.

Example 3: Let $k$ be a smooth positive function on $\mathbb{R}$. Consider the non-linear equation

$$
z_{y y}-\left(k\left(z_{x}\right)\right)^{2} z_{x x}=0
$$

known as the Fermi-Pasta-Ulam equation, or FPU equation, for short. The corresponding hyperbolic exterior differential system on $\mathbb{R}^{5}$ with standard coordinates $x, y, z, p$, and $q$ is

$$
\begin{aligned}
\mathcal{I} & =\left\{d z-p d x-q d y, d p \wedge d x+d q \wedge d y, d q \wedge d x+(k(p))^{2} d p \wedge d y\right\} \\
& =\{d z-p d x-q d y,(d q+k(p) d p) \wedge(d x+k(p) d y),(d q-k(p) d p) \wedge(d x-k(p) d y)\}
\end{aligned}
$$

Let $K(p)$ be an antiderivative of $k(p)$, and note that $K$ is a strictly increasing function $K: \mathbb{R} \rightarrow \mathbb{R}$ which is a diffeomorphism of $\mathbb{R}$ onto its image.

We are going to show that if the range of $K$ is all of $\mathbb{R}$, then $\left(\mathbb{R}^{5}, \mathcal{I}\right)$ is complete in the above sense, in spite of the well-known fact (which we shall discuss later) that solutions to the FPU equation with compactly supported initial data develop singularities in finite time.

Now, the characteristic systems are

$$
\begin{aligned}
& \Xi_{10}=\{d z-p d x-q d y,(d q+k(p) d p),(d x+k(p) d y)\} \\
& \Xi_{01}=\{d z-p d x-q d y,(d q-k(p) d p),(d x-k(p) d y)\}
\end{aligned}
$$

Suppose that $\phi: \mathbb{R} \rightarrow \mathbb{R}^{5}$ is a non-characteristic integral curve of $\mathcal{I}$. Thus,

$$
\phi(t)=\left(x_{0}(t), y_{0}(t), z_{0}(t), p_{0}(t), q_{0}(t)\right)
$$

where $z_{0}^{\prime}(t)=p_{0}(t) x_{0}^{\prime}(t)+q_{0}(t) y_{0}^{\prime}(t)$, but where, also, for all $t$, we have

$$
\left(q_{0}^{\prime}(t) \pm k\left(p_{0}(t)\right) p_{0}^{\prime}(t)\right)^{2}+\left(x_{0}^{\prime}(t) \pm k\left(p_{0}(t)\right) y_{0}^{\prime}(t)\right)^{2}>0
$$

We want to show how to construct $f: \mathbb{R}^{2} \rightarrow \mathbb{R}^{5}$ as in the above Theorem. First, if there is a solution, then there will be functions $A, B, C$ and $D$ on $\mathbb{R}^{2}$ so that

$$
\begin{array}{lll}
d q+k(p) d p & =2 A(s, t) d s \\
d x+k(p) d y & =2 B(s, t) d s & \text { and }
\end{array} \quad \begin{array}{ll}
d q-k(p) d p=2 C(s, t) d t \\
& \\
d x-k(p) d y=2 D(s, t) d t
\end{array}
$$

Since $d(q+K(p))$ is a multiple of $d s$, it follows that if there is to be a solution, then $q+K(p)$ will have to be a function of $s$ alone. Thus

$$
q(s, t)+K(p(s, t))=q(s, s)+K(p(s, s))=q_{0}(s)+K\left(p_{0}(s)\right) .
$$

By similar reasoning using the other characteristic system, we see that

$$
q(s, t)-K(p(s, t))=q_{0}(t)-K\left(p_{0}(t)\right) .
$$

Solving these equations gives the only possibility for the functions $p$ and $q$ if there is to be a solution:

$$
\begin{aligned}
q(s, t) & =\frac{1}{2}\left(q_{0}(s)+q_{0}(t)+K\left(p_{0}(s)\right)-K\left(p_{0}(t)\right)\right) \quad \text { and } \\
K(p(s, t)) & =\frac{1}{2}\left(q_{0}(s)-q_{0}(t)+K\left(p_{0}(s)\right)+K\left(p_{0}(t)\right)\right) .
\end{aligned}
$$

Note that the assumption that the range of $K$ be all of $\mathbb{R}$ implies that this does, indeed define a (unique) function $p: \mathbb{R}^{2} \rightarrow \mathbb{R}$. Now, we also have

$$
\begin{aligned}
d x & =B(s, t) d s+D(s, t) d t \\
k(p) d y & =B(s, t) d s-D(s, t) d t
\end{aligned}
$$

so it follows that $B(s, t)=x_{s}(s, t)$ and $D(s, t)=x_{t}(s, t)$. Substituting this into the second equation gives

$$
d y=(k(p(s, t)))^{-1}\left(x_{s}(s, t) d s-x_{t}(s, t) d t\right)
$$

Set $\lambda(s, t)=\log (k(p(s, t)))$, then differentiating this last equation and multiplying by $e^{\lambda}$ gives the formula

$$
2 x_{s t} d s \wedge d t+d \lambda \wedge\left(x_{s} d s-x_{t} d t\right)=0
$$

and this expands to the linear equation (for $x(s, t)$ )

$$
2 x_{s t}-\lambda_{t} x_{s}+\lambda_{s} x_{t}=0 .
$$

Note also that we have the initial conditions $x(t, t)=x_{0}(t)$ and $x_{s}(t, t)-x_{t}(t, t)=$ $k\left(p_{0}(t)\right) y_{0}^{\prime}(t)$. Since $\lambda$ is known, this equation is a linear hyperbolic equation for $x$
in the $s t$-plane, with initial conditions posed along the non-characteristic line $s=t$. By the usual existence and uniqueness theorems for linear equations, there exists a unique function $x(s, t)$ on the whole $s t$-plane satisfying this equation and the given initial conditions. Once $x$ is known, the fact that it is a solution of the linear equation implies that the 1 -form $(k(p(s, t)))^{-1}\left(x_{s}(s, t) d s-x_{t}(s, t) d t\right)$ is closed. Hence, it is the exterior derivative of a function $y$. By choosing the additive constant correctly, we can make sure that, $y(0,0)=y_{0}(0)$, and then the equations above easily imply that $y(t, t)=y_{0}(t)$ for all $t \in \mathbb{R}$. Finally, the function $z(s, t)$ is constructed in the obvious way:

$$
z(s, t)=z_{0}(0)+\int_{(0,0)}^{(s, t)} p d x+q d y
$$

(the path used for the line integral is immaterial since, by construction, $p d x+q d y$ is a closed 1 -form) and the strip conditions then imply that $z(t, t)=z_{0}(t)$. It is now an elementary matter to check that the map $f: \mathbb{R}^{2} \rightarrow \mathbb{R}^{5}$

$$
f(s, t)=(x(s, t), y(s, t), z(s, t), p(s, t), q(s, t))
$$

pulls back the forms in $\mathcal{I}$ to be zero. It is not, however, obvious that $f$ is an immersion except along some open set containing the line $s=t$.
1.3 Prolongation of hyperbolic systems and their structure equations. In the theory of exterior differential systems, the operation of prolongation plays a central role. This operation is analagous to the process in classical PDE whereby one adjoins the derivatives of the unknown functions as new variables and then adjoins new partial differential equations to ensure that the new unknowns do, in fact, behave like the derivatives of the original unknowns. As usual, for a more complete explanation of the process of prolongation, we refer the reader to $\left[\mathrm{BCG}^{3}\right]$.
1.3.1 Prolongations. We denote by $G_{2}(T M)$ the Grassmann bundle whose fibre over $x \in M$ consists of all 2-planes $E \subset T_{x} M$. Sitting in $G_{2}(T M)$ is the set $G_{2}(\mathcal{I})$ of integral 2-planes of the hyperbolic system $\mathcal{I}$. We then have the base point fibration

where, by our discussion above, each fibre of $\pi$ is a $\mathbb{P}^{1} \times \mathbb{P}^{1}$.
Recall that over $G_{2}(T M)$ there is a tautological Pfaffian system $\mathcal{J}$ whose integral manifolds are the canonical lifts (Gauss maps) of immersed surfaces in $M$,
as expressed by the diagram

where $f_{*}(s)=T_{f(s)}(f(S))$.
For a point $(x, E) \in G_{2}(T M)$ we have by definition

$$
\mathcal{J}_{(x, E)}=\pi^{*}\left(E^{\perp}\right)
$$

where $E \subset T_{x} M$ is a 2-plane and $E^{\perp} \subset T_{x}^{*} M$ its codimension 2 annihilator. The restriction to $G_{2}(\mathcal{I}) \subset G_{2}(T M)$ of the canonical system $\mathcal{J}$ is, by definition, the (first) prolongation $\left(M^{(1)}, \mathcal{I}^{(1)}\right)$ of $(M, \mathcal{I})$; thus

$$
M^{(1)}=G_{2}(\mathcal{I}) \quad \text { and } \quad \mathcal{I}^{(1)}=\left.\mathcal{J}\right|_{M^{(1)}}
$$

We want to elucidate the local structure of the ideal $\mathcal{I}^{(1)}$. To this end, we prove the following structure theorem.

Proposition: The prolongation of a hyperbolic system of class $s$ is a hyperbolic system of class $s+2$.

Proof: Suppose that, on an open set $U$ of $M$, we have chosen an admissable local coframing $(\theta ; \omega)$ for $\mathcal{I}$ as in Section 1.1.2. Thus, on $U$, we have

$$
\mathcal{I}=\left\{\theta^{1}, \ldots, \theta^{s} ; \omega^{1} \wedge \omega^{2}, \omega^{3} \wedge \omega^{4}\right\}
$$

If $E_{0} \in G_{2}(\mathcal{I})$ is an integral element based in $U$, then, by changing basis in the $\omega^{i}$, we can arrange that $\omega^{1} \wedge \omega^{3}$ does not vanish on $E_{0}$. Let $W \subset M^{(1)}$ denote the open subset of 2-dimensional integral elements of $\mathcal{I}$ with base point in $U$ and on which the 2 -form $\omega^{1} \wedge \omega^{3}$ does not vanish. This is, of course, an open neighborhood of $E_{0}$ in $M^{(1)}$. For any $E \in W$, the space $E^{\perp} \subset T_{\pi(E)}^{*} M$ is spanned by $s+21$-forms. Since $\omega_{\pi(E)}^{1}$ and $\omega_{\pi(E)}^{3}$ are linearly independent on $E$, while all of the forms in $\mathcal{I}$ must vanish on $E$, it follows that $E^{\perp}$ has a unique basis of the form

$$
\theta_{\pi(E)}^{1}, \ldots, \theta_{\pi(E)}^{s}, \omega_{\pi(E)}^{2}-h_{20}(E) \omega_{\pi(E)}^{1}, \omega_{\pi(E)}^{4}-h_{02}(E) \omega_{\pi(E)}^{3}
$$

The functions $h_{20}, h_{02}$ thus defined on $W$ define coordinates on the fibers of $W(\subset$ $\left.U^{(1)}\right) \rightarrow U$. Moreover, inspection, combined with the definition of $\mathcal{I}^{(1)}$, shows that the 1 -forms

$$
\begin{aligned}
& \theta_{10}=\omega^{2}-h_{20} \omega^{1} \\
& \theta_{01}=\omega^{4}-h_{02} \omega^{3}
\end{aligned}
$$

lie in the Pfaffian system which generates $\mathcal{I}^{(1)} .{ }^{6}$ Thus, on $W \subset M^{(1)}$, the first prolongation $\mathcal{I}^{(1)}$ is generated by the Pfaffian system

$$
I^{(1)}=\left\{\theta^{1}, \ldots, \theta^{s}, \theta_{10}, \theta_{01}\right\}
$$

By construction

$$
\begin{aligned}
& \omega^{1} \wedge \omega^{2}=\omega^{1} \wedge \theta_{10} \\
& \omega^{3} \wedge \omega^{4}=\omega^{3} \wedge \theta_{01}
\end{aligned}
$$

It follows that the algebraic ideal generated by the sections of $I^{(1)}$ contains all of the forms in $\mathcal{I}$. In particular, we have

$$
d \theta^{1} \equiv \cdots \equiv d \theta^{s} \equiv 0 \bmod \left\{\theta^{1}, \ldots, \theta^{s}, \theta_{10}, \theta_{01}\right\}
$$

Moreover, since $\omega^{2}-h_{20} \omega^{1} \equiv \omega^{4}-h_{02} \omega^{3} \equiv 0 \bmod \left\{\theta^{1}, \ldots, \theta^{s}, \theta_{10}, \theta_{01}\right\}$ and since the 2 -forms $d \omega^{i}$ are well-defined on $U$, it follows that there are (unique) functions $A^{i}$ so that

$$
d \omega^{i} \equiv A^{i} \omega^{1} \wedge \omega^{3} \bmod \left\{\theta^{1}, \ldots, \theta^{s}, \theta_{10}, \theta_{01}\right\}
$$

It follows that

$$
\left.\begin{array}{rl}
d \theta_{10} & \equiv-\left(d h_{20}+\left(A^{2}-h_{20} A^{1}\right) \omega^{3}\right) \wedge \omega^{1} \equiv-\pi_{20 \wedge \omega^{1}} \\
d \theta_{01} \equiv-\left(d h_{02}-\left(A^{4}-h_{02} A^{3}\right) \omega^{1}\right) \wedge \omega^{3} \equiv-\pi_{02} \wedge \omega^{3}
\end{array}\right\} \bmod \left\{\theta^{1}, \ldots, \theta^{s}, \theta_{10}, \theta_{01}\right\}
$$

where we have set $\pi_{20}=d h_{20}+\left(A^{2}-h_{20} A^{1}\right) \omega^{3}$ and $\pi_{02}=d h_{02}-\left(A^{4}-h_{02} A^{3}\right) \omega^{1}$. It follows that on $W$ the ideal $\mathcal{I}^{(1)}$ has the form

$$
\mathcal{I}^{(1)}=\left\{\theta^{1}, \ldots, \theta^{s}, \theta_{10}, \theta_{01} ; \pi_{20 \wedge \omega^{1}}, \pi_{02 \wedge \omega^{3}}\right\}
$$

Thus, $\mathcal{I}^{(1)}$ is a hyperbolic exterior differential system, as desired.
6) Here and elswhere, we adopt the common practice of writing simply $\phi$ instead of $\pi^{*}(\phi)$ to denote the pullback via the submersion $\pi: W \rightarrow U$ whenever this abbreviation will not cause confusion. Since, in this case, $W=U \times \mathbb{R}^{2}$ with $h_{20}$ and $h_{02}$ forming the coordinates on the $\mathbb{R}^{2}$-factor, confusion is almost impossible.

We will let $\mathcal{H}_{s}$ denote the set of hyperbolic exterior differential systems of class $s$. Prolongation then gives a mapping

$$
\mathcal{P}: \mathcal{H}_{s} \rightarrow \mathcal{H}_{s+2}
$$

This process can be iterated, leading to the successive higher prolongations,

$$
\left(M^{(k)}, \mathcal{I}^{(k)}\right) \in \mathcal{H}_{s+2 k}
$$

We now want to remark on the relationship between the integral manifolds of $\mathcal{I}$ and those of $\mathcal{I}^{(1)}$. Given any integral surface $f: S \rightarrow M$ of $\mathcal{I}$, we will denote by $f^{(1)}: S \rightarrow M^{(1)}$ the canonical lift of $f$ defined by the rule

$$
f^{(1)}(s)=f^{\prime}(s)\left(T_{s} S\right)=T_{f(s)} f(S)
$$

Since $f$ is an immersion, so is $f^{(1)}$. Moreover, this latter map is clearly transverse to the fibers of the basepoint fibration $\pi: M^{(1)} \rightarrow M$. By the tautological properties of the canonical system $\mathcal{J}$ on the Grassman bundle $G_{2}(T M)$ it also follows that $f^{(1)}: S \rightarrow M^{(1)}$ is an integral surface of $\mathcal{I}^{(1)}$.

Conversely, any integral surface $g: S \rightarrow M^{(1)}$ of $\mathcal{I}^{(1)}$ which is transverse to the fibers of $\pi: M^{(1)} \rightarrow M$ must be of the form $g=f^{(1)}$ for some unique integral surface $f: S \rightarrow M$ of $\mathcal{I}$; in fact, $f=\pi \circ g$. Thus, the integral surfaces of $\mathcal{I}$ in $M$ are in one-to-one correspondence with the integral surfaces of $\mathcal{I}^{(1)}$ in $M^{(1)}$ which are transverse to the fibers of $\pi$. Obviously, this construction can be repeated, yielding integral surfaces $f^{(k)}: S \rightarrow M^{(k)}$ of $\mathcal{I}^{(k)}$ for all $k \geq 0$.
1.3.2 Relations with the initial value problem. Regarding the initial value problem, it is important to note that a non-characteristic integral curve $\phi: C \rightarrow M$ of $\mathcal{I}$ also has a canonical lifting to a non-characteristic integral curve $\phi^{(1)}: C \rightarrow M^{(1)}$. This lifting is defined by letting $\phi^{(1)}(t)$ be the unique 1 -dimensional integral element of $\mathcal{I}$ which contains $\phi^{\prime}(t)\left(T_{t} C\right)$. Again, every non-characteristic integral curve $\gamma$ : $C \rightarrow M^{(1)}$ of $\mathcal{I}^{(1)}$ which is transverse to the fibers of $\pi$ is of the form $\gamma=\phi^{(1)}$ where $\phi=\pi \circ \gamma$. As we shall see, this construction and its iterates $\phi^{(k)}: C \rightarrow M^{(k)}$ will play an important role in Darboux' method of integration.
1.3.3 Some examples. To illustrate the prolongation construction, consider the exterior differential system associated in the usual way to a second order hyperbolic equation ${ }^{7}$ as discussed in Example 7 in Section 1.1.3

$$
F(x, y, z, p, q, r, s, t)=0
$$

[^2]Here, we are following the classical notation: $p=z_{x}, q=z_{y}, r=z_{x x}, s=z_{x y}$, and $t=z_{y y}$. These are to be regarded as coordinates on the 2-jet space $J^{2}\left(\mathbb{R}^{2}, \mathbb{R}\right)$ endowed with its canonical second order contact ideal $\mathcal{J}$ generated by the three 1 -forms

$$
\begin{aligned}
\theta^{0} & =d z-p d x-q d y \\
\theta^{1} & =d p-r d x-s d y \\
\theta^{2} & =d q-s d x-t d y
\end{aligned}
$$

The pullback of this system to the 7 -dimensional hypersurface $M$ defined by the equation $F=0$ will then be a hyperbolic exterior differential system $(M, \mathcal{I})$ which lies in $\mathcal{H}_{3}$.

In case the equation $F=0$ happens to be a Monge-Ampere equation

$$
\begin{equation*}
E\left(r t-s^{2}\right)+A r+2 B s+C t+D=0 \tag{1}
\end{equation*}
$$

where $A, B, C, D, E$ are functions of $(x, y, z, p, q)$, we showed in the previous section how one may associate a hyperbolic exterior differential system $\left(M_{0}, \mathcal{I}_{0}\right) \in$ $\mathcal{H}_{1}$ to the equation. It turns out, as the reader may check, that the first prolongation of $\left(M_{0}, \mathcal{I}_{0}\right) \in \mathcal{H}_{1}$ contains $(M, \mathcal{I}) \in \mathcal{H}_{3}$ as a dense open set. More precisely, $(M, \mathcal{I})$ is the open subset of $\left(M_{0}^{(1)}, \mathcal{I}_{0}^{(1)}\right)$ consisting of those integral elements on which $d x \wedge d y \neq 0$.

Similarly, a first order hyperbolic system for two functions of two independent variables

$$
\left\{\begin{array}{l}
F(x, y, u, v, p, q, r, s)=0  \tag{1}\\
G(x, y, u, v, p, q, r, s)=0
\end{array}\right.
$$

where $p=u_{x}, q=u_{y}, r=v_{x}$, and $s=v_{y}$ can always be expressed as a hyperbolic exterior differential system $(M, \mathcal{I}) \in \mathcal{H}_{2}$. However, in case (2) is a quasi-linear system, we saw in Section 1.1 .3 that there is an associated hyperbolic exterior differential system $\left(M_{0}, \mathcal{I}_{0}\right) \in \mathcal{H}_{0}$. Again, it turns out that the open subset of the first prolongation of this latter system consisting of the integral elements on which $d x \wedge d y \neq 0$ is the exterior differential system $(M, \mathcal{I}) \in \mathcal{H}_{2}$ canonically associated to (2).
1.3.4 Partial Prolongations. Variations on the prolongation construction are possible and are ocassionally encountered in the theory. Suppose given a hyperbolic exterior differential system $(M, \mathcal{I})$ with the property that the $\left(\mathbb{P}^{1} \cup \mathbb{P}^{1}\right)$-bundle $\Xi \rightarrow M$ can be written as a disjoint union

$$
\Xi=M_{10}^{(1)} \cup M_{01}^{(1)}
$$

where each of $M_{10}^{(1)} \rightarrow M$ and $M_{01}^{(1)} \rightarrow M$ is a $\mathbb{P}^{1}$-bundle. (This can always be arranged by passing to a double cover of $M$ if necessary.) Now, every 2-dimensional
integral element $E \subset T_{x} M$ of $\mathcal{I}$ can be written uniquely in the form $E=L_{10} \oplus L_{01}$ with $L_{10} \in M_{10}^{(1)}$ and $L_{01} \in M_{01}^{(1)}$. It follows that there is a diagram of submersions


It is not difficult to show that there exist canonically constructed hyperbolic exterior differential systems $\mathcal{I}_{10}^{(1)}$ and $\mathcal{I}_{01}^{(1)}$ on $M_{10}^{(1)}$ and $M_{01}^{(1)}$ respectively so that every integral surface $f: S \rightarrow M$ of $\mathcal{I}$ has canonical liftings $f_{10}^{(1)}: S \rightarrow M_{10}^{(1)}$ and $f_{01}^{(1)}: S \rightarrow M_{01}^{(1)}$ which are integral surfaces of $\mathcal{I}_{10}^{(1)}$ and $\mathcal{I}_{01}^{(1)}$ respectively. Thus, these partial prolongations are canonically defined and, for a system of class $s$, we have $\left(M_{10}^{(1)}, \mathcal{I}_{10}^{(1)}\right)$ and $\left(M_{01}^{(1)}, \mathcal{I}_{01}^{(1)}\right)$ in $\mathcal{H}_{s+1}$.

Thus, hyperbolic exterior differential systems of classes $s=0,1,2, \ldots$ form a very natural and interrelated set of exterior differential systems. Although we are primarily interested in the cases $s=0$ and $s=1$, it is convenient to set these in a general context as we have done.
1.3.5 The refined structure equations. We now want to derive more refined structure equations for the higher prolongations which will be fundamental in our later calculation of the conservation laws of $\left(M^{s+4}, \mathcal{I}\right)$. In fact, we will now show that the $k^{\text {th }}$ prolongation satisfies a remarkable set of structure equations.

Proposition: Let $(M, \mathcal{I})$ be a hyperbolic exterior differential system of class s. Then on the manifold $M^{(k)}$ (of dimension $s+4+2 k$ ) there is a coframing

$$
\theta^{1}, \ldots, \theta^{s}, \theta_{10}, \ldots, \theta_{k 0}, \theta_{01}, \ldots, \theta_{0 k}, \pi_{k+1,0}, \omega_{10}, \pi_{0, k+1}, \omega_{01}
$$

such that $\mathcal{I}^{(k)}$ is generated as a differential ideal by $\theta^{1}, \ldots, \theta^{s}, \theta_{10}, \ldots, \theta_{k 0}, \theta_{01}, \ldots$, $\theta_{0 k}$ which satisfies the structure equations



We will discuss the meaning of these equations below. For now, we merely note that they immediately imply the earlier proposition from Section 1.3.1 to the effect that ( $M^{(k)}, \mathcal{I}^{(k)}$ ) is a hyperbolic system; i.e., $d \theta_{k 0}$ and $d \theta_{0 k}$ are decomposable modulo all of the $\theta$ 's. In fact, however, these structure equations go much further in that they give information on $d \theta_{k 0}$ modulo only $\theta^{1}, \ldots, \theta^{s}, \theta_{10}, \ldots, \theta_{k 0}$; i.e., omitting
$\theta_{01}, \ldots, \theta_{0 k}$. In later sections this fact (as well as others derivable from the above structure equations) will play an important role.

