# Hyperbolic Exterior Differential Systems and their Conservation Laws *, Part II ** 

R. Bryant, P. Griffiths And L. Hsu

## Part II Introduction

In Part I of this paper we have introduced the concept of a hyperbolic exterior differential system of class $s$. For $s=0$ these are essentially a geometric formulation of a first order quasi-linear hyperbolic PDE system in two independent and two dependent variables, with the group of contact transformations providing the allowable changes of variables. We