Proof: We are going to follow the notation of the earlier Proposition in Section 1.3.1. Recall that, by the definition of a hyperbolic system, we have, using $\Omega_{10}=\omega_{10} \wedge \theta_{10}$ and $\Omega_{01}=\omega_{01} \wedge \theta_{01}$, that

$$
\begin{equation*}
d \theta^{\alpha} \equiv A_{10}^{\alpha} \omega_{10} \wedge \theta_{10}+A_{01}^{\alpha} \omega_{01} \wedge \theta_{01} \bmod \left\{\theta^{\alpha}\right\} \tag{4}
\end{equation*}
$$

For $k=1$, equation (i) is

$$
d \theta_{10} \equiv-\pi_{20} \wedge \omega_{10}+T_{10} \omega_{01} \wedge \theta_{01} \bmod \left\{\theta^{1}, \ldots, \theta^{s}, \theta_{10}\right\}
$$

Now $\Omega_{10}$ is defined on the original manifold $M$, and, since the 1 -forms $\theta^{1}, \ldots, \theta^{s}$, $\theta_{10}, \omega_{10}, \theta_{01}, \omega_{01}$ are a basis for the semi-basic forms for the projection $M^{(1)} \rightarrow M$, it follows that

$$
d \Omega_{10} \in \Lambda^{3}\left[\theta^{1}, \ldots, \theta^{s}, \theta_{10}, \omega_{10}, \theta_{01}, \omega_{01}\right]
$$

In fact, from the formula for $d \theta_{10}$ given above we have

$$
d \Omega_{10} \equiv-T_{10} \omega_{10} \wedge \omega_{01} \wedge \theta_{01} \bmod \left\{\theta^{1}, \ldots, \theta^{s}, \theta_{10}\right\}
$$

Moreover, $\omega_{10}$ is actually defined on $M=M^{(0)}$ (as opposed to $\theta_{10}$, which is only defined on $M^{(1)}$ even though it is semi-basic for the projection $\left.M^{(1)} \rightarrow M^{(0)}\right)$. Thus we have

$$
d \omega_{10} \in \Lambda^{2}\left[\theta^{1}, \ldots, \theta^{s}, \theta_{10}, \omega_{10}, \theta_{01}, \omega_{01}\right]
$$

Now using a similar argument for $d \omega_{01}$ yields the formulae

$$
\begin{align*}
d \omega_{10} & \equiv S_{10} \omega_{01} \wedge \theta_{01} \bmod \left\{\theta^{1}, \ldots, \theta^{s}, \theta_{10}, \omega_{10}\right\} \\
d \omega_{01} & \equiv S_{01} \omega_{10} \wedge \theta_{10} \bmod \left\{\theta^{1}, \ldots, \theta^{s}, \theta_{01}, \omega_{01}\right\} \tag{5}
\end{align*}
$$

From (5) and

$$
d \Omega_{10}=d \omega_{10} \wedge \theta_{10}-\omega_{10} \wedge d \theta_{10}
$$

we may infer that there exists a 1 -form $\pi_{20}$ on $M^{(1)}$ such that

$$
d \theta_{10} \equiv-\pi_{20} \wedge \omega_{10} \bmod \Lambda^{2}\left[\theta^{1}, \ldots, \theta^{s}, \theta_{10}, \omega_{10}, \theta_{01}, \omega_{01}\right]
$$

Thus

$$
d \theta_{10} \equiv-\pi_{20 \wedge \omega_{10}}+\eta \wedge \omega_{10}+T_{10} \omega_{01} \wedge \theta_{01} \bmod \left\{\theta^{1}, \ldots, \theta^{s}, \theta_{10}\right\}
$$

where $\eta$ is a linear combination of $\theta_{01}$ and $\omega_{01}$. Replacing $\pi_{20}$ by $\pi_{20}-\eta$ and relabeling we have

$$
d \theta_{10} \equiv-\pi_{20} \wedge \omega_{10}+T_{10} \omega_{01} \wedge \theta_{01} \bmod \left\{\theta^{1}, \ldots, \theta^{s}, \theta_{10}\right\}
$$

This establishes (3) for the case $k=1$.

Assuming by induction that (3) holds for some $k \geqq 1$, we shall establish it for $k+1$.

On integral 2-planes $E$ of $\mathcal{I}^{(k)}$ we have

$$
\begin{array}{r}
\left.\pi_{k+1,0} \wedge \omega_{10}\right|_{E}=0 \\
\left.\omega_{10}\right|_{E} \neq 0
\end{array}
$$

so that $M^{(k+1)}$ is locally $M^{(k)} \times \mathbb{R}^{2}$ where $\mathbb{R}^{2}$ has coordinates ( $h_{k+2,0}, h_{0, k+2}$ ) and where

$$
\begin{aligned}
\theta_{k+1,0} & =\pi_{k+1,0}-h_{k+2,0} \omega_{10} \\
\theta_{0, k+1} & =\pi_{0, k+1}-h_{0, k+2} \omega_{01}
\end{aligned}
$$

In fact, the vanishing of $\theta_{k+1,0}$ and $\theta_{0, k+1}$ defines the 2 -planes on which $\pi_{k+1,0} \wedge \omega_{10}$ and $\pi_{0, k+1} \wedge \omega_{01}$ restrict to zero but $\omega_{10} \wedge \omega_{01} \neq 0$. From (1.i) we have on $M^{(k+1)}$

$$
d \theta_{k 0} \equiv-\theta_{k+1,0} \wedge \omega_{10}+T_{k 0} \omega_{01} \wedge \theta_{01} \bmod \left\{\theta^{1}, \ldots, \theta^{s}, \theta_{10}, \ldots, \theta_{k 0}\right\}
$$

We shall write this as

$$
\begin{equation*}
d \theta_{k 0} \equiv-\theta_{k+1,0} \wedge \omega_{10}+T_{k 0} \Omega_{01} \bmod \left\{\theta^{1}, \ldots, \theta^{s}, \theta_{10}, \ldots, \theta_{k 0}\right\} \tag{6}
\end{equation*}
$$

Taking the exterior derivative of this equation and substituting (1.i) for $d \theta_{j 0}$ for $j=1, \ldots, k-1$ together with (6) gives

$$
\begin{equation*}
0 \equiv-d \theta_{k+1,0} \wedge \omega_{10}+\chi \wedge \omega_{01} \wedge \theta_{01} \bmod \left\{\theta^{1}, \ldots, \theta^{s}, \theta_{10}, \ldots, \theta_{k+1,0}\right\} \tag{7}
\end{equation*}
$$

We want to say a word about $\chi$. It arises from the terms $d T_{k 0 \wedge} \Omega_{01}$ and $T_{k 0} d \Omega_{01}$ and the coefficients of $\omega_{01} \wedge \theta_{01}$ in $d \theta^{1}, \ldots, d \theta^{s}, d \theta_{10}, \ldots, d \theta_{k 0}$. Since

$$
d \Omega_{01} \in \Lambda^{3}\left[\theta_{10}, \omega_{10}, \theta_{01}, \omega_{01}\right] \bmod \left\{\theta^{1}, \ldots, \theta^{s}\right\}
$$

we have

$$
d \Omega_{01} \equiv a \omega_{10 \wedge} \wedge \omega_{01} \wedge \theta_{01} \bmod \left\{\theta^{1}, \ldots, \theta^{s}, \theta_{10}\right\}
$$

From (7) we deduce that

$$
\chi \equiv 0 \bmod \left\{\theta^{1}, \ldots, \theta^{s}, \theta_{10}, \ldots, \theta_{k+1,0}, \theta_{01}, \omega_{10}, \omega_{01}\right\}
$$

and hence we infer the existence of a form $\pi_{k+2,0}$ on $M^{(k+1)}$ such that

$$
d \theta_{k+1,0} \equiv-\pi_{k+2,0} \wedge \omega_{10}+T_{k+1,0} \omega_{01} \wedge \theta_{01} \bmod \left\{\theta^{1}, \ldots, \theta^{s}, \theta_{10}, \ldots, \theta_{k+1,0}\right\}
$$

This completes the induction step in the proof of (3).
1.3.6 Interpretations - higher characteristic systems. We would like to discuss some of the meaning of the structure equations (3). First, we give an important definition.

Definition: The $k^{\text {th }}$ characteristic systems of a hyperbolic system are the Pfaffian systems ${ }^{8}$

$$
\begin{aligned}
& \Xi_{10}^{(k)}=\left[\theta^{1}, \ldots, \theta^{s}, \theta_{10}, \ldots, \theta_{k 0}, \pi_{k+1,0}, \omega_{10}\right] \\
& \Xi_{01}^{(k)}=\left[\theta^{1}, \ldots, \theta^{s}, \theta_{01}, \ldots, \theta_{0 k}, \pi_{0, k+1}, \omega_{01}\right] .
\end{aligned}
$$

As explained above, each integral surface $f: S \rightarrow M$ of the original hyperbolic system has its $k^{\text {th }}$-prolongation $f^{(k)}: S \rightarrow M^{(k)}$ which is an integral surface of $\mathcal{I}^{(k)}$, the hyperbolic exterior differential system generated as a differential system by $\theta^{1}, \ldots, \theta^{s}, \theta_{10}, \ldots, \theta_{k 0}, \theta_{01}, \ldots, \theta_{0 k}$ on $M^{(k)}$ (here $k \geqq 1$ ). It follows from (3) that

$$
\pi_{k+1,0} \wedge \omega_{10}=0
$$

on $f^{(k)}(S)$, so that $\Xi_{10}^{(k)}$ induces on $f^{(k)}(S)$ a Pfaffian system of rank one, and similarly for $\Xi_{01}^{(k)}$. Of course, these induce the same two characteristic foliations on solution surfaces to the hyperbolic exterior differential system as the $0^{\text {th }}$ characteristic systems. We will see in Section 1.4 that the geometry - meaning the derived flags, etc. - of the characteristic systems is an important feature of a hyperbolic exterior differential system.

One interpretation of the structure equations (3) is that the characteristic systems $\Xi_{10}^{(k)}$ and $\Xi_{01}^{(k)}$ are coupled only at the first level. To explain this, we pass to the infinite prolongation $M^{(\infty)}$ with coframing ${ }^{9}$

$$
\theta^{1}, \ldots, \theta^{s} ; \theta_{10}, \theta_{20}, \ldots ; \theta_{01}, \theta_{02}, \ldots ; \omega_{10}, \omega_{01}
$$

and write the structure equations (3) as
(i) $d \theta_{k 0} \equiv-\theta_{k+1,0} \wedge \omega_{10}+T_{k 0} \Omega_{01} \bmod \left\{\theta^{1}, \ldots, \theta^{s}, \theta_{10}, \ldots, \theta_{k 0}\right\}$
(ii) $d \theta_{0 k} \equiv-\theta_{0, k+1} \wedge \omega_{01}+T_{0 k} \Omega_{10} \bmod \left\{\theta^{1}, \ldots, \theta^{s}, \theta_{01}, \ldots, \theta_{0 k}\right\}$.
8) The Pfaffian system $\Xi_{10}^{(k)}$ and $\Xi_{01}^{(k)}$ are defined on $M^{(k)}$. When lifted to $M^{(k+i)}$ for any $l \geqq 1$ they are given by

$$
\begin{aligned}
& \Xi_{10}^{(k)}=\left[\theta^{1}, \ldots, \theta^{s}, \theta_{10}, \ldots, \theta_{k+1,0}, \omega_{10}\right] \\
& \Xi_{01}^{(k)}=\left[\theta^{1}, \ldots, \theta^{s}, \theta_{01}, \ldots, \theta_{0, k+1}, \omega_{01}\right] .
\end{aligned}
$$

9) The infinite prolongation $\left(M^{(\infty)}, I^{(\infty)}\right)$ of $(M, \mathcal{I})$ is discussed in Section 2 of $[\mathrm{BG}]_{1}$.

The characteristic systems are

$$
\begin{aligned}
& \Xi_{10}^{(\infty)}=\left[\theta^{1}, \ldots, \theta^{s}, \theta_{10}, \theta_{20}, \ldots, \omega_{10}\right] \\
& \Xi_{01}^{(\infty)}=\left[\theta^{1}, \ldots, \theta^{s}, \theta_{01}, \theta_{02}, \ldots, \omega_{01}\right]
\end{aligned}
$$

and (5) and (8) say that the characteristic systems $\Xi_{10}^{(\infty)}, \Xi_{01}^{(\infty)}$ are Frobenius up to the coupling terms $S_{10} \Omega_{01}, S_{01} \Omega_{10}$ and $T_{k 0} \Omega_{01}, T_{0 k} \Omega_{10}$. It will turn out that the coefficients $T_{k 0}, T_{0 k}$ are the fundamental relative invariants of hyperbolic systems of class $s=0$.

In the case $s=0$ the characteristic systems are disjoint and span all of the 1-forms on $M^{(\infty)}$

$$
\begin{aligned}
& \Xi_{10}^{(\infty)} \cup \Xi_{01}^{(\infty)}=\Omega^{1}\left(M^{(\infty)}\right) \\
& \Xi_{10}^{(\infty)} \cap \Xi_{01}^{(\infty)}=(0)
\end{aligned}
$$

Moreover, as we shall see in Section 1.5 below, there are many interesting cases where $S_{10}=S_{01}=0$ (cf. (5) above). In this case, it follows that: If all the $T_{k 0}$ and $T_{0 k}$ vanish, then each of $\Xi_{10}^{(\infty)}$ and $\Xi_{01}^{(\infty)}$ are Frobenius systems. We point this out here to emphasize the importance of these relative invariants.
1.3.7 Even further refinements. The structure equations discussed above are rather general and apply to hyperbolic systems of any class $s$. For the low values of $s$ and with more information on the structure of the original ideal $\mathcal{I}$, we can introduce further refinements.

Begimning in Section 1.5, we will concentrate mainly on the case $s=0$ and will introduce considerable refinements via the method of equivalence. In the rest of this section, however, we want to comment on how they may also be refined when $s=1$. Such a system is given on a 5 -manifold locally as $\mathcal{I}=\left\{\theta, \Omega_{10}, \Omega_{01}\right\}$. Equation (4) simplifies to

$$
d \theta \equiv A_{10} \Omega_{10}+A_{01} \Omega_{01} \bmod \{\theta\}
$$

If $A_{10}=A_{01}=0$, then the 5 -manifold $M$ is foliated in codimension 1 by the leaves of the system $\theta=0$. Each integral manifold of $\mathcal{I}$ lies in one of these leaves and the system essentially reduces to a 1-parameter family of hyperbolic systems of class $s=0$ (cf. Section 1.4 below).

If, say, $A_{10}=0$ but $A_{01} \neq 0$, then we may normalize so as to have the equation $d \theta \equiv \Omega_{01} \bmod \{\theta\}$. In this case, the non-characteristic initial value problem for the system can be solved using only ODE techniques. Here is how this goes. Since $\theta \wedge d \theta=\theta \wedge \Omega_{01} \neq 0$ but $\theta \wedge(d \theta)^{2}=0$, it follows from the Pfaff-Darboux theorem that every point of $M$ lies in some open set $U$ on which there exists a submersion $f: U \rightarrow \mathbb{R}^{3}$ so that $\theta$ restricted to $U$ is a multiple of $f^{*}(d z-y d x)$.

If $\phi:(0,1) \rightarrow U$ is a non-characteristic integral curve of $\mathcal{I}$, then in particular it follows that $f \circ \phi:(0,1) \rightarrow \mathbb{R}^{3}$ is an immersion and that $P=f^{-1}(f \circ \phi((0,1))) \subset$ $U$ is a smooth 3 -manifold which is an integral manifold of $\theta$ and hence of $\Omega_{01}$. Moreover, $P$ clearly contains $\phi((0,1))$. The 2 -form $\Omega_{10}$ restricts to $P$ to have a Cauchy characteristic and the assumption that $\phi$ is non-characteristic implies that $\phi((0,1))$ is transverse to this characteristic line field. It follows that the union of these Cauchy characteristics passing through $\phi((0,1))$ is an integral surface of $\mathcal{I}$ passing through $\phi((0,1))$. (Of course, by uniqueness, there is only one such integral surface.)

Finally, if $A_{10} A_{01} \neq 0$, we may normalize so as to have $A_{10}=A_{01}=1$; i.e.,

$$
\begin{equation*}
d \theta \equiv \Omega_{10}+\Omega_{01} \bmod \{\theta\} \tag{9}
\end{equation*}
$$

It is easy to show that such a system is locally equivalent to the exterior differential system derived from a hyperbolic Monge-Ampere system as in Example 5 in Section 1.1.3 (cf. Appendix 2 to Section 2 in [ $\left.\mathrm{BG}_{2}\right]$ ). The $k^{\text {th }}$ prolongation then has a coframing

$$
\theta ; \theta_{10}, \ldots, \theta_{k 0}, \omega_{10}, \pi_{k+1,0} ; \theta_{01}, \ldots, \theta_{0 k}, \omega_{01}, \pi_{0, k+1}
$$

satisfying the structure equations ( $3, \mathrm{i}-\mathrm{ii}$ ) and it may also be shown that this coframing can be chosen so that, in addition to (9), we have

$$
\begin{align*}
d \theta_{k 0} & \equiv-\pi_{k+1,0} \wedge \omega_{10} \bmod \left\{\theta, \theta_{10}, \ldots, \theta_{k 0}\right\} \\
d \theta_{0 k} & \equiv-\pi_{0, k+1} \wedge \omega_{01} \bmod \left\{\theta, \theta_{01}, \ldots, \theta_{0 k}\right\} \tag{10}
\end{align*}
$$

Thus, all of the invariants $T_{k 0}$ and $T_{0 k}$ vanish identically and the coupling between the characteristic systems only occurs at the $0^{\text {th }}$ level through equation (9).

### 1.4 Integration by the method of Darboux.

1.4.1 Riemann invariants. Let $(M, \mathcal{I})$ be a hyperbolic system of class $s$. We have discussed how to associate to such a system its prolongations $\left(M^{(k)}, \mathcal{I}^{(k)}\right)$, which are hyperbolic systems of class $s+2 k$, and its characteristic systems

$$
\Xi_{10}^{(k)}, \Xi_{01}^{(k)} \subset \Omega^{1}\left(M^{(k)}\right),
$$

which are Pfaffian systems of rank $s+2+k$. The solution surfaces of $(M, \mathcal{I})$ and the solution surfaces of $\left(M^{(k)}, \mathcal{I}^{(k)}\right)$ are in one-to-one correspondence (where the latter must satisfy a transversality condition), and the characteristic systems induce on each solution surface a pair of foliations by characteristic curves.

As we mentioned in Section 1.2, the characteristic foliations on each solution surface carry no local geometry. However, the characteristic systems $\Xi_{10}^{(k)}$ and $\Xi_{01}^{(k)}$
do, indeed, carry local geometry. In fact the integrability properties of the characteristic systems in the ambient manifolds $M^{(k)}$ play a crucial role in the classical integration methods. In particular, the study of integrable subsystems of the characteristic systems turns out to be very fruitful, and leads directly to the method of Darboux ([Go]), which we describe below.

First, a brief historical perspective. In the early days - beginning over two centuries ago - a primary interest was finding explicit solutions of PDEs. Somewhat later the issue was to prove existence either by finding explicit solutions or by giving on algorithm, based on integration, for finding solutions (cf. the Poisson integral formula, etc.). Soon thereafter, this form of existence proof was extended to the nineteenth century concept of "integration of the equation", which meant to reduce finding the solution to (at most) solving a sequence of ordinary differential equations. In this connection the following notion arose.

Definition: A (generalized) Riemann invariant of $\mathcal{I}$ is a codimension one foliation $\mathcal{F}$ on $M^{(k)}$ such that, for any function $f$ constant on the leaves of $\mathcal{F}$ (we say that $f$ belongs to $\mathcal{F}$ ), its differential df lies in either $\Xi_{10}^{(k)}$ or $\Xi_{01}^{(k)}$.

The importance of Riemann invariants stems from the following fact. Let $f$ belong to a Riemann invariant $\mathcal{F}$ of $\mathcal{I}$, with, say $d f \in \Xi_{10}^{(k)}$, and let $\phi: S \rightarrow M$ be an integral surface of $\mathcal{I}$. Then the pull back function $f \circ \phi^{(k)}: S \rightarrow \mathbb{R}$ clearly has the property that it is constant on each of the curves in the characteristic foliation $\mathcal{F}_{10}$. Thus, $f$ functions as a sort of "conservation law" for characteristic curves on solutions. The knowledge of such functions (when they exist) frequently yields important information about the behavior of solutions to the original system.

We should perhaps say a word about our choice of the fundamental object in the notion of a Riemann invariant. One usually thinks of a Riemann invariant as a function of the variables and their derivatives which is constant on the leaves of one of the characteristic foliations on every solution of the equation. However, if $f$ is a Riemann invariant in this more classical sense, then any function of $f$, say $g \circ f$ where $g: \mathbb{R} \rightarrow \mathbb{R}$ is arbitrary, is also a Riemann invariant, with essentially the same level sets as $f$. However, it turns out that it is the level sets of $f$ rather than the function $f$ itself which is important in applications. On the open set where $d f$ is non-vanishing, knowing these level sets is equivalent to knowing a codimension one foliation. For this reason, we have taken the foliation as the fundamental object, preferring to identify Riemann invariants which determine the same level sets.

The above definition generalizes the classical situations in which the notion of a Riemann invariant is often discussed: a first order hyperbolic system for two unknown functions $u(x, y)$ and $v(x, y)$ and a second order hyperbolic equation for one unknown $z(x, y)$. These give hyperbolic systems of classes $s=2$ and $s=3$ respectively, and the classical Riemann invariants in these cases belong to the special case $k=0$. Their use in producing integral formulas for solutions to certain hyperbolic equations is standard and well known [CH].

Before proceeding, we should like to discuss the possibility that a function $f$ belonging to a Riemann invariant $\mathcal{F}$ might be constant on all solution surfaces. Recall that a hyperbolic system $\mathcal{I}$ is given by a rank $s$ Pfaffian system $I$ together with a pair of 2-forms $\Omega_{10}$ and $\Omega_{01}$ that are well defined and decomposable modulo $\{I\}$. The Pfaffian system $I$ induces a rank $s$ Pfaffian system, still denoted by $I$, on each $M^{(k)}$. This abuse of notation is justified for our present purposes, since it is easy to see that integrable subsystems of $I$ on $M^{(0)}$ and $I$ on $M^{(k)}$ are in one-to-one correspondence. ${ }^{10}$

Recalling our notation $I^{\langle i\rangle}$ for the $i^{\text {th }}$ derived system of the Pfaffian system $I$, let $I^{\langle\infty\rangle}=\bigcap_{i} I^{\langle i\rangle}$ be the largest integrable subsystem of $I$. Since $I^{(\infty)}$ is an integrable subsystem of $\mathcal{I}$, locally $M$ is a product

$$
M=N \times U
$$

where $U \subset \mathbb{R}^{n}\left(n=\operatorname{rank} I^{\langle\infty\rangle}\right)$ has coordinates $t=\left(t^{1}, \ldots, t^{n}\right)$. It is easy to see that $\mathcal{I}$ induces on each $N \times\{t\}$ a hyperbolic system $\mathcal{I}_{t}$ such that the solution surfaces of $\mathcal{I}$ are just the solution surfaces of some $\mathcal{I}_{t}$ for a fixed $t$. Thus, if $I^{\langle\infty\rangle} \neq 0$ we essentially have the situation of a family of hyperbolic systems depending on a parameter.

For this reason we shall make the standing assumption that the infinite derived system $I^{\langle\infty\rangle}$ is trivial. Since it is well known that the infinite derived system $I^{(\infty)}$ contains any integrable subsystem of $I$, it follows from our assumption that $I$ contains no integrable subsystems. Under this assumption we have the result.

Proposition: The Pfaffian systems $\mathcal{I}^{(k)}$ contain no integrable subsystems for $k \geq 0$.

Proof: This follows immediately since the structure equations in the last section clearly imply that, for all $k \geq 0$,

$$
\mathcal{I}^{(k)^{\langle 1\rangle}}=\mathcal{I}^{(k-1)}
$$

Note that $\mathcal{I}^{(0)}=I$ and now we apply induction to conclude that $\mathcal{I}^{(k)^{(\infty)}}=0$.
10) The point is this: In a domain $U \subset \mathbb{R}^{n}$ with coordinates $x=\left(x^{1}, \ldots, x^{n}\right)$, let $\theta^{\alpha}=\theta^{\alpha}(x, d x)$ be a set of everywhere linearly independent one forms. Suppose that we have another connected domain $V \subset \mathbb{R}^{n}$ with coordinates $y=\left(y^{1}, \ldots, y^{n}\right)$ and a linear combination

$$
\theta(x, y, d x)=\sum_{\alpha} f_{\alpha}(x, y) \theta^{\alpha}(x, d x), \quad x \in U, y \in V
$$

satisfying

$$
d \theta=0
$$

Then the $f_{\alpha}(x, y)$ are constant in $y$.

This result implies that no non-constant Riemann invariant can be constant on all integral surfaces of $\mathcal{I}$. To see this, note that, if $f$ is a Riemann invariant of $\mathcal{I}$ defined on $M^{(k)}$ and if $d f_{x} \neq 0$, then $d f$ is not a section of $\mathcal{I}^{(k)}$ on a neighborhood of $x$, and hence that there must exist a nearby point $y \in M^{(k)}$ so that $d f_{y} \notin$ $\mathcal{I}_{y}^{(k)}$. In particular, $d f_{y}$ does not vanish on $\left(\mathcal{I}_{y}^{(k)}\right)^{\perp}$ and hence there exists a 2 dimensional integral element $E \subset\left(\mathcal{I}_{y}^{(k)}\right)^{\perp}$ of $\mathcal{I}^{(k)}$ on which $d f_{y}$ does not vanish. Now $f$ will not be constant on any integral surface $S$ of $\mathcal{I}$ which passes through $y$ and satisfies $T_{y} S=E$. (Such an integral surface always exists by the existence theorem in Section 1.2.3.)

The classical theory of Riemann invariants, as well as the even more classical but less well-known theories of Ampere, Monge, Laplace, Darboux, Goursat, etc (cf. [Go]), thus focuses attention on integrable subsystems of the characteristic systems. We shall see that studying the conservation laws does the same thing.
1.4.2 The method of Darboux. We now introduce the main concept from the classical theory.

Definition: A hyperbolic system ( $M, \mathcal{I}$ ) is integrable in the sense of Darboux at level $k$ if there are rank 2 integrable subsystems

$$
\left\{\begin{array}{l}
\Delta_{10} \subset \Xi_{10}^{(k)} \\
\Delta_{01} \subset \Xi_{01}^{(k)}
\end{array}\right.
$$

which satisfy $\Delta_{10} \cap \mathcal{I}^{(k)}=\Delta_{01} \cap \mathcal{I}^{(k)}=(0)$.
We will now explain the method of Darboux for solving the initial value problem for hyperbolic systems which satisfy this hypothesis.

Theorem: Suppose that $(M, \mathcal{I})$ is integrable in the sense of Darboux at level $k$ and that $f:(0,1) \rightarrow M$ is a non-characteristic integral curve of $\mathcal{I}$. Then there exists an open set $U \subset \mathbb{R}^{2}$ which contains the diagonal interval $\Delta=\{(t, t): 0<t<1\}$ and an integral surface $F: U \rightarrow M$ of $\mathcal{I}$ so that $F(t, t)=f(t)$ for all $0<t<1$. Moreover, $F$ can be chosen so that the characteristic foliations induced on $U$ are the coordinate slices and this makes $F$ unique on some neighborhood of $\Delta$. Finally, $F$ can be constructed from $f$ by a procedure involving ordinary differential equations alone.

Proof: Since $f$ is non-characteristic, the tangent space $T_{f(t)} f((0,1))$ lies in a unique 2-dimensional element of $\mathcal{I}$ which we shall denote by $f^{(1)}(t) \in M^{(1)}$. Now, $f^{(1)}$ : $(0,1) \rightarrow M^{(1)}$ satisfies our hypothesis with respect to $\left(M^{(1)}, \mathcal{I}^{(1)}\right)$, i.e., it is a noncharacteristic integral immersion of $\mathcal{I}^{(1)}$. By repeating this process, we eventually arrive at $f^{(k)}:(0,1) \rightarrow M^{(k)}$, which is a non-characteristic integral immersion of
$\mathcal{I}^{(k)}$. Clearly, it now suffices to prove the theorem for the case $k=0$, and then apply it to the initial data $f^{(k)}$.

Thus, let us suppose that $\Delta_{10} \subset \Xi_{10}$ and $\Delta_{01} \subset \Xi_{01}$ are rank 2 integrable subsystems which satisfy $\Delta_{10} \cap I=\Delta_{10} \cap I=(0)$. Fix a $t_{0} \in(0,1)$ and let $x, y$, $u$, and $v$ be functions on a neighborhood $V$ of $f\left(t_{0}\right)$ which have the property that, on $V$, the system $\Delta_{10}$ is spanned by $d x$ and $d y$ while $\Delta_{01}$ is spanned by $d u$ and $d v$. It now follows from the structure equations and the very definitions of $\Xi_{10}$ and $\Xi_{01}$ that $\mathcal{I}$ in $V$ is generated by the sections of $I$ and the two 2-forms $d x \wedge d y$ and $d u \wedge d v$. Note that, also, by construction, for each $p \in V$ the space $I_{p}^{\perp}$ is transverse to the fibers of the the submersion $(x, y, u, v): V \rightarrow \mathbb{R}^{4}$.

Now, there is a a $\delta>0$ so that $0<t_{0}-\delta<t_{0}+\delta<1$ and so that

$$
f\left(\left(t_{0}-\delta, t_{0}+\delta\right)\right) \subset V
$$

By the assumption that $f$ is non-characteristic, $f$ is transverse to the fibers of both of the mappings $(x, y): V \rightarrow \mathbb{R}^{2}$ and $(u, v): V \rightarrow \mathbb{R}^{2}$. It follows that, by shrinking $\delta$, we may assume that $(x, y) \circ f$ and $(u, v) \circ f$ are smooth embeddings of the interval $\left(t_{0}-\delta, t_{0}+\delta\right)$ into $\mathbb{R}^{2}$. Set

$$
\Gamma_{10}=(x, y) \circ f\left(\left(t_{0}-\delta, t_{0}+\delta\right)\right) \quad \text { and } \quad \Gamma_{01}=(u, v) \circ f\left(\left(t_{0}-\delta, t_{0}+\delta\right)\right)
$$

These are two smooth curves in the $x y$ - and $u v$-planes respectively. Set

$$
M_{\Gamma}=(x, y, u, v)^{-1}\left(\Gamma_{10} \times \Gamma_{01}\right) \subset V \subset M
$$

Now $M_{\Gamma}$ is a smooth manifold of dimension $s+2$ which contains $f\left(\left(t_{0}-\delta, t_{0}+\delta\right)\right)$. Moreover, by constuction, since the image of $M_{\Gamma}$ under the ( $x, y$ ) map and the ( $u, v$ ) map is a curve, it follows that $d x \wedge d y$ and $d u \wedge d v$ both vanish identically on $M_{\Gamma}$. In particular, it follows that the pull back of the system $\mathcal{I}$ to $M_{\Gamma}$ is generated algebraically by the sections of $I$. In other words, I pulls back to be a rank s integrable system $I_{\Gamma}$ on $M_{\Gamma}$.

It follows that there is a unique 2-dimensional leaf $L_{\Gamma}$ of $I_{\Gamma}$ which contains the curve $f\left(\left(t_{0}-\delta, t_{0}+\delta\right)\right)$. (Note that the construction of the leaves of an integrable Pfaffian system can be accomplished by solving ODE alone, PDE methods are not required.) This surface $L_{\Gamma}$ is, by construction, an integral surface of $\mathcal{I}$ which contains $f\left(\left(t_{0}-\delta, t_{0}+\delta\right)\right)$. It is also transverse to the fibers of the submersion $M_{\Gamma} \rightarrow$ $\Gamma_{10} \times \Gamma_{01}$. As a result, by shrinking $\delta$ once more, we may assume that $L_{\Gamma}$ maps diffeomorphically onto the open square

$$
(x, y) \circ f\left(\left(t_{0}-\delta, t_{0}+\delta\right)\right) \times(u, v) \circ f\left(\left(t_{0}-\delta, t_{0}+\delta\right)\right)
$$

We now define $F$ on the open square $\left(t_{0}-\delta, t_{0}+\delta\right) \times\left(t_{0}-\delta, t_{0}+\delta\right)$ by letting $F(s, t)$ be the point of $L_{\Gamma}$ which maps to

$$
(x(f(s)), y(f(s)), u(f(t)), v(f(t))) .
$$

Local uniqueness of this extension is easily established and then an elementary patching argument shows that $F$ can be defined on an entire neighborhood of the "diagonal" interval.

It is important to remark that finding the integrable subsystems of the characteristic systems is a routine matter. In fact, one merely computes the derived systems $\Xi_{10}^{(k)\langle j\rangle}$ and $\Xi_{01}^{(k)\langle j\rangle}$. Since we clearly have, for some $j$ and $j^{\prime}$,

$$
\begin{aligned}
& \Xi_{10}^{(k)}=\Xi_{10}^{(k)\langle 0\rangle} \supseteq \Xi_{10}^{(k)\langle 1\rangle} \supseteq \Xi_{10}^{(k)\langle 2\rangle} \cdots \supseteq \Xi_{10}^{(k)\langle j\rangle}=\Xi_{10}^{(k)\langle j+1\rangle}=\Xi_{10}^{(k)\langle\infty\rangle} \\
& \Xi_{01}^{(k)}=\Xi_{01}^{(k)\langle(0\rangle} \supseteq \Xi_{01}^{(k)\langle 1\rangle} \supseteq \Xi_{01}^{(k)\langle 2\rangle} \cdots \supseteq \Xi_{01}^{(k)\left\langle j^{\prime}\right\rangle}=\Xi_{01}^{(k)\left\langle j^{\prime}+1\right\rangle}=\Xi_{01}^{(k)\langle\infty\rangle}
\end{aligned}
$$

and since any integrable subsystem of a Pfaffian system lies inside its last derived system, the test for integrability by the method of Darboux can be carried out effectively, using only differentiations and algebraic manipulations.
1.4.3 An example - the f-Gordon equation. Our main interest in Darboux integrability in this paper is its effect on the computation of conservation laws (to be defined in Section 2.1 below). However, classically, the importance of Darboux' method was that it often lead to explicit formulas for solutions to important PDEs. We will illustrate this by an example drawn from the classical literature.

First, however, we would like to remark that the integration of the MongeAmpere equation $z_{x x} z_{y y}-z_{x y}^{2}=-1$ accomplished in Example 2 of Section 1.2 was done by the method of Darboux, for this equation happens to be integrable by the method of Darboux at level 0, as can be seen immediately by the formulae

$$
\begin{aligned}
& \Xi_{10}=\{d z-p d x-q d y, d(p-y), d(q+x)\} \\
& \Xi_{01}=\{d z-p d x-q d y, d(p+y), d(q-x)\}
\end{aligned}
$$

This is, of course, not surprising because, as we saw in Section 1.1, this system is (globally) equivalent to the hyperbolic system generated by the classical wave equation $z_{x y}=0$.

We now turn to a more interesting example. Consider the so-called $f$-Gordon equation $z_{x y}=f(z)$. As this is an equation of Monge-Ampere type, we may construct a corresponding hyperbolic system of the form $\left(\mathbb{R}^{5}, \mathcal{I}\right) \in \mathcal{H}_{1}$ where, with the usual notation,

$$
\mathcal{I}=\{d z-p d x-q d y,(d p-f(z) d y) \wedge d x,(d q-f(z) d x) \wedge d y\}
$$

The characteristic systems are

$$
\begin{aligned}
& \Xi_{10}=\{d z-p d x-q d y, d p-f(z) d y, d x\} \\
& \Xi_{01}=\{d z-p d x-q d y, d q-f(z) d x, d y\}
\end{aligned}
$$

We may now compute the derived flag for these characteristic systems. Assuming that $f^{\prime}(z) \neq 0$, which we shall for the rest of this section, the only integrable subsystems of $\Xi_{10}$ and $\Xi_{01}$ are of rank one and are generated by $d x$ and $d y$, respectively.

We will now determine the conditions on $f$ that the $f$-Gordon equation be integrable by the method of Darboux at level one. As explained in Section 1.3, the first prolongation $\left(M^{(1)}, \mathcal{I}^{(1)}\right)$ of the exterior differential system $(M, \mathcal{I})$ associated to the $f$-Gordon equation is obtained locally by introducing new coordinates $r$ ( $=h_{20}$ ) and $t\left(=h_{02}\right)$ and setting

$$
\begin{aligned}
\theta_{10} & =d p-f(z) d y-r d x \\
\theta_{01} & =d q-f(z) d x-t d y
\end{aligned}
$$

From

$$
\left.\begin{array}{l}
d \theta_{10} \equiv-\left(d r-f^{\prime}(z) p d y\right) \wedge d x \\
d \theta_{01} \equiv-\left(d t-f^{\prime}(z) q d x\right) \wedge d y
\end{array}\right\} \bmod \theta
$$

we infer that

$$
\begin{aligned}
& \Xi_{10}^{(1)}=\left[d x, d z-q d y, d p-f(z) d y, d r-f^{\prime}(z) p d y\right] \\
& \Xi_{01}^{(1)}=\left[d y, d q-p d x, d q-f(z) d x, d t-f^{\prime}(z) q d x\right] .
\end{aligned}
$$

First, we compute that $\Xi_{10}^{(1)\langle 1\rangle}=\left[d x, d p-f(z) d y, d r-f^{\prime}(z) p d y\right]$. Now it is not difficult to compute that

$$
\left.\begin{array}{rl}
d(d p-f(z) d y) & \equiv-f^{\prime}(z) d z \wedge d y \\
d\left(d r-f^{\prime}(z) p d y\right) & \equiv-f^{\prime \prime}(z) p d z \wedge d y
\end{array}\right\} \bmod \Xi_{10}^{(1)(1\rangle}
$$

while of course $d(d x)=0$. Since we have assumed that $f^{\prime}$ is non-vanishing, we see that

$$
\begin{aligned}
\Xi_{10}^{(1)\langle 2\rangle} & =\left[d x,\left(d r-f^{\prime}(z) p d y\right)-\left(f^{\prime \prime}(z) / f^{\prime}(z)\right) p(d p-f(z) d y)\right] \\
& =\left[d x, d r-\left(f^{\prime \prime}(z) / f^{\prime}(z)\right) p d p-\left(f^{\prime}(z)^{2}-f^{\prime \prime}(z) f(z)\right) p / f^{\prime}(z) d y\right]
\end{aligned}
$$

We can now compute that in order for the third derived system to have rank two, it is necessary and sufficient that $f^{\prime}(z)^{2}-f^{\prime \prime}(z) f(z)=0$. The general solution of this relation is

$$
f(z)=A e^{B z}
$$

for some constants $A$ and $B$. Since we are assuming that $f^{\prime}(z)$ is non-zero, neither $A$ nor $B$ can vanish. By scaling $z$ and $x$ and $y$, we may then easily reduce to the case

$$
f(z)=e^{z}
$$

In this case

$$
\begin{aligned}
& \Xi_{10}^{(1)\langle 2\rangle}=\Xi_{10}^{(1)\langle\infty\rangle}=[d x, d r-p d p]=\left[d x, d\left(r-\frac{1}{2} p^{2}\right)\right] \\
& \Xi_{01}^{(1)(2\rangle}=\Xi_{01}^{(1)(\infty)}=[d y, d t-q d q]=\left[d y, d\left(t-\frac{1}{2} q^{2}\right)\right] .
\end{aligned}
$$

This proves the classical result that Liouville's equation $z_{x y}=e^{z}$ is integrable by the method of Darboux.

In fact, it is a theorem of Lie [Li] that, up to local equivalence, this equation and the wave equation $z_{x y}=0$ are the only $f$-Gordon equations integrable by the method of Darboux at any level.

The explicit expressions above show that on solution surfaces we have

$$
\begin{aligned}
& d\left(r-\frac{1}{2} p^{2}\right) \wedge d x=0 \\
& d\left(t-\frac{1}{2} q^{2}\right) \wedge d y=0 .
\end{aligned}
$$

These relations may be used to show that the general solution of the Liouville equation on any rectangle of the form

$$
\mathcal{R}=\left\{(x, y) \in \mathbb{R}^{2} \mid a<x<b, c<y<d\right\}
$$

can be written in the form

$$
\begin{equation*}
e^{z}=\frac{2 X^{\prime}(x) Y^{\prime}(y)}{(X(x)+Y(y))^{2}} \tag{1}
\end{equation*}
$$

where $X:(a, b) \rightarrow \mathbb{R}$ and $Y:(c, d) \rightarrow \mathbb{R}$ are arbitrary smooth maps subject to the open conditions that $X^{\prime}(x) Y^{\prime}(y)>0$ and $X(x)+Y(y)>0$ for all $(x, y) \in \mathcal{R}$. The calculation is a little complicated, perhaps because the more fundamental equation is the $s=0$ Liouville system

$$
\begin{aligned}
u_{y} & =e^{v} \\
v_{x} & =e^{u}
\end{aligned}
$$

As we shall show in the Section 1.5, this latter system is more easily integrated by the method of Darboux, giving (1) above as an immediate consequence.

Finally, we would like to mention an example, (cf. Chapter 3 of Darboux [Da]), of an equation integrable by the method of Darboux at level $k$ but not at level $k-1$. This is the linear equation

$$
z_{x y}=-\frac{k(k+1)}{(x-y)^{2}} z
$$

defined on the half-plane $x>y$. For $k$ not an integer, it turns out that this equation cannot be integrated by the method of Darboux, but for $k \geq 0$ an integer, it turns out that applying the method of Darboux yields the general solution in the form

$$
z(x, y)=(x-y)^{k+1} \frac{\partial^{2 k}}{\partial x^{k} \partial y^{k}}\left(\frac{X(x)-Y(y)}{x-y}\right)
$$

where $X$ and $Y$ are arbitrary functions of one variable.
1.4.4 Semi-integrability by Darboux' method. In closing this section, we want to remark that there is actually a generalization of the method of Darboux which only depends on there being a transverse rank 2 integrable subsystem in one of the characteristic systems in order to solve the initial value problem via ODE methods. Since this will not play an important role in this paper, we only give a sketch of the method.

Let us say that a hyperbolic system $(M, \mathcal{I})$ is semi-integrable in the sense of Darboux if there exists an integrable rank 2 subsystem $\Delta \subset \Xi_{10}$ which satisfies $\Xi_{10}=\Delta \oplus I$. In this case, it easily follows that any point of $M$ lies in an open set $U$ on which $\mathcal{I}$ can be generated in the form

$$
\mathcal{I}=\left\{\theta^{1}, \ldots, \theta^{s} ; d x \wedge d y, \omega^{3} \wedge \omega^{4}\right\}
$$

Suppose, now that we are given a non-characteristic integral curve $\phi:(0,1) \rightarrow U$ of $\mathcal{I}$. The hypothesis that $\phi$ be non-characteristic implies that the map $(x, y) \circ \phi$ : $(0,1) \rightarrow \mathbb{R}^{2}$ is an immersion. By shrinking domains appropriately, we can assume that this map is an embedding, which we shall. Set $\Gamma=(x, y) \circ \phi((0,1)) \subset \mathbb{R}^{2}$, and let

$$
M_{\Gamma}=(x, y)^{-1}(\Gamma)
$$

Then $M_{\Gamma}$ is a smooth hypersurface in $M$ which contains $\phi((0,1))$. Let $\mathcal{I}_{\Gamma}$ denote the induced exterior differential system on $M_{\Gamma}$. If we let an overbar denote the pull back of forms from $M$ to $M_{\Gamma}$, then we see that

$$
\mathcal{I}_{\Gamma}=\left\{\bar{\theta}^{1}, \ldots, \bar{\theta}^{s} ; \bar{\omega}^{3} \wedge \bar{\omega}^{4}\right\} .
$$

It follows that there exists a non-zero vector field $X$ on $M_{\Gamma}$ (unique up to scalar multiples) which satisfies

$$
\bar{\theta}^{1}(X)=\cdots=\bar{\theta}^{s}(X)=\bar{\omega}^{3}(X)=\bar{\omega}^{4}(X)=0
$$

(The reader familiar with the theory of exterior differential systems will recognize $X$ as spanning the Cauchy characteristic distribution on $\mathcal{I}_{\Gamma}$.) Note that, by our assumption that $\phi$ be non-characteristic, the 1 -forms $\phi^{*}\left(\omega^{3}\right)$ and $\phi^{*}\left(\omega^{4}\right)$ do not vanish simultaneously at any point of $(0,1)$. Thus, it follows that for every $t$, the vectors $X(\phi(t))$ and $\phi^{\prime}(t)(\partial / \partial t)$ are linearly independent. It then follows that there is an open neighborhood $\mathcal{R}$ of $(0,1) \times\{0\}$ in $(0,1) \times(-1,1)$ so that the mapping $f: \mathcal{R} \rightarrow M$ defined by

$$
f(t, s)=\exp _{s X}(\phi(t)), \quad(t, s) \in \mathcal{R}
$$

is well-defined and an immersion. By its very construction, $f$ is an integral surface of $\mathcal{I}$ which solves the initial value problem for $\phi$.

Clearly, this method can be generalized to the case where, for some $k \geq 0$, the ideal $\left(M^{(k)}, \mathcal{I}^{(k)}\right)$ is semi-integrable by the method of Darboux. Thus, in all of these cases, the initial value problem can be solved by ODE methods.
1.5 Structure equations for hyperbolic systems of class $s=0$. From the general theory of conservation laws for exterior differential systems (cf. [ BG ${ }_{1}$ ]), it is known that the symbol of an exterior differential system $\mathcal{I}$ determines an algebraic normal form for its conservation laws. The space $\mathcal{C}$ of conservation laws of $\mathcal{I}$ is then isomorphic to the space of closed forms in this algebraic normal form. In Chapter 2 of this paper, we shall need to do some rather explicit computations for hyperbolic systems of class $s=0$ using this normal form. In this section, as preparation for those computations, we are going to use the extra assumption $s=0$ to refine the general structure equations derived in Section 1.3 above.
1.5.1 Symmetry and non-degeneracy. Recall that, according to our definitions, a hyperbolic system $\mathcal{I}$ of class $s=0$ on a 4 -manifold $M$ is locally generated by a pair of decomposable 2-forms $\Omega_{10}$ and $\Omega_{01}$ on $M$ which are defined up to non-zero multiples and with the property that $\Omega_{10 \wedge} \Omega_{01} \neq 0$. As usual, we shall let $\Xi_{10}$ and $\Xi_{01}$ denote the characteristic systems.

We first want to say a word about the "generality" of hyperbolic systems with $s=0$. Working locally, we can imagine a hyperbolic system with $s=0$ as a pair of 2 -plane distributions on a neighborhood of the origin in $\mathbb{R}^{4}$. Since the bundle $G_{2}\left(T \mathbb{R}^{4}\right)$ of 2-planes in the tangent spaces at points of $\mathbb{R}^{4}$ is a smooth bundle of fiber dimension 4, it follows that, locally, a choice of a hyperbolic system with $s=$ 0 depends on the choice of $4+4=8$ functions of four variables, these functions being subject only to some open conditions which ensure that the two distributions are transverse. On the other hand, we want to identify two such hyperbolic systems if they differ only by some diffeomorphism of $\mathbb{R}^{4}$. Since a local diffeomorphism of $\mathbb{R}^{4}$ depends on a choice of 4 functions of four variables, it seems reasonable to guess that the "moduli space" of equivalence classes of local hyperbolic systems with $s=0$ modulo diffeomorphisms "depends" on $8-4=4$ arbitrary functions of four variables. In particular, we should expect there to be differential invariants attached to a hyperbolic system with $s=0$ just as the Riemannian curvature tensor is attached to a Riemannian metric. In the next subsection, we will develop a mechanism for computing these invariants, analogous to the construction of the Levi-Civita connection and its curvature in Riemannian geometry.

The simplest hyperbolic exterior differential system of class $s=0$ occurs when both $\Xi_{10}$ and $\Xi_{01}$ are integrable, and we will henceforth refer to this case as the trivial case. In the trivial case, we may choose local coordinates $(x, y, u, v)$ such that the system $\mathcal{I}$ is generated by

$$
\Omega_{10}=d u \wedge d x \quad \text { and } \quad \Omega_{01}=d v \wedge d y
$$

This is the exterior differential system arising from the (trivial) PDE system $u_{y}=$ $v_{x}=0$, whose solutions are $u=u(x)$ and $v=v(y)$. This is essentially the $s=0$ version of the classical wave equation in characteristic coordinates.

Before going on to study the non-trivial cases, we want to first remark on a general hypothesis that we will be assuming in order to simplify the exposition. Strictly speaking, we are considering a structure which has slightly more information than a hyperbolic system with $s=0$. In fact, we are also imposing a choice of which of the two characteristic systems we want to call $\Xi_{10}$ and which we want to call $\Xi_{01}$. Thus, we might think of the structure we are studying primarily as a pair $\Xi=\left(\Xi_{10}, \Xi_{01}\right)$ of transverse rank 2 Pfaffian systems on $M^{4}$. In some sense, the pair ${ }^{*} \Xi=\left(\Xi_{01}, \Xi_{10}\right)$ should be thought of as the "opposite" structure, the operation $\Xi \mapsto^{*} \Xi$ defining an involution on the space of structures that we are studying.

From the structure equations below, we will extract certain so-called "relative invariants" of the pair $\Xi$. In modern terminology, a relative invariant is a "natural" section $\sigma_{\Xi}$ of a "natural" line bundle $L_{\Xi}$ associated to $\Xi$ where "natural" means that, whenever one has a diffeomorphism $f: M_{1} \rightarrow M_{2}$ which induces an isomorphism $f^{*}\left(\Xi_{2}\right)=\Xi_{1}$ of hyperbolic systems with $s=0$, there is also canonically determined an isomorphism $f^{*} L_{\Xi_{2}}=L_{\Xi_{1}}$ which satisfies $f^{*} \sigma_{\Xi_{2}}=\sigma_{\Xi_{1}}$. (We will give examples of such invariants in the next subsection.)

In the present case, the involution $\Xi \mapsto{ }^{*} \Xi$ clearly exchanges each relative invariant with an 'opposite' relative invariant * $\sigma$. Thus, the relative invariants are naturally grouped into pairs (or, ocassionally, singlets when a relative invariant happens to be its own opposite).

Definition: A hyperbolic exterior differential system of class $s=0$ is symmetric in case the relative invariants in each pair $\left(\sigma,{ }^{*} \sigma\right)$ are either both zero or both non-zero.

For example, for a symmetric system, the integrability properties of the two characteristic systems will be the same (in particular, the derived flags will have the same ranks). For simplicity, we shall concentrate on symmetric systems in this paper. Thus, henceforth, we shall assume that all of our systems are symmetric in this sense.

In the non-trivial case, our general assumption that the system be symmetric implies that each of $\Xi_{10}$ and $\Xi_{01}$ will be non-integrable. ${ }^{11}$ We shall say that a hyperbolic system with $s=0$ is non-degenerate if this condition is satisfied. For the rest of this section, we shall assume that all of our hyperbolic systems are non-degenerate.

[^3]1.5.2 A G-structure for the non-degenerate case and its invariants. Let us now assume that we have a non-degenerate hyperbolic system $(M, \mathcal{I})$ of class $s=0$. The first derived system of each characteristic system then has rank 1. This suggests that we consider local coframings $\left(\eta^{1}, \eta^{2}, \eta^{3}, \eta^{4}\right)$ on $M$ satisfying the conditions
\[

$$
\begin{equation*}
\Xi_{10}=\left\{\eta^{1}, \eta^{2}\right\} \quad \Xi_{01}=\left\{\eta^{3}, \eta^{4}\right\} \tag{1}
\end{equation*}
$$

\]

and

$$
\begin{equation*}
d \eta^{1} \equiv 0 \bmod \eta^{1}, \eta^{2} \quad d \eta^{3} \equiv 0 \bmod \eta^{3}, \eta^{4} \tag{2}
\end{equation*}
$$

Thus, $\eta^{1}$ and $\eta^{3}$ span the first derived systems of the characteristic systems $\Xi_{10}$ and $\Xi_{01}$, respectively. If $\left(\bar{\eta}^{1}, \bar{\eta}^{2}, \bar{\eta}^{3}, \bar{\eta}^{4}\right)$ is another coframing on the same domain with properties (1) and (2), then the formula for transition between the two coframings takes the form

$$
\left(\begin{array}{c}
\bar{\eta}^{1} \\
\bar{\eta}^{2} \\
\bar{\eta}^{3} \\
\bar{\eta}^{4}
\end{array}\right)=\left(\begin{array}{cccc}
a_{1}^{1} & 0 & 0 & 0 \\
a_{1}^{2} & a_{2}^{2} & 0 & 0 \\
0 & 0 & a_{3}^{3} & 0 \\
0 & 0 & a_{3}^{4} & a_{4}^{4}
\end{array}\right)\left(\begin{array}{c}
\eta^{1} \\
\eta^{2} \\
\eta^{3} \\
\eta^{4}
\end{array}\right)
$$

where the $a_{j}^{i}$ are arbitrary subject to the obvious condition that $a_{i}^{i} \neq 0$ (or else the transition matrix would not be invertible).

Now, for any coframing which satisfies (1) and (2), there must exist functions $A$ and $C$ so that

$$
\begin{equation*}
d \eta^{2} \equiv A \eta^{3} \wedge \eta^{4} \bmod \eta^{1}, \eta^{2} \quad d \eta^{4} \equiv C \eta^{1} \wedge \eta^{2} \bmod \eta^{3}, \eta^{4} \tag{3}
\end{equation*}
$$

By our assumptions, $\Xi_{10}$ and $\Xi_{01}$ are non-integrable. It follows from this and (2) that neither $A$ nor $C$ can vanish. Using this, we can construct a coframing, say $\left(\bar{\eta}^{1}, \bar{\eta}^{2}, \bar{\eta}^{3}, \bar{\eta}^{4}\right)=\left(C \eta^{1}, \eta^{2}, A \eta^{3}, \eta^{4}\right)$ which satisfies properties (1) and (2) and also the equations $A=C=1$. We shall say that a coframing is 1-adapted to $\mathcal{I}$ if it satisfies the conditions (1), (2), and (3) with $A=C=1$.

If $\eta=\left(\eta^{1}, \eta^{2}, \eta^{3}, \eta^{4}\right)$ is a 1 -adapted coframing on a domain $U \subset M$, then any other coframing on $U$, say $\bar{\eta}=\left(\bar{\eta}^{1}, \bar{\eta}^{2}, \bar{\eta}^{3}, \bar{\eta}^{4}\right)$ is seen to be 1 -adapted to $\mathcal{I}$ if and only if there exist functions $a_{1}^{2}, a_{3}^{4}, a_{2}^{2} \neq 0$, and $a_{4}^{4} \neq 0$ on $U$ so that

$$
\left(\begin{array}{c}
\bar{\eta}^{1} \\
\bar{\eta}^{2} \\
\bar{\eta}^{3} \\
\bar{\eta}^{4}
\end{array}\right)=\left(\begin{array}{cccc}
a_{4}^{4} / a_{2}^{2} & 0 & 0 & 0 \\
a_{1}^{2} & a_{2}^{2} & 0 & 0 \\
0 & 0 & a_{2}^{2} / a_{4}^{4} & 0 \\
0 & 0 & a_{3}^{4} & a_{4}^{4}
\end{array}\right)\left(\begin{array}{c}
\eta^{1} \\
\eta^{2} \\
\eta^{3} \\
\eta^{4}
\end{array}\right) .
$$

Such "transition matrices" take values in a certain 4-dimensional lower triangular subgroup of GL $(4, \mathbb{R})$ which we shall henceforth denote $G$. Thus, the local coframings which are 1 -adapted to $\mathcal{I}$ are the local sections of a principal $G$-bundle $B \rightarrow M$ which is a subbundle of the bundle of all coframes of $M$. In other words, $B$ is a $G$ structure on $M$ in the usual sense. We will refer to $B$ as the $G$-structure associated
to (or determined by) the non-degenerate hyperbolic exterior differential system $\mathcal{I}$. Moreover, one can clearly recover $\mathcal{I}$ from a knowledge of $B$.

Now we shall apply the equivalence method ${ }^{12}$ to the $G$-structure $B$ in order to understand its invariants. Accordingly, we write the structure equations on $B$ in the form

$$
d\left(\begin{array}{c}
\omega^{1}  \tag{4}\\
\omega^{2} \\
\omega^{3} \\
\omega^{4}
\end{array}\right)=-\left(\begin{array}{cccc}
\phi_{44}-\phi_{22} & 0 & 0 & 0 \\
\phi_{21} & \phi_{22} & 0 & 0 \\
0 & 0 & \phi_{22}-\phi_{44} & 0 \\
0 & 0 & \phi_{43} & \phi_{44}
\end{array}\right) \wedge\left(\begin{array}{c}
\omega^{1} \\
\omega^{2} \\
\omega^{3} \\
\omega^{4}
\end{array}\right)+\left(\begin{array}{c}
T^{1} \\
T^{2} \\
T^{3} \\
T^{4}
\end{array}\right)
$$

where, in the terminology of the equivalence method, the $\phi_{i j}$ are the pseudoconnection forms and the $T^{i}$ are the torsion terms (which are semi-basic ${ }^{13}$ ). These forms are not uniquely determined by these equations, and, following the usual method of equivalence, we now want to understand how modifications of the pseudo-connection forms can be employed to simplify the torsion terms.

Now, by the defining properties of the $G$-structure $B$, we have

$$
\begin{array}{llll}
d \omega^{1} \equiv 0 & \bmod \omega^{1}, \omega^{2} & T^{1} \equiv 0 & \bmod \omega^{1}, \omega^{2} \\
d \omega^{2} \equiv \omega^{3} \wedge \omega^{4} & \bmod \omega^{1}, \omega^{2} & \text { so } & T^{2} \equiv \omega^{3} \wedge \omega^{4}
\end{array} \bmod \omega^{1}, \omega^{2} .
$$

It follows that there exist 1 -forms $\chi_{1}, \chi_{2}, \chi_{3}$, and $\chi_{4}$ which are linear combinations of the $\omega^{i}$ so that

$$
\begin{aligned}
& T^{2}=\omega^{3} \wedge \omega^{4}+\chi_{1} \wedge \omega^{1}+\chi_{2} \wedge \omega^{2} \\
& T^{4}=\omega^{1} \wedge \omega^{2}+\chi_{3} \wedge \omega^{3}+\chi_{4} \wedge \omega^{4}
\end{aligned}
$$

The equations for $d \omega^{2}$ and $d \omega^{4}$ can therefore be written in the form

$$
\begin{aligned}
& d \omega^{2}=-\left(\phi_{21}-\chi_{1}\right) \wedge \omega^{1}-\left(\phi_{22}-\chi_{2}\right) \wedge \omega^{2}+\omega^{3} \wedge \omega^{4} \\
& d \omega^{4}=-\left(\phi_{43}-\chi_{3}\right) \wedge \omega^{3}-\left(\phi_{44}-\chi_{4}\right) \wedge \omega^{4}+\omega^{1} \wedge \omega^{2} .
\end{aligned}
$$

It follows that we may assume that the $\phi_{i j}$ have been chosen so that

$$
T^{2}=\omega^{3} \wedge \omega^{4} \quad \text { and } \quad T^{4}=\omega^{1} \wedge \omega^{2}
$$

12) The general equivalence method is explained in Appendix 1 to Section 2 of [BG 2]. Fortunately, however, the full complexity of the method will not be needed in this simple case.
13) I.e., these terms have the form $T^{i}=T_{j k}^{i} \omega^{j} \wedge \omega^{k}$ for some functions $T_{j k}^{i}=-T_{k j}^{i}$ on $B$.
so we assume this from now on. This condition still does not determine the $\phi_{i j}$ since making the replacements

$$
\left(\begin{array}{c}
\phi_{21} \\
\phi_{22} \\
\phi_{43} \\
\phi_{44}
\end{array}\right) \mapsto\left(\begin{array}{c}
\phi_{21}+a_{1} \omega^{1}+a_{2} \omega^{2} \\
\phi_{22}+a_{2} \omega^{1}+a_{3} \omega^{2} \\
\phi_{43}+c_{3} \omega^{3}+c_{4} \omega^{4} \\
\phi_{44}+c_{4} \omega^{3}+c_{1} \omega^{4}
\end{array}\right)
$$

in the above equations will clearly not affect $T^{2}$ or $T^{4}$. However, the above congruences on $T^{1}$ and $T^{3}$ imply that

$$
\begin{aligned}
& T^{1} \equiv T_{13}^{1} \omega^{1} \wedge \omega^{3}+T_{14}^{1} \omega^{1} \wedge \omega^{4} \bmod \omega^{2} \\
& T^{3} \equiv T_{31}^{3} \omega^{3} \wedge \omega^{1}+T_{32}^{3} \omega^{3} \wedge \omega^{2} \bmod \omega^{4}
\end{aligned}
$$

and the above replacements can be chosen so that $T_{13}^{1}=T_{14}^{1}=T_{31}^{3}=T_{32}^{3}=0$. Note that the only replacements of the above form which preserve these latter conditions are ones with $a_{2}=a_{3}=c_{4}=c_{1}=0$.

The upshot of this discussion is that, for the $G$-structure we have associated to a non-degenerate hyperbolic system with $s=0$, there is a choice of pseudoconnection so that the structure equations take the form

$$
\begin{align*}
\left(\begin{array}{c}
d \omega^{1} \\
d \omega^{2} \\
d \omega^{3} \\
d \omega^{4}
\end{array}\right)= & -\left(\begin{array}{cccc}
\phi_{44}-\phi_{22} & 0 & 0 & 0 \\
\phi_{21} & \phi_{22} & 0 & 0 \\
0 & 0 & \phi_{22}-\phi_{44} & 0 \\
0 & 0 & \phi_{43} & \phi_{44}
\end{array}\right) \wedge\left(\begin{array}{l}
\omega^{1} \\
\omega^{2} \\
\omega^{3} \\
\omega^{4}
\end{array}\right) \\
& +\left(\begin{array}{c}
\omega^{2} \wedge\left(p_{1} \omega^{1}+p_{3} \omega^{3}+p_{4} \omega^{4}\right) \\
\omega^{3} \wedge \omega^{4} \\
\omega^{4} \wedge\left(q_{3} \omega^{3}+q_{1} \omega^{1}+q_{2} \omega^{2}\right) \\
\omega^{1} \wedge \omega^{2}
\end{array}\right) \tag{5}
\end{align*}
$$

Moreover, with the structure equations in this form, the 1 -forms $\phi_{22}$ and $\phi_{44}$ are unique, the form $\phi_{21}$ is determined up to the addition of a multiple of $\omega^{1}$, and the form $\phi_{43}$ is determined up to the addition of a multiple of $\omega^{3}$. As we shall see in the next subsection, no further reduction of this $G$-structure can be made without making some non-vanishing assumptions on the invariants.

To complete the discussion of the structure equations, it will be necessary to compute their "Bianchi identities" by differentiating the equations in (5). ${ }^{14} \mathrm{We}$ will not give the details of the calculations (which are straightforward, if tedious), but shall describe the results. First of all, differentiation of the equations (5) and
14) The reader may want to skip the remainder of this subsection on first reading, instead going on directly to Section 1.5.3. These "Bianchi" calculations are somewhat technical and will be more meaningful once the reader cant see that they are needed.
reduction of the results modulo various combinations of the $\omega^{i}$ shows that there are relations of the form

$$
\left.\begin{array}{rl}
d p_{1} & \equiv p_{1} \phi_{22}-q_{2} \phi_{43}  \tag{6}\\
d p_{3} & \equiv p_{3}\left(3 \phi_{22}-2 \phi_{44}\right)+p_{4} \phi_{43} \\
d p_{4} & \equiv p_{4}\left(2 \phi_{22}\right) \\
d q_{3} & \equiv q_{3} \phi_{44}-p_{4} \phi_{21} \\
d q_{1} & \equiv q_{1}\left(3 \phi_{44}-2 \phi_{22}\right)+q_{2} \phi_{21} \\
d q_{2} & \equiv q_{2}\left(2 \phi_{44}\right)
\end{array}\right\} \bmod \omega^{1}, \omega^{2}, \omega^{3}, \omega^{4} .
$$

We shall use the notation $\nabla p_{4}$ to mean $d p_{4}-2 p_{4} \phi_{22}$, i.e., the semi-basic part of the exterior derivative of $p_{4}$, and similarly for the other quantities.

If we now introduce "curvature" 2 -forms $\Phi_{22}, \Phi_{44}, \Phi_{21}$, and $\Phi_{43}$ by the equations

$$
\begin{align*}
& d \phi_{22}=-\phi_{21} \wedge\left(p_{3} \omega^{3}+p_{4} \omega^{4}\right)+q_{3} \omega^{1} \wedge \omega^{2}+\frac{1}{2} p_{3} q_{1} \omega^{2} \wedge \omega^{4}+\Phi_{22} \\
& d \phi_{44}=-\phi_{43 \wedge}\left(q_{1} \omega^{1}+q_{2} \omega^{2}\right)+p_{1} \omega^{3} \wedge \omega^{4}+\frac{1}{2} q_{1} p_{3} \omega^{4} \wedge \omega^{2}+\Phi_{44} \\
& d \phi_{21}=-\phi_{21} \wedge\left(\phi_{44}-2 \phi_{22}-p_{1} \omega^{2}\right)+\Phi_{21}  \tag{7}\\
& d \phi_{43}=-\phi_{43 \wedge}\left(\phi_{22}-2 \phi_{44}-q_{3} \omega^{4}\right)+\Phi_{43}
\end{align*}
$$

the exterior derivatives of the equations (5) become

$$
\begin{align*}
& 0=\left(\Phi_{22}-\Phi_{44}\right) \wedge \omega^{1}-\left(\nabla p_{1} \wedge \omega^{1}+\nabla p_{3} \wedge \omega^{3}+\nabla p_{4} \wedge \omega^{4}-q_{3} p_{3} \omega^{3} \wedge \omega^{4}\right) \wedge \omega^{2} \\
& 0=-\Phi_{21 \wedge \omega^{1}}-\Phi_{22} \wedge \omega^{2}-\omega^{1} \wedge \omega^{2} \wedge \omega^{3} \\
& 0=\left(\Phi_{44}-\Phi_{22}\right) \wedge \omega^{3}-\left(\nabla q_{3} \wedge \omega^{3}+\nabla q_{1} \wedge \omega^{1}+\nabla q_{2} \wedge \omega^{2}-p_{1} q_{1} \omega^{1} \wedge \omega^{2}\right) \wedge \omega^{4}  \tag{8}\\
& 0=-\Phi_{43} \wedge \omega^{3}-\Phi_{44} \wedge \omega^{4}-\omega^{3} \wedge \omega^{4} \wedge \omega^{1} .
\end{align*}
$$

(Note that because $\phi_{21}$ and $\phi_{43}$ are not canonical, the expression $\nabla p_{3}$ is actually only well-defined modulo $\omega^{3}$. However, since this term only occurs wedged with $\omega^{3}$, the resulting term is well defined. A similar comment applies to the other ambiguities caused by the ambiguity in the pseudo-connection.) The identities (8) now give relations among the coefficients of the derivatives of the primary invariants (i.e., the torsion coefficients) and the curvature coefficients. It is not useful to write these out here; the form (8) will suffice for our purposes.

A little exterior algebra shows that the relations (8) imply that $\Phi_{22}$ and $\Phi_{44}$ are semi-basic 2 -forms, i.e., they are quadratic expressions in the $\omega^{i}$. In fact,

$$
\begin{equation*}
\Phi_{22}=-\kappa_{1} \wedge \omega^{1}-\kappa_{2} \wedge \omega^{2} \quad \text { and } \quad \Phi_{44}=-\kappa_{3} \wedge \omega^{3}-\kappa_{4} \wedge \omega^{4} \tag{9}
\end{equation*}
$$

where

$$
\kappa_{i}=k_{i j} \omega^{j}
$$

for functions $k_{i j}$, suitably skew-symmetrized so as to be well-defined. Using this plus some more exterior algebra, it follows that there are 1-forms $\phi_{211}$ and $\phi_{433}$ so that

$$
\begin{align*}
& \Phi_{21}=-\left(\kappa_{1}-\omega^{3}\right) \wedge \omega^{2}-\phi_{211} \wedge \omega^{1} \\
& \Phi_{43}=-\left(\kappa_{3}-\omega^{1}\right) \wedge \omega^{4}-\phi_{433} \wedge \omega^{3} \tag{10}
\end{align*}
$$

1.5.3 Relative invariants. In order to interpret the coefficients in the torsion terms, it is important to understand how the quantities $p_{1}, p_{3}, p_{4}$ and $q_{1}, q_{2}, q_{3}$ vary on the fibres of $B \rightarrow M$. This information can be read off from the relations (6).

It follows from (6) that $p_{4}$ and $q_{2}$ are what is known in the classical literature as relative invariants-i.e., they are well-defined as sections of suitable line bundles over $M$. For example, the expression $\sigma=p_{4}\left(\omega^{3} \wedge \omega^{4}\right)^{2}$ is a well-defined section of the square of the determinant bundle of the characteristic system $\Xi_{01}$. Its opposite (in the sense of the involution discussed in Section 1.5.1) is ${ }^{*} \sigma=q_{2}\left(\omega^{1} \wedge \omega^{2}\right)^{2}$, a section of the 'opposite' bundle, i.e., the square of the determinant bundle of the characteristic system $\Xi_{10}$. Moreover, calculation using (6) shows that, for example, $\tau=\left(p_{4} q_{1}+q_{2} q_{3}\right)\left(\omega^{1} \wedge \omega^{2}\right)^{3}$ is a well-defined section of the cube of the determinant bundle of $\Xi_{10}$ with 'opposite' invariant ${ }^{*} \tau=\left(q_{2} p_{3}+p_{4} p_{1}\right)\left(\omega^{3} \wedge \omega^{4}\right)^{3}$.

Thus, even though $p_{4}$ and $q_{2}$ are not well-defined on $M$, their zero loci make sense on $M$. Note also that if $p_{4}$ (resp. $q_{2}$ ) vanishes identically, then $p_{3}$ and $q_{3}$ (resp. $q_{1}$ and $p_{1}$ ) become relative invariants, a fact to which we shall return later.
1.5.4 Structure reduction in the generic case. According to (6), on the open set $M^{*} \subset M$ which is the complement of the zero loci of the relative invariants $p_{4}$ and $q_{2}$, there exists a $G_{1}$-substructure $B_{1} \subset B$ defined by the equations

$$
\left(p_{4}\right)^{2}=1, \quad\left(q_{2}\right)^{2}=1, \quad p_{3}=0, \quad q_{1}=0
$$

where $G_{1}$ is the group consisting of the diagonal matrices in $G$ whose diagonal entries are each $\pm 1$. Thus, up to a finite group ambiguity (of order 4), this $G_{1-}$ structure defines a canonical coframing on $M^{*}$. The invariants of this coframing are then invariants of the original hyperbolic system. Note that any symmetry of the hyperbolic system $\mathcal{I}$ must preserve the open set $M^{*}$ and, on this open set, must preserve the $G_{1}$-structure $B_{1}$. Since preserving this latter structure is essentially equivalent to preserving a coframing on $M^{*}$, it follows that the group of symmetries of $\mathcal{I}$ on $M^{*}$ is of dimension at most 4.

Conversely, it is not hard to show that hyperbolic systems satisfying these conditions with a 4 -parameter symmetry group do exist: Let $H$ be a Lie group of dimension 4 and choose any basis ( $\eta^{1}, \ldots, \eta^{4}$ ) of its left-invariant 1 -forms. Form the ideal $\mathcal{I}=\left\{\eta^{1} \wedge \eta^{2}, \eta^{3} \wedge \eta^{4}\right\}$. This will yield a hyperbolic system which clearly
does have (at least) a 4-parameter symmetry group. If the group $H$ is sufficiently "generic" among 4-dimensional Lie groups and the basis $\left(\eta^{i}\right)$ is chosen sufficiently generically, it can be shown that the resulting system will have its relative invariants $p_{4}$ and $q_{2}$ be non-zero, so that its group of symmetries is exactly of dimension 4.

What the corresponding PDE look like is a very interesting question. Also particularly interesting is the problem of knowing whether the ideals $(H, \mathcal{I})$ are complete for the initial value problem in the sense of Section 1.2.3.
1.5.5 Normal forms. We now want to interpret the vanishing of the torsion coefficients in (5) in terms of integrability of various bundles intrinsically associated to the original hyperbolic system $\mathcal{I}$ and use this to derive (local) normal forms in various special cases. We will then use the structure equations to develop a test for 'linearizability' and to classify the systems which are, in some sense, the most "homogeneous" among non-degenerate hyperbolic systems with $s=0$.

We begin our first interpretation by noting that the rank 2 Pfaffian system $\Theta=\left\{\omega^{1}, \omega^{3}\right\}$ spanned by the first derived systems of $\Xi_{10}$ and $\Xi_{01}$ is well-defined. Indeed, from the structure equations, the 2-form

$$
\Omega=\omega^{1} \wedge \omega^{3}
$$

itself is well-defined, since the scalings of $\omega^{1}$ and $\omega^{3}$ cancel. Of course, $\Omega$ is a relative invariant, being a section of the determinant bundle of $\Theta$ and having the property that ${ }^{*} \Omega=-\Omega$. The integrability of $\Omega$ has the following interpretation:

Proposition: For any non-degenerate hyperbolic system with $s=0$, the system $\Theta=\left\{\omega^{1}, \omega^{3}\right\}$ is Frobenius if and only if $p_{4}=q_{2}=0$. Moreover, $\mathcal{I}$ has the property that $p_{4}$ and $q_{2}$ vanish identically if and only if every point of $M$ has a neighborhood $U$ on which there exists a coordinate system $(x, y, u, v): U \rightarrow \mathbb{R}^{4}$ and functions $A, B, C$, and $D$ on $U$ satisfying $A B \neq 1, C_{v} \neq 0, D_{u} \neq 0$ as well as $A_{v}=B_{u}=0$ so that, on $U$,

$$
\mathcal{I}=\{(d u-C d y) \wedge(d x-A d y),(d v-D d x) \wedge(d y-B d x)\}
$$

Proof: The fact that the system $\Theta$ is differentially closed if and only if $p_{4}$ and $q_{2}$ vanish is immediate from the structure equations (5). It remains to verify that the promised coordinate system exists and has the properties claimed for it. First, since $\Theta$ is Frobenius, it follows that every point of $M$ has a neighborhood $U$ on which there exist functions $x$ and $y$ so that $\Theta=\{d x, d y\}$. Thus, for a 1-adpated coframing on this neighborhood, we must have $\eta^{1}$ and $\eta^{3}$ be linear combinations of $d x$ and $d y$. By making a linear change of variables in $x$ and $y$, we can assume that $\eta^{1} \wedge d y$ and $\eta^{3} \wedge d x$ are non-zero. It follows that there are functions $A$ and $B$ so
that $\eta^{1}$ is a multiple of $d x-A d y$ and $\eta^{3}$ is a multiple of $d y-B d x$. Since $\eta^{1} \wedge \eta^{3}$ is non-zero, it follows that $A B \neq 1$.

Next, the systems $\left\{\eta^{1}, \eta^{2}, \eta^{3}\right\}$ and $\left\{\eta^{1}, \eta^{3}, \eta^{4}\right\}$ are clearly Frobenius on $U$, so there must be functions $u$ and $v$ so that

$$
\left\{\eta^{1}, \eta^{2}, \eta^{3}\right\}=\{d x, d y, d u\} \quad \text { and } \quad\left\{\eta^{1}, \eta^{3}, \eta^{4}\right\}=\{d x, d y, d v\}
$$

It follows that there are functions $C$ and $D$ so that

$$
\begin{aligned}
& \Xi_{10}=\left\{\eta^{1}, \eta^{2}\right\}=\{d x-A d y, d u-C d y\} \\
& \Xi_{01}=\left\{\eta^{3}, \eta^{4}\right\}=\{d y-B d x, d v-D d x\}
\end{aligned}
$$

Now, by (2), it follows that $A_{v}=B_{u}=0$, and the non-degeneracy assumption implies that neither $C_{v}$ nor $D_{u}$ can vanish.

Finally, any hyperbolic system which locally can be put in the form we have just derived is clearly a non-degenerate hyperbolic system (with $s=0$ ) for which $\Theta$ is Frobenius.

As we remarked before, in the case where $p_{4}$ and $q_{2}$ vanish identically, so that $\Theta$ is integrable, then $p_{3}$ and $q_{1}$ become relative invariants. If we have information about their vanishing or non-vanishing, we can refine the normal form given above:

Proposition: Suppose that $\left(M^{4}, \mathcal{I}\right)$ is a non-degenerate hyperbolic system satisfying the condition that $p_{4}$ and $q_{2}$ vanish identically. Any point in the open set in $M$ where $p_{3}$ and $q_{1}$ are non-zero lies in a neighborhood on which there exists a coordinate system $(x, y, u, v): U \rightarrow \mathbb{R}^{4}$ and two functions $C$ and $D$ with $C_{v}$ and $D_{u}$ non-zero so that, on $U$,

$$
\mathcal{I}=\{(d u-C d y) \wedge(d x-u d y),(d v-D d x) \wedge(d y-v d x)\}
$$

On the other hand, $p_{3}$ and $q_{1}$ vanish identically on a neighborhood of a point in $M$ if and only if that point lies in a neighborhood $U$ with coordinates and functions $C$ and $D$ as above so that, on $U$

$$
\mathcal{I}=\{(d u-C d y) \wedge d x,(d v-D d x) \wedge d y\}
$$

Proof: Construct a local coordinate system of the type guaranteed by the first proposition. Because $p_{4}=q_{2}=0$, the structure equations imply

$$
\omega^{1} \wedge d \omega^{1}=p_{3} \omega^{1} \wedge \omega^{2} \wedge \omega^{3} \quad \text { and } \quad \omega^{3} \wedge d \omega^{3}=q_{1} \omega^{3} \wedge \omega^{4} \wedge \omega^{1} .
$$

Suppose first, that $p_{3}$ and $q_{1}$ are non-zero at a point in $M$ (and hence on a neighborhood of this point), then using the fact that $\omega^{1}=\lambda(d x-A d y)$ for some non-zero
function $\lambda$, we compute that $\omega^{1} \wedge d \omega^{1}=\lambda^{2} d x \wedge d y \wedge d A=p_{3} \omega^{1} \wedge \omega^{2} \wedge \omega^{3}$. It follows that $\left\{\omega^{1}, \omega^{2}, \omega^{3}\right\}=\{d x, d y, d A\}$ and hence that we may take $u=A$. Applying the same argument to $\omega^{3}$, and using the assumption that $q_{1} \neq 0$, we see that we may take $v=B$. This is the first statement we wanted to prove.

On the other hand, if $p_{3}$ and $q_{1}$ vanish identically on a neighborhood of the point in question, then these structure equations imply that $\left\{\omega^{1}\right\}$ and $\left\{\omega^{3}\right\}$ are each integrable separately. It then follows that we can choose our initial functions $x$ and $y$ so that $\omega^{1}$ is a multiple of $d x$ and $\omega^{3}$ is a multiple of $d y$. The rest of the construction proceeds as before.

The coordinate systems constructed in the course of the proofs of the above two propositions are not canonical. However, an examination of the proofs shows that the ambiguity in the choice of coordinates is, first, the choice of $x$ and $y$ subject to the condition that $\Omega=\lambda d x \wedge d y$, which involves a choice of 2 arbitrary functions of two variables, and then a choice of $u$ and $v$ subject to conditions which determine each of these two functions up to a choice of an arbitrary function of three variables. Thus, the "coordinate ambiguity" (sometimes known as the "gauge group") in the above normal forms depends only on functions of three variables. Since the normal forms involve 2 arbitrary functions of four variables (namely, $C$ and $D$ ), it is reasonable to say that the "moduli space" of hyperbolic systems with $p_{4}=q_{2}=0$ "depends" on 2 arbitrary functions of four variables. This is not entirely unexpected, of course, because the conditions $p_{4}=q_{2}=0$ represent two conditions on the 4 arbitary functions of four variables on which the "moduli space" of general hyperbolic systems with $s=0$ "depends". What is, perhaps, surprising is that imposing the further conditions $p_{3}=q_{1}=0$ does not lower this "generality". The space of such structures still "depends" on 2 arbitrary functions of four variables.

We would also like to note that the normal form of the first proposition is quite useful for doing calculations. For example, it is easy to calculate that, in an open set $U$ with local coordinates as in that proposition, the coframing

$$
\begin{aligned}
& \eta^{1}=D_{u}(d x-A d y) \\
& \eta^{2}=(d u-C d y) \\
& \eta^{3}=C_{v}(d y-B d x) \\
& \eta^{4}=(d v-D d x)
\end{aligned}
$$

is 1-adpated to $\mathcal{I}$. Thus, for example, we have the following simple formula:

$$
\Omega=(1-A B) D_{u} C_{v} d x \wedge d y
$$

In particular, $\Omega$ will be closed, a condition equivalent to the conditions $p_{4}=q_{2}=$ $p_{1}=q_{3}=0$, if and only if

$$
\left((1-A B) D_{u} C_{v}\right)_{u}=\left((1-A B) D_{u} C_{v}\right)_{v}=0
$$

In other words $(1-A B) D_{\psi} C_{v}$ must be a (non-zero) function of $x$ and $y$ only. Note that this condition still leaves 1 arbitrary function of four variables free (either $D$ or $C$ can still be chosen arbitrarily subject only to the condition that $D_{u} C_{v} \neq 0$ ), another suprising "function count", given that the closure of $\Omega$ is four conditions on the invariants. Even imposing the further condition that $p_{3}=q_{1}=0$, so that we can take $A=B=0$ in the normal form, still leaves $C$ and $D$ subject only to the single condition that $D_{u} C_{v}$ should be a function of $x$ and $y$ only.
1.5.6 Linear systems. As another example of the use of the invariants of $B$ to understand normal forms, we want to give a characterization of linear systems of PDE in terms of these invariants. Before stating the characterization, it is useful to first get an idea of what we might want to prove by computing the invariants for the general linear first order hyperbolic system for two functions of two variables.

Consider the general such PDE system

$$
\mathbf{A}\binom{u_{x}}{v_{x}}+\mathbf{B}\binom{u_{y}}{v_{y}}+\mathbf{C}\binom{u}{v}=\binom{0}{0},
$$

where $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ are 2-by-2 matrices with entries which are functions of $x$ and $y$. The assumption that this system is hyperbolic is equivalent to the condition that $A$ and $B$ be everywhere linearly independent matrices which are simultaneously diagonalizable, i.e., there should exist invertible matrices $\mathbf{P}$ and $\mathbf{Q}$ (with entries which are functions of $x$ and $y$ ) so that

$$
\mathbf{P A Q}=\left(\begin{array}{cc}
a_{1} & 0 \\
0 & a_{2}
\end{array}\right) \quad \text { and } \quad \mathbf{P B Q}=\left(\begin{array}{cc}
b_{1} & 0 \\
0 & b_{2}
\end{array}\right)
$$

(of course, $a_{1} b_{2}-a_{2} b_{1} \neq 0$ ). This implies that we can make a change of dependent variables $u$ and $v$, writing

$$
\binom{u}{v}=\mathbf{Q}\binom{z}{w}
$$

and then the above equations reduce to the form

$$
\binom{a_{1} z_{x}+b_{1} z_{y}}{a_{2} w_{x}+b_{2} w_{y}}+\mathbf{P}^{-1} \mathbf{C Q}\binom{z}{w}=0
$$

Now, every point of the $x y$-plane has a neighborhood on which there exist coordinates $s$ and $t$ so that

$$
a_{1} \frac{\partial}{\partial x}+b_{1} \frac{\partial}{\partial y}=a \frac{\partial}{\partial t} \quad \text { and } \quad a_{2} \frac{\partial}{\partial x}+b_{2} \frac{\partial}{\partial y}=b \frac{\partial}{\partial s}
$$

for some non-zero functions $a$ and $b{ }^{15}$ Making this change of coordinates, the above equation takes the form

$$
\binom{z_{t}}{w_{s}}=\left(\begin{array}{ll}
c_{11} & c_{12} \\
c_{21} & c_{22}
\end{array}\right)\binom{z}{w}
$$

15) Just choose $s$ with $d s \neq 0$ to be constant on the flow lines of the first vector field and $t$ with $d t \neq 0$ to be constant on the flow lines of the second vector field.
where the $c_{i j}$ are functions of $s$ and $t$. Now setting $c_{11}=-f_{s}$ and $c_{22}=-g_{t}$ and then defining $u=e^{f} z$ and $v=e^{g} w$, the system reduces to the form

$$
u_{t}=P v \quad \text { and } \quad v_{s}=Q u
$$

for some functions $P$ and $Q$ of $s$ and $t$.
The corresponding exterior differential system on $M=\mathbb{R}^{4}$ with coordinates $s, t, u$, and $v$ is generated by

$$
\mathcal{I}=\{(d u-P v d t) \wedge d s,(d v-Q u d s) \wedge d t\}
$$

It is easy to see that this is a non-degenerate system if and only if $P(s, t)$ and $Q(s, t)$ are non-zero. By reversing $s$ and $t$ if necessary, we may even assume that $P$ and $Q$ are positive, which we shall do from now on.

In order to understand the form of the invariants for the general first order linear hyperbolic system for two functions of two variables, it is therefore enough to understand the form of the invariants for this latter system. Note that the second proposition above implies that such a system must have all of its primary invariants equal to zero: $p_{4}=q_{2}=p_{3}=q_{1}=p_{1}=q_{3}=0$. Let $B \rightarrow M$ be the $G$-structure associated to the above $\mathcal{I}$. Consider the 1 -adapted coframing $\eta$ with components

$$
\begin{aligned}
\eta^{1} & =Q d s \\
\eta^{2} & =(d u-P v d t) \\
\eta^{3} & =P d t \\
\eta^{4} & =(d v-Q u d s)
\end{aligned}
$$

as a (global) section of $B$. Of course, we have $\eta^{*}\left(\omega^{i}\right)=\eta^{i}$ by the tautological properties of the forms $\omega^{i}$. Let us use $\varphi_{i j}$ to denote $\eta^{*}\left(\phi_{i j}\right)$. The first and third structure equations give

$$
\begin{aligned}
& \left(\varphi_{22}-\varphi_{44}\right) \wedge Q d s=\left(\varphi_{22}-\varphi_{44}\right) \wedge \eta^{1}=d \eta^{1}=Q_{t} d t \wedge d s \\
& \left(\varphi_{44}-\varphi_{22}\right) \wedge P d t=\left(\varphi_{44}-\varphi_{22}\right) \wedge \eta^{3}=d \eta^{3}=P_{s} d s \wedge d t
\end{aligned}
$$

while the second and fourth coupled with the above formulae give

$$
\begin{aligned}
& -\varphi_{22} \wedge \eta^{2}=d \eta^{2}+\varphi_{21} \wedge \eta^{1}-\eta^{3} \wedge \eta^{4} \equiv 0 \bmod \eta^{1} \\
& -\varphi_{44 \wedge} \wedge \eta^{4}=d \eta^{4}+\varphi_{43} \wedge \eta^{3}-\eta^{1} \wedge \eta^{2} \equiv 0 \bmod \eta^{3}
\end{aligned}
$$

which, in particular, imply that $\varphi_{22}=f_{221} \eta^{1}+f_{222} \eta^{2}$ and $\varphi_{44}=f_{443} \eta^{3}+f_{333} \eta^{4}$ for some functions $f_{i i j}$. Substituting these formulae into the preceding formulae and then solving yields

$$
\varphi_{22}=-\left(P_{s} / P\right) d s \quad \text { and } \quad \varphi_{44}=-\left(Q_{t} / Q\right) d t
$$

From this, we get the formulae

$$
\eta^{*}\left(\Phi_{22}\right)=(\log P)_{s t} d s \wedge d t \quad \text { and } \quad \eta^{*}\left(\Phi_{44}\right)=-(\log Q)_{s t} d s \wedge d t
$$

In particular, it follows that in (9), we must have $k_{i j}=0$ for all $i j$-pairs except possibly 13 and 31 . It is this last observation which provides the key to characterizing linear systems.

Proposition: A non-degenerate hyperbolic system $\left(M^{4}, \mathcal{I}\right)$ satisfies $p_{4}=q_{2}=$ $p_{3}=q_{1}=p_{1}=q_{3}=0$ and $\Phi_{22}+\Phi_{44}=F \omega^{1} \wedge \omega^{3}$ for some function $F$ if and only if $\mathcal{I}$ is locally the hyperbolic system associated to a linear first order hyperbolic system for two functions of two variables. In particular, if $\mathcal{I}$ satisfies these hypotheses, then every point of $M$ lies in a coordinate chart $(x, y, u, v): U \rightarrow \mathbb{R}^{4}$ in which $\mathcal{I}$ has generators of the form

$$
\mathcal{I}=\left\{\left(d u-e^{2 \lambda} v d y\right) \wedge d x,\left(d v-e^{2 \mu} u d x\right) \wedge d y\right\}
$$

for some functions $\lambda$ and $\mu$ of $x$ and $y$ alone.

Proof: We have already shown that a hyperbolic system which arises from a linear first order hyperbolic PDE system for two functions of two variables satisfies these invariant conditions. It remains to prove the converse. Consider the structure equations (5-10), substituting the identities $p_{4}=q_{2}=p_{3}=q_{1}=p_{1}=q_{3}=0$. The equations (8) reduce to

$$
\begin{aligned}
& 0=\left(\Phi_{22}-\Phi_{44}\right) \wedge \omega^{1} \\
& 0=-\Phi_{21 \wedge \omega^{1}}-\Phi_{22 \wedge \omega^{2}-\omega^{1} \wedge \omega^{2} \wedge \omega^{3}} \\
& 0=\left(\Phi_{44}-\Phi_{22}\right) \wedge \omega^{3} \\
& 0=-\Phi_{43} \wedge \omega^{3}-\Phi_{44} \wedge \omega^{4}-\omega^{3} \wedge \omega^{4} \wedge \omega^{1} .
\end{aligned}
$$

The first and third of these equations together imply that $\Phi_{22}-\Phi_{44}$ is a multiple of both $\omega^{1}$ and $\omega^{3}$ and hence of $\omega^{1} \wedge \omega^{3}$. On the other hand, by hypothesis, $\Phi_{22}+\Phi_{44}=$ $F \omega^{1} \wedge \omega^{3}$, thus implying

$$
\begin{aligned}
d \phi_{22} & =k_{13} \omega^{1} \wedge \omega^{3} \\
d \phi_{44} & =k_{31} \omega^{3} \wedge \omega^{1} .
\end{aligned}
$$

We are now going to show that every point of $M$ lies in a local coordinate system as in the statement of the Proposition. To produce this coordinate system, we proceed as follows: First, note that because the primary invariants are zero, it follows that $\omega^{1}$ and $\omega^{3}$ are separately integrable. Now fix a point of $B$ and choose functions $x$ and $y$ on a neighborhood $V$ of the point so that $\omega^{1}$ is a multiple
of $d x$ and $\omega^{3}$ is a multiple of $d y$. By suitably restricting the domain $V$, we can assume that the map $(x, y): V \rightarrow \mathbb{R}^{2}$ is a submersion onto a rectangle of the form $\mathcal{R}=\left(x_{0}, x_{1}\right) \times\left(y_{0}, y_{1}\right)$ in the $x y$-plane and that the fibers of this map are connected and contractible.

By construction, the 2 -forms $d \phi_{22}$ and $d \phi_{44}$ are closed multiples of $d x \wedge d y$ and hence are of the form $d \phi_{22}=L(x, y) d x \wedge d y$ and $d \phi_{44}=M(x, y) d x \wedge d y$ for some functions $L$ and $M$ on $\mathcal{R} .{ }^{16}$ Now, there clearly exist functions $\lambda$ and $\mu$ on $\mathcal{R}$ so that $L=2 \lambda_{x y}$ and $M=-2 \mu_{x y}$. It follows then that

$$
d\left(\phi_{22}+\lambda_{x} d x-\lambda_{y} d y\right)=d\left(\phi_{44}-\mu_{x} d x+\mu_{y} d y\right)=0
$$

Thus, there exist functions $s$ and $t$ on $V$ so that

$$
\begin{aligned}
& \phi_{22}=d s-\lambda_{x} d x+\lambda_{y} d y \\
& \phi_{44}=d t+\mu_{x} d x-\mu_{y} d y
\end{aligned}
$$

Now, using the structure equations and the fact that $\omega^{1} \wedge d x=0$, we compute

$$
\begin{aligned}
d\left(e^{t-s-\mu-\lambda} \omega^{1}\right) & =e^{t-s-\mu-\lambda}\left(-\phi_{44}+\phi_{22}+d t-d s-d \mu-d \lambda\right) \wedge \omega^{1} \\
& =-2 e^{t-s-\mu-\lambda}\left(\mu_{x}+\lambda_{x}\right) d x \wedge \omega^{1}=0 .
\end{aligned}
$$

Thus, the non-vanishing 1 -form $e^{t-s-\mu-\lambda} \omega^{1}$ is closed and a multiple of $d x$. Thus, $e^{t-s-\mu-\lambda} \omega^{1}=X^{\prime}(x) d x=d X \neq 0$ for some function $X$ on $\mathcal{R}$. Replacing $x$ by $X$, we can assume that $X=x$, so we do this. Using a similar argument applied to $\omega^{3}$ and replacing $y$ by a function $Y$ if necessary we can arrange

$$
\begin{aligned}
& \omega^{1}=e^{s-t+\mu+\lambda} d x \\
& \omega^{3}=e^{t-s+\mu+\lambda} d y
\end{aligned}
$$

In particular, we now have $\Omega=e^{2(\lambda+\mu)} d x \wedge d y$. (Note that this change of variables in $x$ and $y$ will likely change the boundaries of the rectangle $\mathcal{R}$, but this is not important.) Now we can compute that

$$
d\left(e^{f+\lambda} \omega^{1} \wedge \omega^{3} \wedge \omega^{2}\right)=\left(d s-\phi_{22}\right) \wedge \omega^{1} \wedge \omega^{3} \wedge \omega^{2}=0
$$

It follows that there is a function $u$, unique up to addition of an arbitrary function of $x$ and $y$ so that

$$
e^{s+\lambda} \omega^{1} \wedge \omega^{3} \wedge \omega^{2}=\Omega \wedge d u
$$

16) For convenience, for the rest of the proof, we will simplify our notation by simply writing $F$ instead of $F(x, y)$ or $(x, y)^{*}(F)$ as notation for the pullback via $(x, y)$ of a function $F$ on $\mathcal{R}$.

Thus, there exists a function $P$ on $V$ so that

$$
\omega^{2} \equiv e^{-s-\lambda}(d u-P d y) \bmod d x
$$

Similarly, there exist functions $v$ and $Q$ so that

$$
\omega^{4} \equiv e^{-t-\mu}(d v-Q d x) \bmod d y
$$

Now, using the structure equations, we compute that

$$
0=\omega^{1} \wedge\left(d \omega^{2}+\phi_{22} \wedge \omega^{2}-\omega^{3} \wedge \omega^{4}\right)=e^{-t+\mu} d x \wedge d y \wedge d\left(P-e^{2 \lambda} v\right)
$$

It follows that $P=e^{2 \lambda} v+P_{0}$ where $P_{0}$ is a function of $x$ and $y$ alone. Similarly, we see that $Q=e^{2 \mu} u+Q_{0}$ where $Q_{0}$ is a function of $x$ and $y$ alone. Now, by the existence theory for linear hyperbolic PDE, there exist functions $u_{0}$ and $v_{0}$ on $\mathcal{R}$ which satisfy the equations

$$
\left(u_{0}\right)_{y}=e^{2 \lambda} v_{0}+P_{0} \quad \text { and } \quad\left(v_{0}\right)_{x}=e^{2 \lambda} u_{0}+Q_{0}
$$

Replacing $u$ and $v$ by $u+u_{0}$ and $v+v_{0}$, we get a new coordinate system where $P_{0}=Q_{0}=0$. Now the functions $x, y, u$, and $v$ are constant on the fibers of the submersion $V \rightarrow M$ and furnish the desired coordinate system on an open neighborhood of the base point in $M$ of the point in $B$ that we initially fixed. But, by construction,

$$
\mathcal{I}=\left\{\omega^{1} \wedge \omega^{2}, \omega^{3} \wedge \omega^{4}\right\}=\left\{\left(d u-e^{2 \lambda} v d y\right) \wedge d x,\left(d v-e^{2 \mu} u d x\right) \wedge d y\right\}
$$

as we wanted to show.
Example. Let us illustrate this result by applying it to the FPU equation

$$
z_{y y}-\left(k\left(z_{x}\right)\right)^{2} z_{x x}=0
$$

introduced as Example 3 in Section 1.2.3. Recall that $k$ is assumed to be a smooth positive function on $\mathbb{R}$. Since there is no explicit $z$-dependence in this equation, we can associate to it the $s=0$ exterior differential system $\mathcal{I}$ on $x y p q$-space defined by

$$
\mathcal{I}=\{(d q+k(p) d p) \wedge(d x+k(p) d y),(d q-k(p) d p) \wedge(d x-k(p) d y)\}
$$

(Since $x$ and $y$ appear linearly in the generating forms, it is not surprising that this system should turn out to be linear.) The condition that this system be nondegenerate is easily seen to be that $k^{t}(p) \neq 0$, so we assume this from now on. It is then easy to compute that the coframing

$$
\begin{aligned}
& \eta^{1}=k^{\prime}(p) /\left(4 k(p)^{2}\right)(k(p) d p+d q) \\
& \eta^{2}=(k(p) d y+d x) \\
& \eta^{3}=k^{\prime}(p) /\left(4 k(p)^{2}\right)(k(p) d p-d q) \\
& \eta^{4}=(k(p) d y-d x)
\end{aligned}
$$

is 1 -adapted. Since $\eta^{1}$ and $\eta^{3}$ are clearly integrable and since $\Omega=\eta^{1} \wedge \eta^{3}$ is clearly closed, we must have $p_{4}=q_{2}=p_{3}=q_{1}=p_{1}=q_{3}=0$. Further computation then reveals that

$$
\begin{aligned}
d \phi_{22} & =K \eta^{1} \wedge \eta^{3} \\
d \phi_{44} & =K \eta^{3} \wedge \eta^{1}
\end{aligned}
$$

where

$$
K=\frac{4\left(k^{\prime \prime \prime}(p) k^{\prime}(p) k(p)^{2}-3 k^{\prime \prime}(p) k^{\prime}(p)^{2} k(p)-k^{\prime \prime}(p)^{2} k(p)^{2}+4 k^{\prime}(p)^{4}\right)}{k^{\prime}(p)^{4}}
$$

In particular, $\Phi_{22}+\Phi_{44} \equiv 0$, so all the conditions for linearity are fulfilled.
Before ending this subsection, we would like to comment on the geometric meaning of the invariants in the linear case. As the structure equations derived in the course of the proof make clear, the quantities

$$
\begin{aligned}
\Omega & =\omega^{1} \wedge \omega^{3}=e^{2 \lambda+2 \mu} d x \wedge d y=-\Omega \\
g & =\omega^{1} \circ \omega^{3}=e^{2 \lambda+2 \mu} d x \circ d y={ }^{*} g \\
\frac{1}{2}\left(\Phi_{22}-\Phi_{44}\right) & =K \omega^{1} \wedge \omega^{3}=(\lambda+\mu)_{x y} d x \wedge d y \\
\frac{1}{2}\left(\Phi_{22}+\Phi_{44}\right) & =F \omega^{1} \wedge \omega^{3}=(\lambda-\mu)_{x y} d x \wedge d y
\end{aligned}
$$

are invariants of the system

$$
\mathcal{I}=\left\{\left(d u-e^{2 \lambda} v d y\right) \wedge d x,\left(d v-e^{2 \mu} u d x\right) \wedge d y\right\}
$$

Note, in particular, that $K$ is the curvature of the pseudo-Riemannian metric $g$.
The cases where $K$ and $F$ are constant are particularly interesting since these are precisely the cases where the (pseudo-)group of local automorphisms of the structure acts transitively on $B$. Depending on the signum of $K$, there are three models for complete pseudo-Riemannian metrics of constant curvature:

$$
g_{K}=\left\{\begin{array}{cl}
\frac{d x \circ d y}{\cos ^{2}(c(x+y))} & \text { where }|c(x+y)|<\pi / 2 \text { and } K=c^{2}>0 \\
\frac{d x \circ d y}{\frac{d x \circ d y}{\cos ^{2}(c(x-y))}} & \text { if } K=0, \\
\text { where }|c(x-y)|<\pi / 2 \text { and } K=-c^{2}<0
\end{array}\right.
$$

For each allowable value of $K$, there is a 1-parameter family of inequivalent homogeneous linear hyperbolic systems corresponding to the value of $F$. For example, when $K=0$, we get the systems

$$
\mathcal{I}_{0, F}=\left\{\left(d u-e^{F x y} v d y\right) \wedge d x,\left(d v-e^{-F x y} u d x\right) \wedge d y\right\}
$$

while when $K=c^{2}>0$ and $F=K \beta$, we get the systems

$$
\mathcal{I}_{K, \beta}=\left\{\left(d u-\frac{v}{\cos ^{1+\beta}(c(x+y))} d y\right) \wedge d x,\left(d v-\frac{u}{\cos ^{1-\beta}(c(x+y))} d x\right) \wedge d y\right\} .
$$

The case of $K<0$ is similar. (Note, by the way, that the geometric difference between the $K>0$ and $K<0$ cases is that, in the former, a "time-like" curve (i.e., one on which the metric restricts to be negative) can cross all of the null curves while a "space-like" one cannot. In the latter, the reverse is true.)

The corresponding linear PDE, in various coordinate systems and in the $s=1$ version as well, were studied extensively by Euler and Poisson. ${ }^{17}$

As our final remark about these linear systems with $K$ and $F$ constant, we note these systems are precisely the ones for which the group of symmetries acts transitively on $B$. In particular, in these cases, there can be no canonical subbundle of $B$ which is preserved under all symmetries. Hence, there cannot be any canonical structure reduction in these cases. This shows that, without making some assumptions about the torsion terms in (5) or the curvature terms in (9) or (10), there will be no canonical reduction of the $G$-structure.
1.5.7 The first prolongation. We are now going to apply the structure equations derived so far to study the first prolongation of $(M, \mathcal{I})$. Our goal in this subsection is to prove that any non-degenerate hyperbolic system with $s=0$ which is integrable by the method of Darboux at level 1 is locally equivalent to one of two possible hyperbolic systems.

First, we want to describe a natural submersion $B \rightarrow M^{(1)}$ which will be used to express the characteristic systems on $M^{(1)}$ in terms of the structure equations on $B$. First, we give an invariant description: Let $E \subset T B$ be the codimension 2 distribution defined by the equations $\omega^{2}=\omega^{4}=0$. By its very construction, for each $b \in B$, the image subspace

$$
\pi^{(1)}(b)=\pi_{*}\left(E_{b}\right) \subset T_{\pi(b)} M
$$

is a 2 -dimensional integral element of $\mathcal{I}$ and hence is an element of $M^{(1)}$. Thus, we have defined a mapping $\pi^{(1)}: B \rightarrow M^{(1)}$. As we shall see, this is a submersion
17) For example, see Chap. 3 of Livre 4 of [Da], where Darboux studies the equation

$$
z_{x y}=\frac{\beta^{\prime}}{(x-y)} z_{x}-\frac{\beta}{(x-y)} z_{y}
$$

for $z$ as a function of $x$ and $y$ in the hall-plane $x-y>0$. As this is a second order equation with no explicit $z$ dependence, the methods of Section 1.1 show how to associate to it a hyperbolic system of class $s=0$. This system will clearly be linear and one easily sees that, as $\beta$ and $\beta^{\prime}$ vary, this gives a two parameter family of linearizable systems with $K$ and $F$ constant,
onto the open set of integral elements of $\mathcal{I}$ on which the 2 -form $\Omega=\omega^{1} \wedge \omega^{3}$ is nonzero and the $\pi^{(1)}$-pullback of the Pfaffian system $\mathcal{I}^{(1)}$ is just the Pfaffian system generated by the 1 -forms $\omega^{2}$ and $\omega^{4}$.

Explicitly, one can see this as follows: If $\eta=\left(\eta^{i}\right)$ is a 1-adapted coframing on an open set $U \subset M$, then on $B_{U}=\pi^{-1}(U) \subset B$, there exist unique functions $s_{2}$, $s_{4}, r_{2} \neq 0$, and $r_{4} \neq 0$, so that

$$
\left(\begin{array}{c}
\omega^{1} \\
\omega^{2} \\
\omega^{3} \\
\omega^{4}
\end{array}\right)=\left(\begin{array}{cccc}
r_{2} / r_{4} & 0 & 0 & 0 \\
-s_{2} / r_{2} & 1 / r_{2} & 0 & 0 \\
0 & 0 & r_{4} / r_{2} & 0 \\
0 & 0 & -s_{4} / r_{4} & 1 / r_{4}
\end{array}\right)\left(\begin{array}{c}
\eta^{1} \\
\eta^{2} \\
\eta^{3} \\
\eta^{4}
\end{array}\right)
$$

(In fact, the map $\left(\pi, r_{2}, r_{4}, s_{2}, s_{4}\right): B_{U} \rightarrow U \times \mathbb{R}^{*} \times \mathbb{R}^{*} \times \mathbb{R} \times \mathbb{R}$ is a diffeomorphism.) Substituting these formulae into the structure equations (4) and expanding shows that there are congruences

$$
\left.\begin{array}{rl}
\phi_{22} & \equiv\left(1 / r_{2}\right) d r_{2} \\
\phi_{44} & \equiv\left(1 / r_{4}\right) d r_{4} \\
\phi_{21} & \equiv\left(r_{4} / r_{2}^{2}\right) d s_{2} \\
\phi_{43} & \equiv\left(r_{2} / r_{4}^{2}\right) d s_{4}
\end{array}\right\} \bmod \eta^{1}, \eta^{2}, \eta^{3}, \eta^{4}
$$

If $\left(\mathbf{e}_{1}, \mathbf{e}_{2}, \mathbf{e}_{3}, \mathbf{e}_{4}\right)$ is the frame field on $U$ dual to the coframing $\eta$, then one computes that, for $x=\pi(b)$,

$$
\pi^{(1)}(b)=\operatorname{span}\left\{\mathbf{e}_{1}(x)+s_{2}(b) \mathbf{e}_{2}(x), \mathbf{e}_{3}(x)+s_{4}(b) \mathbf{e}_{4}(x)\right\}
$$

which is clearly an integral element of $\mathcal{I}$. Moreover, it follows that

$$
\left(\pi^{(1)}\right)^{*}\left(\mathcal{I}^{(1)}\right)=\left\{\eta^{2}-s_{2} \eta^{1}, \eta^{4}-s_{4} \eta^{3}\right\}=\left\{\omega^{2}, \omega^{4}\right\}
$$

as we wanted to show.
Now, from (5), and the definitions given in Section 1.3.6, the structure equations

$$
\begin{aligned}
d \omega^{2} & \equiv-\phi_{21} \wedge \omega^{1}+\omega^{3} \wedge \omega^{4} \bmod \omega^{2} \\
d \omega^{4} & \equiv-\phi_{43} \wedge \omega^{3}+\omega^{1} \wedge \omega^{2} \bmod \omega^{4}
\end{aligned}
$$

yield these formulae for the characteristic systems:

$$
\begin{aligned}
& \Xi_{10}^{(1)}=\left\{\omega^{2}, \omega^{1}, \phi_{21}\right\}, \\
& \Xi_{01}^{(1)}=\left\{\omega^{4}, \omega^{3}, \phi_{43}\right\} .
\end{aligned}
$$

By the structure equations (5), (7), and (10), we see that

$$
\begin{aligned}
d \omega^{2}-\omega^{3} \wedge \omega^{4} & \equiv d \omega^{1} \equiv d \phi_{21} \equiv 0 \bmod \omega^{2}, \omega^{1}, \phi_{21} \\
d \omega^{4}-\omega^{1} \wedge \omega^{2} & \equiv d \omega^{3} \equiv d \phi_{43} \equiv 0 \bmod \omega^{4}, \omega^{3}, \phi_{43}
\end{aligned}
$$

so it follows that

$$
\begin{aligned}
& \Xi_{10}^{(1)\langle 1\rangle}=\left\{\omega^{1}, \phi_{21}\right\}, \\
& \Xi_{01}^{(1)\langle 1\rangle}=\left\{\omega^{3}, \phi_{43}\right\} .
\end{aligned}
$$

Now, if Darboux' method is to succeed at level 1 (i.e., if both the level 1 characteristic systems are to contain completely integrable subsystems of rank 2), then both of these latter systems must be completely integrable. However, by the structure equations (5), (7), and (10) we see that

$$
\left.\begin{array}{rl}
d \omega^{1} & \equiv-\left(p_{3} \omega^{3}+p_{4} \omega^{4}\right) \wedge \omega^{2} \\
d \phi_{21} & \equiv-\left(\kappa_{1}-\omega^{3}\right) \wedge \omega^{2}
\end{array}\right\} \bmod \omega^{1}, \phi_{21}
$$

and

$$
\left.\begin{array}{rl}
d \omega^{3} & \equiv-\left(q_{1} \omega^{1}+q_{2} \omega^{2}\right) \wedge \omega^{4} \\
d \phi_{43} & \equiv-\left(\kappa_{3}-\omega^{1}\right) \wedge \omega^{4}
\end{array}\right\} \bmod \omega^{3}, \phi_{43}
$$

Thus, it follows that the systems $\Xi_{10}^{(1)\langle 1\rangle}$ and $\Xi_{01}^{(1)\langle 1\rangle}$ are completely integrable if and only if

$$
p_{3}=p_{4}=q_{1}=q_{2}=0 \quad \text { and } \quad\left(k_{13}-1\right)=\left(k_{31}-1\right)=k_{14}=k_{32}=0
$$

We are now going to show that, up to diffeomorphism, there are essentially only two systems satisfying these conditions. To do this, we assume that the above equations hold and set, according to (6),

$$
\begin{aligned}
d p_{1} & =p_{1} \phi_{22}+\nabla p_{1}=p_{1} \phi_{22}+p_{11} \omega^{1}+p_{12} \omega^{2}+p_{13} \omega^{3}+p_{14} \omega^{4} \\
d q_{3} & =q_{3} \phi_{44}+\nabla q_{3}=q_{3} \phi_{44}+q_{33} \omega^{3}+q_{34} \omega^{4}+q_{31} \omega^{1}+q_{32} \omega^{2} .
\end{aligned}
$$

Now, on account of the above vanishing assumptions, equations (7) and (8) simplify dramatically. In fact, (8) becomes

$$
\begin{aligned}
& 0=\left(\Phi_{22}-\Phi_{44}\right) \wedge \omega^{1}-\nabla p_{1} \wedge \omega^{1} \wedge \omega^{2} \\
& 0=-\Phi_{22 \wedge \omega^{2}}-\omega^{1} \wedge \omega^{2} \wedge \omega^{3} \\
& 0=\left(\Phi_{44}-\Phi_{22}\right) \wedge \omega^{3}-\nabla q_{3} \wedge \omega^{3} \wedge \omega^{4} \\
& 0=-\Phi_{44} \wedge \omega^{4}-\omega^{3} \wedge \omega^{4} \wedge \omega^{1} .
\end{aligned}
$$

These equations can now be solved for $\Phi_{22}$ and $\Phi_{44}$, yielding the identity $p_{14}=q_{32}$ and the formulae

$$
\begin{aligned}
& \Phi_{22}=\omega^{1} \wedge \omega^{3}+p_{13} \omega^{2} \wedge \omega^{3}+(b+f) \omega^{2} \wedge \omega^{4} \\
& \Phi_{44}=\omega^{3} \wedge \omega^{1}+q_{31} \omega^{4} \wedge \omega^{1}+(b-f) \omega^{4} \wedge \omega^{2}
\end{aligned}
$$

where we have written $2 b$ for the common value of $q_{32}$ and $p_{14}$ and $f$ is a yet to be determined function. Now, a straightforward differentiation using the structure equations so far yields

$$
0=d\left(d\left(\phi_{22}-\phi_{44}\right)\right) \equiv-\left(p_{13} \phi_{21}+q_{31} \phi_{43}\right) \wedge \omega^{1} \wedge \omega^{3} \bmod \omega^{2}, \omega^{4}
$$

so it follows that we must have $p_{13}=q_{31}=0$. We can now compute

$$
\left.\begin{array}{l}
0=d\left(d \phi_{22}\right) \equiv-(f+b) \phi_{21} \wedge \omega^{1} \wedge \omega^{4} \\
0=d\left(d \phi_{44}\right) \equiv-(f-b) \phi_{21} \wedge \omega^{1} \wedge \omega^{4}
\end{array}\right\} \bmod \omega^{2}, \omega^{3}
$$

Thus, $b=f=0$. We now compute

$$
\begin{aligned}
& 0=d\left(d q_{3}\right) \equiv q_{34} \omega^{1} \wedge \omega^{2} \bmod \omega^{3}, \omega^{4} \\
& 0=d\left(d p_{1}\right) \equiv p_{12} \omega^{3} \wedge \omega^{4} \bmod \omega^{1}, \omega^{2}
\end{aligned}
$$

which implies $q_{34}=0$ and $p_{12}=0$. Finally, we compute

$$
\begin{aligned}
& 0=d\left(d \phi_{22}\right)=\left(q_{33}-p_{1}\right) \omega^{1} \wedge \omega^{2} \wedge \omega^{3} \\
& 0=d\left(d \phi_{44}\right)=\left(p_{11}-q_{3}\right) \omega^{1} \wedge \omega^{3} \wedge \omega^{4}
\end{aligned}
$$

which yields $q_{33}=p_{1}$ and $p_{11}=q_{3}$.
At this point, we have structure equations

$$
d\left(\begin{array}{c}
\omega^{1}  \tag{11}\\
\omega^{2} \\
\omega^{3} \\
\omega^{4}
\end{array}\right)=-\left(\begin{array}{cccc}
\phi_{44}-\phi_{22} & 0 & 0 & 0 \\
\phi_{21} & \phi_{22} & 0 & 0 \\
0 & 0 & \phi_{22}-\phi_{44} & 0 \\
0 & 0 & \phi_{43} & \phi_{44}
\end{array}\right) \wedge\left(\begin{array}{c}
\omega^{1} \\
\omega^{2} \\
\omega^{3} \\
\omega^{4}
\end{array}\right)+\left(\begin{array}{c}
-p_{1} \omega^{1} \wedge \omega^{2} \\
\omega^{3} \wedge \omega^{4} \\
-q_{3} \omega^{3} \wedge \omega^{4} \\
\omega^{1} \wedge \omega^{2}
\end{array}\right)
$$

with

$$
\begin{align*}
d \phi_{22} & =\omega^{1} \wedge \omega^{3}+q_{3} \omega^{1} \wedge \omega^{2} \\
d \phi_{44} & =\omega^{3} \wedge \omega^{1}+p_{1} \omega^{3} \wedge \omega^{4} \tag{12}
\end{align*}
$$

and

$$
\begin{align*}
d p_{1} & =p_{1} \phi_{22}+q_{3} \omega^{1} \\
d q_{3} & =q_{3} \phi_{44}+p_{1} \omega^{3} . \tag{13}
\end{align*}
$$

From these equations, we see that the 2-forms $\Psi_{10}=q_{3} \omega^{1} \wedge \omega^{2}$ and $\Psi_{01}=p_{1} \omega^{3} \wedge \omega^{4}$ are well-defined on $M$. From the equations (13), we see that if there is a point $x_{0} \in$ $M$ where both of these 2 -forms vanish, then both $q_{3}$ and $p_{1}$ vanish on the fiber above $x_{0}$ and hence on all the connected components of $B$ which intersect this fiber. In particular, $\Psi_{10}$ and $\Psi_{01}$ must vanish identically on the entire connected component of $M$ which contains $x_{0}$.

Now, the equations (11-13) contain, in particular, the equations

$$
\begin{aligned}
& d \omega^{1}=-\left(\phi_{44}-\phi_{22}\right) \wedge \omega^{1}-p_{1} \omega^{1} \wedge \omega^{2} \\
& d \omega^{3}=-\left(\phi_{22}-\phi_{44}\right) \wedge \omega^{3}-q_{3} \omega^{3} \wedge \omega^{4}
\end{aligned}
$$

and

$$
d\left(\phi_{44}-\phi_{22}\right)=-2 \omega^{1} \wedge \omega^{3}-q_{3} \omega^{1} \wedge \omega^{2}+p_{1} \omega^{3} \wedge \omega^{4}
$$

The significance of these equations is that they imply that there is a canonical $\mathbb{R}^{*}$-bundle $F \rightarrow M$ with connection form $\theta=\phi_{44}-\phi_{22}$ with the following property: To every integral manifold $S \subset M$ of the exterior differential system with independence condition ( $\mathcal{I}, \omega^{1} \wedge \omega^{3}$ ), the bundle $F$ restricts to be the coframe bundle of the "characteristic Lorentzian metric" $g=\omega^{1} \circ \omega^{3}$ and its connection form restricts to be the Levi-Civita connection form of this metric. Moreover, this characteristic metric has constant curvature and is locally isometric to the complete metric of this curvature, namely

$$
g_{0}=\frac{d x \circ d y}{\cos ^{2}(x+y)}
$$

Let us first dispose of the connected components of $M$ where $\Psi_{10}$ and $\Psi_{01}$ vanish identically. Over such a component, the above structure equations simplify considerably. Since all of the primary invariants (even $q_{3}$ and $p_{1}$ ) are now zero, and since, by inspection, we have $\Phi_{22}+\Phi_{44}=0$, it follows by our linearization result that the system $\mathcal{I}$ represents a linearizable system of PDE. Moreover, $F=0$ and $K=1$. Thus, the system is locally equivalent to the constant curvature linear example

$$
\mathcal{I}_{1,0}=\left\{\left(d u-\frac{v}{\cos (x+y)} d y\right) \wedge d x,\left(d v-\frac{u}{\cos (x+y)} d x\right) \wedge d y\right\}
$$

defined in the domain in $x y u v$-space given by the inequality $|x+y|<\pi / 2$.
Applying the method of Darboux and going through the calculations then shows that every solution of the system

$$
u_{y}(x, y)=\sec (x+y) v(x, y) \quad v_{x}(x, y)=\sec (x+y) u(x, y)
$$

can be written in the form

$$
\begin{aligned}
& u(x, y)=g(y) \sec (x+y)+f(x) \tan (x+y)+f^{\prime}(x) \\
& v(x, y)=f(x) \sec (x+y)+g(y) \tan (x+y)+g^{\prime}(y)
\end{aligned}
$$

for some functions of one variable $f$ and $g$ and that these are unique up to a replacement of the form

$$
(f(x), g(y)) \mapsto(f(x)+a \sin (x)+b \cos (x), g(y)+a \cos (y)+b \sin (y))
$$

where $a$ and $b$ are constants. We leave to the reader the task of showing how to determine the functions $f$ and $g$ from non-characteristic initial data.

Example (continued): Before going on to consider the non-linear possibility, let us consider the FPU equation that we earlier saw was linear. We computed that

$$
K=\frac{4\left(k^{\prime \prime \prime}(p) k^{\prime}(p) k(p)^{2}-3 k^{\prime \prime}(p) k^{\prime}(p)^{2} k(p)-k^{\prime \prime}(p)^{2} k(p)^{2}+4 k^{\prime}(p)^{4}\right)}{k^{\prime}(p)^{4}}
$$

and now know that $K=1$ is the necessary and sufficient condition that this equation be integrable by the method of Darboux at level 1 . Now, the equation $K \equiv 1$ is a third order differential equation for $k(p)$. Inspection shows that if $k(p)$ is a positive solution to this equation, then so is $c_{2} k\left(c_{1} p+c_{0}\right)$ for any three constants $c_{0}, c_{1} \neq 0$, and $c_{2}>0$. The transformations $k(p) \mapsto c_{2} k\left(c_{1} p+c_{0}\right)$ with $c_{1} \neq 0$ and $c_{2}>0$ form a 3 -parameter group $\Gamma$ which acts effectively on the space of positive solutions of this third order equation. Up to equivalence under this group action, only two of these solutions have a positive dimensional stabilizer, namely

$$
k(p)=p^{-2} \quad \text { and } \quad k(p)=p^{-\frac{2}{3}}
$$

both only defined for $p \neq 0$. It is not hard to show that, aside from these two socalled "singular solutions", up to the action of $\Gamma$ there are only two other solutions. The first is the unique function $k: \mathbb{R} \rightarrow(0,1)$ defined implicitly by the equation

$$
p=\frac{2}{\sqrt{k(p)}}+\log \left(\frac{1-\sqrt{k(p)}}{1+\sqrt{k(p)}}\right)
$$

and the second is the unique function $k:(-\infty, 0) \rightarrow(1, \infty)$ defined implicitly by the equation

$$
p=\frac{2}{\sqrt{k(p)}}+\log \left(\frac{\sqrt{k(p)}-1}{\sqrt{k(p)}+1}\right)
$$

Thus, this gives the complete list of FPU equations which are integrable by the method of Darboux at the first level.

As an explicit example of the use of the method of Darboux in this case, let us consider the case $k(p)=p^{-2}$. Consider the differential system $\mathcal{I}$ defined in the region $p>0$ in $x y p q$-space by the 2 -forms

$$
\mathcal{I}=\left\{\left(d q+p^{-2} d p\right) \wedge\left(d x+p^{-2} d y\right),\left(d q-p^{-2} d p\right) \wedge\left(d x-p^{-2} d y\right)\right\}
$$

which is the differential system associated to the equations

$$
\begin{aligned}
p_{y}-q_{x} & =0 \\
q_{y}+\left(p^{-3} / 3\right)_{x} & =0
\end{aligned}
$$

which model the dynamics of a polytropic perfect gas [FX]. The initial conditions are specified along $y=0$ in the form

$$
\begin{aligned}
& p(x, 0)=p_{0}(x)>0 \\
& q(x, 0)=q_{0}(x) .
\end{aligned}
$$

Applying the method of Darboux then leads to the following recipe for solving this initial value problem. Define a new function $s$ by setting

$$
s(x)=\int_{0}^{x} \frac{1}{2} p_{0}(\xi) d \xi
$$

Note that, because $p_{0}>0$, the function $s$ is strictly increasing. If we now reparametrize the initial curve $(x, y, p, q)=\left(x, 0, p_{0}(x), q_{0}(x)\right)$ in terms of $s$, we can write

$$
(x, y, p, q)=\left(2 \alpha(s), 0,1 / \alpha^{\prime}(s), \beta^{\prime}(s)\right)
$$

for some functions $\alpha$ and $\beta$ defined on the range of $s$. Then the integral surface of $\mathcal{I}$ containing this curve is given by

$$
\begin{aligned}
& x(s, t)=\alpha(s)+\alpha(t)+\beta(t)-\beta(s)+\frac{1}{2}(s-t)\left(\beta^{\prime}(s)+\beta^{\prime}(t)+\alpha^{\prime}(t)-\alpha^{\prime}(s)\right) \\
& y(s, t)=2(s-t) /\left(\alpha^{\prime}(s)+\alpha^{\prime}(t)+\beta^{\prime}(t)-\beta^{\prime}(s)\right) \\
& p(s, t)=2 /\left(\alpha^{\prime}(s)+\alpha^{\prime}(t)+\beta^{\prime}(t)-\beta^{\prime}(s)\right) \\
& q(s, t)=\frac{1}{2}\left(\beta^{\prime}(s)+\beta^{\prime}(t)+\alpha^{\prime}(t)-\alpha^{\prime}(s)\right)
\end{aligned}
$$

where the four functions are defined in the region in the st-plane defined by the inequality

$$
\left(\alpha^{\prime}(s)+\alpha^{\prime}(t)+\beta^{\prime}(t)-\beta^{\prime}(s)\right)>0 .
$$

Note that this certainly includes the line $s=t$ which corresponds to the original initial curve.

Now, it is well-known that, under suitable hypotheses on the initial data, solutions of an FPU equation will develop "shocks". In our terminology, this corresponds to the failure of the solution surface constructed above to be representable as a graph, i.e., in the form $p=P(x, y)$ and $q=Q(x, y)$ for some functions $P$ and $Q$ defined on the whole $x y$-plane. Of course, this will generally happen when the $\operatorname{map}(s, t) \mapsto(x(s, t), y(s, t))$ fails to be a diffeomorphism from the st-plane to the
$x y$-plane. In order to understand this behavior, it is helpful to simplify the above formulae by introducing new functions $a$ and $b$ by $a=\beta+\alpha$ and $b=\beta-\alpha$ so that the above formulae become

$$
\begin{aligned}
& x(s, t)=a(t)-b(s)+\frac{1}{2}(s-t)\left(a^{\prime}(t)+b^{\prime}(s)\right) \\
& y(s, t)=2(s-t) /\left(a^{\prime}(t)-b^{\prime}(s)\right) \\
& p(s, t)=2 /\left(a^{\prime}(t)-b^{\prime}(s)\right) \\
& q(s, t)=\frac{1}{2}\left(a^{\prime}(t)+b^{\prime}(s)\right)
\end{aligned}
$$

The condition for characteristic completeness is then that we have $a^{\prime}(t)>b^{\prime}(s)$ for all $s$ and $t$ in the range of definition. In other words, the graph of $a^{\prime}$ must lie strictly above the graph of $b^{\prime}$. One easily computes that

$$
d x \wedge d y=-2 \frac{\left(a^{\prime}(t)-b^{\prime}(s)-(t-s) a^{\prime \prime}(t)\right)\left(a^{\prime}(t)-b^{\prime}(s)-(t-s) b^{\prime \prime}(s)\right)}{\left(a^{\prime}(t)-b^{\prime}(s)\right)^{2}} d s \wedge d t
$$

It follows that, in order that $d x \wedge d y$ not vanish (so that the mapping $(s, t) \mapsto$ $(x(s, t), y(s, t))$ is at least a local diffeomorphism), we must have $a^{\prime}(t)>b^{\prime}(s)+$ $(t-s) b^{\prime \prime}(s)$ for all $s$ and $t$ in the range of definition, i.e., the graph of $a^{\prime}$ must lie above all the tangent lines to the graph of $b^{\prime}$, as well as $a^{\prime}(t)+(s-t) a^{\prime \prime}(t)>b^{\prime}(s)$ for all $s$ and $t$ in the range of definition, i.e., the graph of $b^{\prime}$ must lie below all the tangent lines to the graph of $a^{\prime}$. These conditions clearly cannot be met for any non-constant functions $a$ and $b$ which satisfy the conditions $a^{\prime}(t) \equiv a_{0}$ and $b^{\prime}(s) \equiv b_{0}$ for all $s$ and $t$ satisfying $|s|,|t|>M$, in other words, for initial data which are compactly supported perturbations of constant initial data. Thus, this analysis recovers the well-known fact (see, for instance, [La]) that solutions of this FPU equation with initial condition which are "small" perturbations of constant initial data must develop shocks.

Now let us turn to the analysis of the components of $M$ on which $\Psi_{10}$ and $\Psi_{01}$ do not vanish simultaneously. Thus, restricting our attention to one such component and calling it $M$ for convenience, we see that $q_{3}$ and $p_{1}$ do not vanish simultaneously at any point of $B$. Now, it follows from (13) that at any point of $B$ where $q_{3}$ vanishes, its differential $d q_{3}$ is a non-zero multiple of $\omega^{3}$. In particular, the locus $q_{3}=0$ is a smooth hypersurface in $B$ which contains all of the fibers of $B \rightarrow M$ which it intersects. Moreover, since $d q_{3}=p_{1} \omega^{3}$ along this locus, it follows that this hypersurface is a union of leaves of the foliation of $B$ defined by $\omega^{3}=0$. In particular the image of this hypersurface in $M$, which is the locus where $\Psi_{10}$ vanishes, is a countable union of closed leaves of the characteristic foliation defined by $\omega^{3}=0$. A similar picture prevails for the locus where $\Psi_{01}$ vanishes. It is a countable union of closed leaves of the characteristic foliation defined by $\omega^{1}=0$.

Suppose, first of all, that we are at a point $z$ of $M$ where neither $\Psi_{10}$ nor $\Psi_{01}$ vanish. Then, by the structure equations, we can choose a coframing $\eta=$ $\left(\eta^{1}, \eta^{2}, \eta^{3}, \eta^{4}\right)$ on a neighborhood of $z$ which is a section of $B$ and also satisfies $p_{1}=q_{3}=1$. This coframing then satisfies the following equations (which are clearly specializations of (11-13)).

$$
d\left(\begin{array}{c}
\eta^{1} \\
\eta^{2} \\
\eta^{3} \\
\eta^{4}
\end{array}\right)=-\left(\begin{array}{cccc}
\varphi_{44}-\varphi_{22} & 0 & 0 & 0 \\
\varphi_{21} & \varphi_{22} & 0 & 0 \\
0 & 0 & \varphi_{22}-\varphi_{44} & 0 \\
0 & 0 & \varphi_{43} & \varphi_{44}
\end{array}\right) \wedge\left(\begin{array}{c}
\eta^{1} \\
\eta^{2} \\
\eta^{3} \\
\eta^{4}
\end{array}\right)+\left(\begin{array}{c}
-\eta^{1} \wedge \eta^{2} \\
\eta^{3} \wedge \eta^{4} \\
-\eta^{3} \wedge \eta^{4} \\
\eta^{1} \wedge \eta^{2}
\end{array}\right)
$$

with

$$
\begin{aligned}
& d \varphi_{22}=\eta^{1} \wedge \eta^{3}+\eta^{1} \wedge \eta^{2} \\
& d \varphi_{44}=\eta^{3} \wedge \eta^{1}+\eta^{3} \wedge \eta^{4}
\end{aligned}
$$

and

$$
\begin{aligned}
& 0=\varphi_{22}+\eta^{1} \\
& 0=\varphi_{44}+\eta^{3}
\end{aligned}
$$

and the ideal $\mathcal{I}$ is generated by $\left\{\eta^{1} \wedge \eta^{2}, \eta^{3} \wedge \eta^{4}\right\}$.
Since $\eta^{1}$ and $\eta^{3}$ are integrable but $d \eta^{1} \neq 0$ and $d \eta^{3} \neq 0$, we may introduce coordinates $(x, y, p, q)$ with $p$ and $q$ positive, such that

$$
\eta^{1}=\frac{d x}{p} \quad \text { and } \quad \eta^{3}=\frac{d y}{q}
$$

From the structure equations, we have

$$
\Omega_{10}=\eta^{1} \wedge \eta^{2}=-d \eta^{1}-\eta^{1} \wedge \eta^{3}=\left(\frac{d p}{p}+\frac{d y}{q}\right) \wedge \frac{d x}{p}
$$

and similarly

$$
\Omega_{01}=\eta^{3} \wedge \eta^{4}=-d \eta^{3}-\eta^{3} \wedge \eta^{1}=\left(\frac{d q}{q}+\frac{d x}{p}\right) \wedge \frac{d y}{q}
$$

Setting $p=e^{-u}$ and $q=e^{-v}$ gives

$$
\begin{aligned}
& \Omega_{10}=-\left(d u-e^{v} d y\right) \wedge e^{u} d x \\
& \Omega_{01}=-\left(d v-e^{u} d x\right) \wedge e^{v} d y
\end{aligned}
$$

and therefore the system models the $s=0$ Liouville system

$$
\begin{aligned}
u_{y} & =e^{v} \\
v_{x} & =e^{u}
\end{aligned}
$$

Now, although the coordinate chart on a neighborhood of $z$ that we have constructed is not canonical, the 1 -forms $\eta^{1}$ and $\eta^{3}$ are actually well-defined. Any two local coordinate charts $(x, y, p, q)$ and $(\bar{x}, \bar{y}, \bar{p}, \bar{q})$ with $p, q, \bar{p}$, and $\bar{q}$ all positive and satisfying

$$
\eta^{1}=\frac{d x}{p}=\frac{d \bar{x}}{\bar{p}} \quad \text { and } \quad \eta^{3}=\frac{d y}{q}=\frac{d \bar{y}}{\bar{q}}
$$

must be related on some (possibly smallex) neighborhood of $z$ by relations of the form

$$
(x, y, p, q)=\left(X(\bar{x}), Y(\bar{y}), X^{\prime}(\bar{x}) \bar{p}, Y^{\prime}(\bar{y}) \bar{q}\right)
$$

where $X$ and $Y$ are functions of one variable with positive derivative. Thus, on the open set in $M$ which is the complement of the loci $\Psi_{10}=0$ and $\Psi_{01}=0$, the hyperbolic structure is actually locally homogeneous and induces a special atlas of charts which carry the system $\mathcal{I}$ into the $s=0$ Liouville system.

Now let us consider the case where $z \in M$ lies in the locus where $\Psi_{10}$ vanishes. Let $y$ be any function on a neighborhood of $z$ satisfying $y(z)=0$ but $(d y)_{z} \neq 0$ and with the property that $d y \wedge \omega^{3}=0$. Then we may restrict to smaller neighborhood $V$ of $z$ with the property that $y=0$ defines the zero locus of $\Psi_{10}$ in $V$. Regarding $y$ as a function on $B_{V}$ by pull-back, we see that $q_{3}$ and $y$ have the same zero locus in $B_{V}$ and have non-vanishing differentials there. It follows that the ratio $r=q_{3} / y$ is a smooth non-vanishing function on $B_{V}$ and calculation shows that

$$
d r \equiv r \phi_{44} \bmod \omega^{3}
$$

It follows that we can choose a section $\left(\eta^{1}, \eta^{2}, \eta^{3}, \eta^{4}\right)$ of $B_{V} \rightarrow V$ which lies in the locus $p_{1}=r=1$. In other words, this section will have $p_{1}=1$ and $q_{3}=y$. Thus, we will have structure equations

$$
d\left(\begin{array}{c}
\eta^{1} \\
\eta^{2} \\
\eta^{3} \\
\eta^{4}
\end{array}\right)=-\left(\begin{array}{cccc}
\varphi_{44}-\varphi_{22} & 0 & 0 & 0 \\
\varphi_{21} & \varphi_{22} & 0 & 0 \\
0 & 0 & \varphi_{22}-\varphi_{44} & 0 \\
0 & 0 & \varphi_{43} & \varphi_{44}
\end{array}\right) \wedge\left(\begin{array}{c}
\eta^{1} \\
\eta^{2} \\
\eta^{3} \\
\eta^{4}
\end{array}\right)+\left(\begin{array}{c}
-\eta^{1} \wedge \eta^{2} \\
\eta^{3} \wedge \eta^{4} \\
-y \eta^{3} \wedge \eta^{4} \\
\eta^{1} \wedge \eta^{2}
\end{array}\right)
$$

with

$$
\begin{aligned}
& d \varphi_{22}=\eta^{1} \wedge \eta^{3}+y \eta^{1} \wedge \eta^{2} \\
& d \varphi_{44}=\eta^{3} \wedge \eta^{1}+\eta^{3} \wedge \eta^{4}
\end{aligned}
$$

and

$$
\begin{aligned}
0 & =\varphi_{22}+y \eta^{1} \\
d y & =y \varphi_{44}+\eta^{3}
\end{aligned}
$$

and the ideal $\mathcal{I}$ is generated by $\left\{\eta^{1} \wedge \eta^{2}, \eta^{3} \wedge \eta^{4}\right\}$. Now, since $\eta^{1}$ is integrable, but $d \eta^{1} \neq 0$, it is easy to see that there exist functions $x$ and $p>0$ on a possibly smaller neighborhood of $z$ so that $\eta^{1}=p d x$, and by construction, there exists
a function $Q$ on this neighborhood so that $\eta^{3}=Q d y$. However, looking at the equation $d y=y \varphi_{44}+\eta^{3}$ on the locus $y=0$ shows that we must have $Q=1$ on this locus. Thus, we may write $Q=1+y q$, where $q$ is some smooth function on a neighborhood of $z$. Substituting this into the equations just above shows that

$$
\varphi_{22}=-y p d x \quad \text { and } \quad \varphi_{44}=-q d x
$$

Substituting these relations into the formulae for $d \varphi_{22}$ and $d \varphi_{44}$ and then solving for $\eta^{1} \wedge \eta^{2}$ and $\eta^{3} \wedge \eta^{4}$ yields the expressions

$$
\eta^{1} \wedge \eta^{2}=d x \wedge(d p-p q d y) \quad \text { and } \quad \eta^{3} \wedge \eta^{4}=d y \wedge(d q-p(1+y q) d x)
$$

It follows that the four functions $x, y, p$, and $q$ form a coordinate system on a neighborhood of $z$. Moreover, the system in $x y p q$-space defined by

$$
\mathcal{J}=\{d x \wedge(d p-p q d y), d y \wedge(d q-p(1+y q) d x)\}
$$

is easily seen to be a non-degenerate hyperbolic system away from the hypersurfaces defined by $p=0$ and $1+y q=0$.

There is a similar normal form for this non-linear system in a neighborhood of a point on the locus where $\Psi_{01}$ vanishes. Details will be left to the reader.

In conclusion, we have shown that there are essentially only two types of nondegenerate hyperbolic systems with $s=0$ which are integrable by the method of Darboux at level one. The first type is linear and the second is non-linear, being locally equivalent to the $s=0$ Liouville system away from a hypersurface. We shall refer to it henceforth as the extended $s=0$ Liouville system. It would be an interesting project to classify the global systems (if any) of the latter type which are characteristically complete in the sense of Section 1.2.3.

Now from our results so far, we see that the method of Darboux can be used to integrate the $s=0$ Liouville system. In fact we may explicitly carry out the integration, as follows.

On $M$ we use the coframing

$$
\omega^{1}=d x, \quad \omega^{2}=d u-e^{v} d y, \quad \omega^{3}=d y, \quad \omega^{4}=d v-e^{u} d x
$$

Then on $M^{(1)}$ we have the coframing

$$
\omega_{10}=\omega^{1}, \quad \theta_{10}=\omega^{2}-h_{20} \omega^{1}, \quad \omega_{01}=\omega^{3}, \quad \theta_{01}=\omega^{4}-h_{02} \omega^{3}
$$

and

$$
\begin{aligned}
& \pi_{20}=d h_{20}-e^{u+v} d y \\
& \pi_{02}=d h_{02}-e^{u+v} d x
\end{aligned}
$$

Differentiating, we obtain

$$
\begin{aligned}
& d \theta_{10}=-\pi_{20 \wedge} \omega_{10}+e^{v} \omega_{01} \wedge \theta_{01} \\
& d \theta_{01}=-\pi_{02} \wedge \omega_{01}+e^{u} \omega_{10} \wedge \theta_{10}
\end{aligned}
$$

and

$$
\begin{aligned}
& d \pi_{20} \equiv e^{u+v} \omega_{01} \wedge\left(\theta_{10}+\theta_{01}\right) \bmod \omega_{10} \\
& d \pi_{02} \equiv e^{u+v} \omega_{10} \wedge\left(\theta_{10}+\theta_{01}\right) \bmod \omega_{01}
\end{aligned}
$$

By definition the characteristic systems are

$$
\begin{aligned}
& \Xi_{10}^{(1)}=\left[\omega_{10}, \theta_{10}, \pi_{20}\right] \\
& \Xi_{01}^{(1)}=\left[\omega_{01}, \theta_{01}, \pi_{02}\right],
\end{aligned}
$$

and clearly

$$
\left.\begin{array}{rl}
d \theta_{10} & \equiv e^{v} \quad \omega_{01} \wedge \theta_{01} \\
d \pi_{20} & \equiv e^{u+v} \omega_{01} \wedge \theta_{01}
\end{array}\right\} \bmod \Xi_{10}^{(1)}
$$

with similar formulas holding for the other characteristic system. It follows that the first derived systems are

$$
\begin{aligned}
\Xi_{10}^{(1)\langle 1\rangle} & =\left[\omega_{10}, \pi_{20}-e^{u} \theta_{10}\right] \\
& =\left[d x, d\left(h_{20}-e^{u}\right)\right]
\end{aligned}
$$

and similarly

$$
\Xi_{01}^{(1)\langle 1\rangle}=\left[d y, d\left(h_{02}-e^{v}\right)\right] .
$$

Using this result we may integrate the $s=0$ Liouville system, as follows: Since

$$
d\left(h_{20}-e^{u}\right) \wedge d x=d y \wedge d\left(h_{02}-e^{v}\right)=0
$$

on solutions we may set

$$
\begin{aligned}
& h_{20}-e^{u}=\alpha^{\prime}(x) \\
& h_{02}-e^{v}=\beta^{\prime}(y)
\end{aligned}
$$

for functions $\alpha(x)$ and $\beta(y)$. The equations

$$
\theta_{10}=\theta_{01}=0
$$

then yield

$$
\begin{aligned}
& d u=\left(\alpha^{\prime}(x)+e^{u}\right) d x+e^{y} d y \\
& d v=\left(\beta^{\prime}(y)+e^{v}\right) d y+e^{u} d x
\end{aligned}
$$

and so we must solve the ODEs

$$
\begin{aligned}
& u_{x}=\alpha^{\prime}(x)+e^{u} \\
& v_{y}=\beta^{\prime}(y)+e^{v}
\end{aligned}
$$

For this we set

$$
\begin{gathered}
U=u-\alpha(x)=\log f \\
V=v-\beta(y)=\log g
\end{gathered}
$$

and consider functions $a(x), b(x)$ which satisfy

$$
\left\{\begin{array}{l}
a^{\prime}(x)=-e^{\alpha(x)} \\
b^{\prime}(y)=-e^{\beta(y)}
\end{array}\right.
$$

Then the above ODE system is

$$
\begin{aligned}
U_{x} & =e^{\alpha(x)} e^{U} \\
V_{y} & =e^{\beta(y)} e^{V}
\end{aligned}
$$

or, equivalently,

$$
\begin{aligned}
& \frac{f_{x}}{f^{2}}=-a^{\prime}(x) \\
& \frac{g_{y}}{g^{2}}=-b^{\prime}(y)
\end{aligned}
$$

These may be integrated to give

$$
\begin{aligned}
& e^{u} e^{-\alpha}=f=\frac{1}{a(x)+\tilde{b}(y)} \\
& e^{v} e^{-\beta}=g=\frac{1}{\tilde{a}(x)+b(y)} .
\end{aligned}
$$

Substituting in the expressions for $d u$ and $d v$ allows us to set $\tilde{a}=a, \tilde{b}=b$ and finally

$$
\begin{aligned}
& e^{u}=\frac{-a^{\prime}(x)}{a(x)+b(y)} \\
& e^{v}=\frac{-b^{\prime}(y)}{a(x)+b(y)}
\end{aligned}
$$

for the general solution to the $s=0$ Liouville system. We note that $a(x), b(y)$ are arbitrary subject to $a^{\prime}(x)<0$ and $b^{\prime}(y)<0$.

Remark: To close the loop with the previous discussion at the end of Section 1.4 of the $s=1$ Liouville equation, we may differentiate the first equation with respect to $y$ to obtain

$$
e^{u+v}=\frac{a^{\prime}(x) b^{\prime}(y)}{(a(x)+b(y))^{2}}
$$

which since

$$
(u+v)_{x y}=2 e^{u+v}
$$

gives the general solution to the $s=1$ Liouville equation mentioned in Section 1.4.3.

### 1.6 Hyperbolic Euler-Lagrange systems of class $s=0$.

1.6.1 Bi-symplectic structures. Among the hyperbolic exterior differential systems of class $s=0$ are the Euler-Lagrange systems that we introduced as Example 4 in Section 1.1.3. In this section, we shall develop some of the special properties of these systems, culminating in a variant of Noether's Theorem, which will describe the relationship between symmetries and classical conservation laws for these EulerLagrange systems. (The notion of conservation laws for general hyperbolic exterior differential systems will be developed more fully in Section 2.1.)

We begin by recalling the construction of hyperbolic Euler-Lagrange systems of class $s=0$. Let $M$ be a 4 -manifold and let $\Phi$ be a symplectic 2 -form on $M$. Associated to any other 2 -form $\Lambda$ on $M$, we want to consider the functional $F_{\mathrm{A}}$ on immersed $\Phi$-Lagrangian surfaces $S \subset M$ which are oriented and compact (possibly with boundary) which is defined by the rule

$$
\begin{equation*}
F_{\Lambda}(S)=\int_{S} \Lambda \tag{1}
\end{equation*}
$$

In Example 4 in Section 1.1.3, we saw that the critical points $S$ of this functional relative to variations that leave fixed the boundary of $S$ are the integral surfaces of the exterior differential system

$$
\Phi=\Psi=0
$$

Here, $\Psi=d \psi$ where $\psi$ is defined to be the (unique) 1-form satisfying $d \Lambda=\Phi \wedge \psi$. We shall call $\mathcal{E}(\Lambda)=\{\Phi, \Psi\}$ the Euler-Lagrange system associated to the functional $F_{\Lambda}$.

Note that if $\tilde{\Lambda}=\Lambda+d \gamma+f \Phi$ for some 1-form $\gamma$ and some function $f$, then for any compact, oriented $\Phi$-Lagrangian surface $S$, we will have

$$
F_{\bar{\Lambda}}(S)=F_{\Lambda}(S)+\int_{\partial S} \gamma
$$

It follows that $S$ is critical for $F_{\bar{\Lambda}}$ with respect to variations through $\Phi$-Lagrangian surfaces fixing the boundary if and only if it is critical for $F_{\Lambda}$ with respect to the
same variations. Thus, it would not be surprising if $\Lambda$ and $\tilde{\Lambda}$ gave rise to the same Euler-Lagrange system.

In fact, this is precisely what happens. Let $\mathcal{I}_{\Phi}$ be the differential ideal generated by the symplectic form $\Phi$, and recall that the characteristic cohomology group $\bar{H}^{2}\left(M, \mathcal{I}_{\Phi}\right)$ is by definition $H^{2}\left(\Omega^{*}(M) / \mathcal{I}_{\Phi}\right)$ where the differential on the quotient complex $\Omega^{*}(M) / \mathcal{I}_{\Phi}$ is induced by $d$. Note that $\Lambda$ and $\tilde{\Lambda}$ determine the same cohomology class in $\bar{H}^{2}\left(M, \mathcal{I}_{\Phi}\right)$ if and only if there exist a 1 -form $\gamma$ and a function $f$ so that $\tilde{\Lambda}=\Lambda+d \gamma+f \Phi$. In this case, we will have

$$
d \tilde{\Lambda}=d \Lambda+d f \wedge \Phi=(\psi+d f) \wedge \Phi
$$

so we can take $\tilde{\psi}=\psi+d f$, implying $\tilde{\Psi}=d \tilde{\psi}=d \psi=\Psi$. In particular, $\Psi$ depends only on the characteristic cohomology class $[\Lambda]_{\Phi} \in \bar{H}^{2}\left(M, \mathcal{I}_{\Phi}\right)$ defined by $\Lambda$. In fact, this construction can be carried further to show that, on any open set $U \subset M$ satisfying $H^{2}(U, \mathbb{R})=H^{3}(U, \mathbb{R})=0$, there is actually an isomorphism

$$
\bar{H}^{2}\left(U, \mathcal{I}_{\Phi}\right) \simeq\left\{\Psi \in \Omega^{2}(U) \mid \Phi \wedge \Psi=0, d \Psi=0\right\}
$$

defined by the obvious assignment $[\Phi] \mapsto \Psi$.
Throughout this section we shall make the assumption that the exterior differential system $\mathcal{E}(\Lambda)$ is non-degenerate in the sense that the 2 -form $\Psi$ satisfies the condition that $\Psi \wedge \Psi$ is nowhere vanishing. This is equivalent to the assumption that the associated PDE system have non-degenerate symbol.

Indeed, taking the exterior derivative of the equation $d \Lambda=\Phi \wedge \psi$ yields

$$
\Phi_{\wedge} \wedge \Psi=0 .
$$

Hence, in order that $\{\Phi, \Psi\}$ span a hyperbolic pencil at each point one must have

$$
\Psi \wedge \Psi=-f^{2} \Phi \wedge \Phi
$$

for some non-vanishing function $f$ (which we may take to be positive), while the condition that $\{\Phi, \Psi\}$ span an elliptic pencil at each point is that

$$
\Psi \wedge \Psi=f^{2} \Phi \wedge \Phi
$$

for some non-vanishing function $f$ (which we could also take to be positive).
The properties of these pairs ( $\Phi, \Psi$ ) are of sufficient interest to warrant giving the separate

DEFINITION: A bi-symplectic structure $(M ; \Phi, \Psi)$ on a 4 -manifold $M$ is a pair ( $\Phi, \Psi$ ) of everywhere linearly independent symplectic forms. The bi-symplectic
structure is said to be special in case $\Phi_{\wedge} \Psi=0$ and is said to be hyperbolic (respectively, elliptic) if the pencil generated by $\Phi$ and $\Psi$ is everywhere hyperbolic (respectively, elliptic).

Example 1: Prescribed Gauss curvature. In an open set $M \subset \mathbb{R}^{4}$, consider the standard symplectic form

$$
\Phi=d p \wedge d x+d q \wedge d y
$$

If $f$ and $g$ are any non-vanishing (smooth) functions of two variables, then the 2-form

$$
\Psi=f(p, q) d p \wedge d q-g(x, y) d x \wedge d y
$$

is closed, non-degenerate, and satisfies

$$
\Phi \wedge \Psi=0 .
$$

On solution surfaces to $\Phi=\Psi=0$ of the form

$$
(x, y) \rightarrow(x, y, p(x, y), q(x, y))
$$

we obviously have $p=z_{x}$ and $q=z_{y}$ for some function $z(x, y)$ which satisfies the PDE

$$
\begin{equation*}
f\left(z_{x}, z_{y}\right)\left(z_{x x} z_{y y}-z_{x y}^{2}\right)=g(x, y) \tag{2}
\end{equation*}
$$

A special case is when

$$
f=\frac{1}{\left(1+p^{2}+q^{2}\right)^{2}}
$$

Then we may think of $(x, y, z(x, y))$ as the graph of a piece of surface $\Sigma$ in Euclidean space $\mathbb{E}^{3}$, and the left hand side of (2) is the Gauss curvature of $\Sigma$.

Note that this defines an elliptic bi-symplectic structure if $g$ is positive and a hyperbolic bi-symplectic structure if $g$ is negative. Especially noteworthy is the case when $g=-c^{2}<0$ for some constant $c$. Then this system models the equation for constant negative curvature surfaces. In this case, the decomposable 2-forms in the pencil generated by $\Phi$ and $\Psi$ are

$$
\begin{aligned}
& \Omega_{10}=\left(\frac{d p}{1+p^{2}+q^{2}}+c d y\right) \wedge\left(\frac{d q}{1+p^{2}+q^{2}}-c d x\right) \\
& \Omega_{01}=\left(\frac{d p}{1+p^{2}+q^{2}}-c d y\right) \wedge\left(\frac{d q}{1+p^{2}+q^{2}}+c d x\right)
\end{aligned}
$$

It may be checked that the characteristic curves on integral surfaces are the usual asymptotic curves of elementary differential geometry.
1.6.2 A structural symmetry. For a special bi-symplectic structure the roles of $\Phi$ and $\Psi$ are obviously symmetric, which suggests that hyperbolic Euler-Lagrange systems of class $s=0$ might have a corresponding structural symmetry. We shall now see that this is indeed the case.

Proposition: Suppose that $(M ; \Phi, \Psi)$ is a special bi-symplectic structure on a manifold $M$ satisfying $H^{2}(M)=H^{3}(M)=0$. Then there exist 2-forms $\Lambda$ and $\Omega$ on $M$ so that the exterior differential system generated by the equations

$$
\Phi=\Psi=0
$$

is the Euler-Lagrange system both of the functional $F_{\mathrm{A}}$ defined on $\Phi$-Lagrangian surfaces and of the functional $F_{\Omega}$ defined on $\Psi$-Lagrangian surfaces.

Proof: Since $H^{2}(M)=0$, there are 1 -forms $\varphi$ and $\psi$ such that

$$
\left\{\begin{array}{l}
\Phi=d \varphi \\
\Psi=d \psi
\end{array}\right.
$$

from which it follows that

$$
d(\Phi \wedge \psi)=0
$$

since $\Phi \wedge \Psi=0$. Thus, since $H^{3}(M)=0$, there is a 2-form $\Lambda$ so that

$$
\Phi \wedge \psi=d \Lambda
$$

and we conclude that

$$
\begin{equation*}
\Phi=\Psi=0 \tag{3}
\end{equation*}
$$

is the Euler-Lagrange system for the functional

$$
F_{\Lambda}(S)=\int_{S} \Lambda
$$

on $\Phi$-Lagrangian surfaces $S$.
Symmetrically, since $d(\Psi \wedge \varphi)=0$ and $H^{3}(M)=0$, there exists a 2-form $\Omega$ so that

$$
\Psi_{\wedge \varphi}=d \Omega
$$

Then (3) is also the Euler-Lagrange system for the functional

$$
F_{\Omega}(R)=\int_{R} \Omega
$$

on $\Psi$-Lagrangian surfaces $R$.

The importance of this proposition is that it gives a sufficient criterion for an exterior differential system $\mathcal{I}$ generated by a pencil of 2 -forms on a 4 -manifold to be expressible as a (non-degenerate) Euler-Lagrange system; namely it must be generated by a bi-symplectic structure.

Example 2: Systems defined by conservation laws. We want to illustrate this situation. To put this example in context, recall that, by the last proposition in Section 1.1.4, a (real-analytic) hyperbolic Euler-Lagrange system is (at least, locally) the exterior differential system associated to a hyperbolic system of conservation laws. In this example, we want to look at the converse situation. We will determine conditions that a translation-invariant hyperbolic system of conservation laws

$$
\begin{aligned}
u_{t}+(f(u, v))_{x} & =0 \\
v_{t}+(g(u, v))_{x} & =0
\end{aligned}
$$

might satisfy in order to be an Euler-Lagrange system. Using the notation from Section 1.1, we can express this pair of PDE as the exterior differential system

$$
\begin{aligned}
& \Phi=-d u \wedge d x+d f \wedge d t=d(-u d x+f d t) \\
& \Psi=-d v \wedge d x+d g \wedge d t=d(-v d x+g d t)
\end{aligned}
$$

The conditions

$$
\Phi \wedge \Phi \neq 0, \quad \Psi \wedge \Psi \neq 0, \quad \Phi \wedge \Psi=0
$$

are, respectively,

$$
f_{v} \neq 0, \quad g_{u} \neq 0, \quad f_{u}=g_{v}
$$

By the last of these relations we have

$$
f=F_{v}, \quad g=F_{u}
$$

for a function $F(u, v)$, and the first two inequalities give

$$
F_{v v} \neq 0, \quad F_{u u} \neq 0 .
$$

To determine the functional $F_{\Lambda}$, we set

$$
\psi=-v d x+F_{u} d t
$$

so that $d \psi=\Psi$ and seek to determine a 2 -form $\Lambda$ so that

$$
d \Lambda=\psi \wedge \Phi
$$

The right hand side is

$$
\begin{aligned}
v d F_{v} \wedge d x \wedge d t-F_{u} d u \wedge d x \wedge d t & =v d F_{v} \wedge d x \wedge d t-d F \wedge d x \wedge d t+F_{v} d v \wedge d x \wedge d t \\
& =d\left(\left(v F_{v}-F\right) d x \wedge d t\right)
\end{aligned}
$$

To better understand this we may locally introduce new coordinates $U$ and $V$ by the relations

$$
\begin{aligned}
U & =-u \\
V & =F_{v}
\end{aligned}
$$

so that $u=-U$ and $v=H(U, V)$. Then we have

$$
\begin{aligned}
& \Phi=d U \wedge d x+d V \wedge d t \\
& \Lambda=(V H(U, V)-F(-U, H(U, V))) d x \wedge d t
\end{aligned}
$$

Then $\Phi$-Lagrangian surfaces which are locally graphs over $x t$-space are given by $U=W_{x}, V=W_{t}$ for a function $W(x, t)$, and

$$
\Lambda=\left(W_{t} H\left(W_{x}, W_{t}\right)-F\left(-W_{x}, H\left(W_{x}, W_{t}\right)\right)\right) d x \wedge d t=L\left(W_{x}, W_{t}\right) d x \wedge d t
$$

defines a first order functional on $W(x, t)$.
For a familiar specific case we may take

$$
F=\frac{1}{2}\left(u^{2}+v^{2}\right)
$$

in which case $U=-u, V=v$ and the Lagrangian is

$$
L=\frac{1}{2}\left(V_{t}^{2}-V_{x}^{2}\right) .
$$

The hyperbolic PDE is, not surprisingly, the $s=0$ wave equation

$$
\begin{aligned}
& u_{t}+v_{x}=0 \\
& v_{t}+u_{x}=0
\end{aligned}
$$

More interesting is the general case when

$$
F=\frac{1}{\lambda}\left(u^{\lambda}+v^{\lambda}\right), \quad \lambda \neq 0,1,2 .
$$

Then the functional is a constant times

$$
\int\left(W_{x}^{\lambda}+(\lambda-1) W_{t}^{\lambda / \lambda-1}\right) d x \wedge d t
$$

The hyperbolic PDE system is

$$
\begin{aligned}
& u_{t}+(\lambda-1) v^{\lambda-2} v_{x}=0 \\
& v_{t}+(\lambda-1) u^{\lambda-2} u_{x}=0
\end{aligned}
$$

For $\lambda=3$, this is a pair of coupled Burgers'-type equations that, after adjusting constants, becomes

$$
\begin{aligned}
& u_{t}+v v_{x}=0 \\
& v_{t}+u u_{x}=0
\end{aligned}
$$

In these specific examples, the symmetry between $\Phi$ and $\Psi$ is accomplished by exchanging the roles of $u$ and $v$ and carrying out the same computations.
Special case: The Fermi-Pasta-Ulam equation gives rise to the $s=0$ hyperbolic system generated by

$$
\begin{aligned}
& \Phi=d p \wedge d x+d q \wedge d y \\
& \Psi=d q \wedge d x-d h \wedge d y
\end{aligned}
$$

where $h(p)=k^{2}(p)$ in the notation of Example 3 in Section 1.2. This system is special bi-symplectic and can be expressed as an Euler-Lagrange system by taking in the above formulae.

$$
F(p, q)=q-H(p)
$$

where $H^{\prime}(p)=h(p)$. Working through the above recipe, we see that the FPU equation is the equation for critical points of the functional

$$
z(x, y) \mapsto \int\left(H\left(z_{x}\right)+\frac{1}{2} z_{y}^{2}\right) d x \wedge d y
$$

1.6.3 An analog of Noether's Theorem. We will now explore the relationship between symmetries and conservation laws for (hyperbolic) Euler-Lagrange systems I. (As will be explained in Section 2.1, a 'conservation law' for $\mathcal{I}$ can be thought of as a closed 2 -form which lies in the ideal $\mathcal{I}$ and thus we will make this identification without further comment.)

The usual statement of this relationship is some version of the classical theorem of E. Noether which asserts that there is an isomorphism

$$
\left\{\begin{array}{l}
\text { symmetries of } \\
\text { a variational } \\
\text { problem }
\end{array}\right\} \simeq\left\{\begin{array}{l}
\text { conservation laws } \\
\text { for the variational } \\
\text { equations }
\end{array}\right\}
$$

The result we shall give below is of this general type but differs from this particular statement in a number of important specifics.

Let $(\Phi, \Psi)$ define a hyperbolic, special bi-symplectic structure on a 4-manifold $M$. We have seen that $\Phi=\Psi=0$ gives the Euler-Lagrange system for a pair of functionals

$$
\int_{S} \Lambda, \quad \int_{R} \Omega
$$

defined, respectively, on $\Phi$ - and $\Psi$-Lagrangian surfaces. Thus our formulation of Noether's Theorem will need to treat $\Phi$ and $\Psi$ symmetrically. In addition, because the hyperbolic system generated by $\Phi$ and $\Psi$ always has at least these two 2forms as independent conservation laws while there is no a priori reason that it should have symmetries, we may suspect that the correct version of the above isomorphism should instead be with a quotient space of all conservation laws. With these observations in mind, we set

$$
\mathfrak{G}=\{\text { Lie algebra of vector fields } v \text { on } M \text { that preserve each of } \Phi \text { and } \Psi\}
$$

$\langle\Phi, \Psi\rangle=\{$ constant linear combinations of $\Phi$ and $\Psi\}$
$\mathcal{C}_{0}=\{$ closed 2-forms which are linear combinations of $\Phi$ and $\Psi\}$
$\overline{\mathcal{C}}_{0}=\mathcal{C}_{0} /\langle\Phi, \Psi\rangle$.
To relate the definition of $\mathfrak{G}$ to symmetries of a variational problem, let us suppose that we have found 2 -forms $\Lambda$ and $\Omega$ as described in the Proposition of the previous subsection. We denote by $\left(M, \Phi,[\Lambda]_{\Phi}\right)$ the data consisting of the symplectic manifold $(M, \Phi)$ together with the equivalence class $[\Lambda]_{\Phi} \in \bar{H}^{2}\left(M, \mathcal{I}_{\Phi}\right)$. We make the corresponding interpretation of $\left(M, \Psi,[\Omega]_{\Psi}\right)$. Then the following proposition should be expected, in view of the explicit identification of $H^{2}\left(M, \mathcal{I}_{\Phi}\right)$ made in Section 1.6.1.

Proposition: If $v \in \mathfrak{G}$, then $v$ is an infinitesimal symmetry of each of the data ( $M, \Phi,[\Lambda]_{\Phi}$ ) and $\left(M, \Psi,[\Omega]_{\Psi}\right)$. Conversely, if a vector field $v$ is an infinitesimal symmetry of either of the data $\left(M, \Phi,[\Lambda]_{\Phi}\right)$ or $\left(M, \Psi,[\Omega]_{\Psi}\right)$, then $v \in \mathfrak{G}$.

Proof: Let $v \in \mathfrak{G}$. From $d \Lambda=\psi \wedge \Phi$ and $d \psi=\Psi$ we infer that

$$
\mathcal{L}_{v} \psi=d h
$$

for some function $h$, and then a short calculation gives

$$
d\left(\mathcal{L}_{v}(\Lambda-h \Phi)\right)=0
$$

It follows that

$$
\mathcal{L}_{v} \Lambda=h \Phi+d \lambda
$$

for some 1 -form $\lambda$, which is the same as

$$
\mathcal{L}_{v}\left([\Lambda]_{\Phi}\right)=0 .
$$

Applying a similar argument to $\mathcal{L}_{v} \varphi$, the proposition now follows.

We can now state and prove our version of Noether's Theorem.
Proposition: There is a natural isomorphism

$$
\begin{equation*}
\eta: \mathfrak{G} \rightarrow \overline{\mathcal{C}}_{0} \tag{*}
\end{equation*}
$$

Proof: To define $\eta$, we write locally, as was done above,

$$
\left\{\begin{array}{l}
\Phi=d \varphi \\
\Psi=d \psi
\end{array}\right.
$$

for 1 -forms $\varphi$ and $\psi$, which are well-defined up to transformations

$$
\left\{\begin{array}{l}
\varphi \rightarrow \bar{\varphi}=\varphi+d G  \tag{4}\\
\psi \rightarrow \bar{\psi}=\psi+d H
\end{array}\right.
$$

for functions $G$ and $H$. For $v \in \mathfrak{G}$ we have

$$
\mathcal{L}_{v} \Phi=\mathcal{L}_{u} \Psi=0
$$

from which it follows that

$$
\begin{align*}
& \mathcal{L}_{v} \varphi=d g  \tag{5}\\
& \mathcal{L}_{v} \psi=d h
\end{align*}
$$

for suitable functions $g$ and $h$. We set

$$
\begin{equation*}
\eta(v)=(v-\varphi) \Psi+(v\lrcorner \psi) \Phi-(g \Psi+h \Phi) . \tag{6}
\end{equation*}
$$

First, we show that $\eta$ is well-defined: The functions $g$ and $h$ are well-defined up to constants by (5). Thus, $\eta$ is well-defined modulo $\langle\Phi, \Psi\rangle$ once we know $\varphi$ and $\psi$. These are in turn well-defined up to a substitution (4). Then

$$
\begin{aligned}
& \mathcal{L}_{v} \bar{\varphi}=\mathcal{L}_{v}(\varphi+d G)=d\left(g+\mathcal{L}_{v} G\right) \\
& \mathcal{L}_{v} \bar{\psi}=\mathcal{L}_{v}(\psi+d H)=d\left(h+\mathcal{L}_{v} H\right)
\end{aligned}
$$

which gives

$$
\begin{aligned}
& \bar{g}=g+\mathcal{L}_{v} G \\
& \bar{h}=h+\mathcal{L}_{v} H
\end{aligned}
$$

and

$$
\begin{aligned}
(v\lrcorner \bar{\varphi}-\bar{g}) \Psi & \left.=(v\lrcorner \varphi+\mathcal{L}_{v} G-g-\mathcal{L}_{v} G\right) \Psi \\
& =(v\lrcorner \varphi-g) \Psi
\end{aligned}
$$

showing that $\eta$ is well-defined.

Second, we show that $\eta(v)=0$ is closed: We have

$$
\begin{aligned}
d[(v\lrcorner \varphi-g) \Psi+(v\lrcorner \psi-h) \Phi] & =(d(v\lrcorner \varphi-g)) \wedge \Psi+(d(v\lrcorner \psi-h)) \wedge \Phi \\
& =-(v\lrcorner \Phi) \wedge \Psi-(v\lrcorner \Psi) \wedge \Phi \\
& =0-v\lrcorner(\Phi \wedge \Psi) \\
& =0
\end{aligned}
$$

since $\Phi_{\wedge} \Psi=0$.
Third, we show that $\eta$ is injective: If $\eta(v)=0$ then

$$
\left\{\begin{array}{l}
v\lrcorner \varphi-g=a \\
v\lrcorner \psi-h=b
\end{array}\right.
$$

for constants $a$ and $b$. The exterior derivative of the first equation gives

$$
\begin{aligned}
0 & \left.=\mathcal{L}_{v} \varphi-v\right\lrcorner \Phi-d g \\
& =v\lrcorner \Phi
\end{aligned}
$$

which implies that $v=0$.
Fourth and finally, we show that $\eta$ is surjective: We may choose a local coframe $\omega^{1}, \omega^{2}, \omega^{3}, \omega^{4}$ so that

$$
\begin{aligned}
& \Phi=\omega^{1} \wedge \omega^{2}+\omega^{3} \wedge \omega^{4} \\
& \Psi=\omega^{1} \wedge \omega^{2}-\omega^{3} \wedge \omega^{4} .
\end{aligned}
$$

For any function $F$ we set

$$
d F=\sum_{i=1}^{4} F_{i} \omega^{i}
$$

A conservation law of level zero is a closed 2-form

$$
A \Phi+C \Psi
$$

From

$$
d A \wedge \Phi+d C \wedge \Psi=0
$$

we infer that

$$
\left\{\begin{array}{l}
A_{1}-C_{1}=0  \tag{7}\\
A_{2}-C_{2}=0 \\
A_{3}+C_{3}=0 \\
A_{4}+C_{4}=0
\end{array}\right.
$$

We want to find a vector field $v$ such that if we define $g$ and $h$ by

$$
\left\{\begin{array}{l}
A=v\lrcorner \psi-h  \tag{8}\\
B=v\lrcorner \varphi-g
\end{array}\right.
$$

then (5) holds. In fact (5) and (8) imply

$$
\left\{\begin{array}{l}
d A=v \Psi  \tag{9}\\
d B=v
\end{array}\right.
$$

and the exterior derivatives of these equation give

$$
\begin{aligned}
& 0=d(v\lrcorner \Psi)=\mathcal{L}_{v} \Psi \\
& 0=d(v\lrcorner \Phi)=\mathcal{L}_{v} \Phi
\end{aligned}
$$

as required.
We note that for $g$ and $h$ defined by (8), equations (9) imply equations (5). Now there are unique vector fields $v_{A}$ and $v_{B}$ satisfying

$$
\begin{aligned}
& \left.d A=v_{A}\right\lrcorner \Psi \\
& \left.d B=v_{B}\right\lrcorner \Phi
\end{aligned}
$$

We then have to show that $v_{A}=v_{B}$, and this is just a restatement of (7).
Example 3: Suppose that

$$
\Phi=d p \wedge d x+d q \wedge d y
$$

and that we have a translation-invariant Lagrangian

$$
\Lambda=L(p, q) d x \wedge d y
$$

as in Section 1.1 above. Then the Euler-Lagrange form is given by

$$
\Psi=L_{p p} d p \wedge d y+L_{p q}(d q \wedge d y+d x \wedge d p)+L_{q q} d x \wedge d q
$$

and we may work through the mapping $\left(^{*}\right)$ to obtain

$$
\begin{aligned}
& \eta(\partial / \partial x)=p \Psi-L_{q} \Phi \\
& \eta(\partial / \partial y)=q \Psi-L_{p} \Phi
\end{aligned}
$$

These conservation laws are similar to linear momenta.
Example 2 (continued): Consider the hyperbolic Euler-Lagrange system generated by

$$
\begin{aligned}
& \Phi=-d u \wedge d x+d F_{v} \wedge d t \\
& \Psi=-d v \wedge d x+d F_{u} \wedge d t
\end{aligned}
$$

where $F(u, v)$ is a function satisfying $F_{u u} F_{v v} \neq 0$. This system admits translation symmetries and therefore has conservation laws corresponding to linear momenta. These are

$$
\begin{aligned}
& \eta(\partial / \partial x)=\Xi \\
&=u\left(-d v \wedge d x+d F_{u} \wedge d t\right)+v\left(-d u \wedge d x+d F_{v} \wedge d t\right) \\
& \eta(\partial / \partial t)=\Lambda=-F_{v}\left(-d v \wedge d x+d F_{u} \wedge d t\right)-F_{u}\left(-d u \wedge d x+d F_{v} \wedge d t\right)
\end{aligned}
$$

Suppose we assume that $F$ is homogeneous of degree $\mu$, i.e., that

$$
F(\alpha u, \alpha v)=\alpha^{\mu} F(u, v)
$$

Then $F_{u}$ and $F_{v}$ are homogeneous of degree $\mu-1$ and the 1-parameter group of dilation symmetries

$$
(u, v, x, y) \rightarrow\left(\alpha u, \alpha v, \frac{x}{\alpha}, \frac{t}{\alpha^{\mu-1}}\right)
$$

preserves both $\Phi$ and $\Psi$. The corresponding conservation law is
$\Gamma=\left(u x-(\mu-1) t F_{v}\right)\left(-d v \wedge d x+d F_{u} \wedge d t\right)+\left(v x-(\mu-1) t F_{u}\right)\left(-d u \wedge d x+d F_{v} \wedge d t\right)$.

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Robert Bryant
Department of Mathematics
Duke University
Durham, NC 27708, USA
Phillip Griffiths
Institute for Advanced Study
Princeton, NJ 08540, USA
Lucas Hsu
School of Mathematics
Institute for Advanced Study
Princeton, NJ 08540, USA


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[^1]:    3) We should remark that unsymmetric systems for which $\operatorname{dim} C_{0}=\infty$ have also been characterized. The results will be presented elsewhere.
[^2]:    7) The general procedure for cononically associating an exterior differential system to a PDE system is explained in Chapter 1 of $\left[\mathrm{BCG}^{3}\right]$. In the first example here, $(x, y, z, p, q, r, s, t)$ are coordinates in the jet space $J^{2}\left(\mathbb{R}^{2}, \mathbb{R}\right)$ and in the second example $(x, y, u, v, p, q, r, s)$ are coordinates in $J^{1}\left(\mathbb{R}^{2}, \mathbb{R}^{2}\right)$. In both cases the PDE defines a submanifold $M$ of the jet space and the exterior differential system is generated as a differential system by the restriction to $M$ of the contact system.
[^3]:    11) In any case, if one of the systems is integrable, then the system is semi-integrable in the sense of Darboux and, as we pointed ont in Section 1.4, the initial value problem be solved by ODE methods anyway.
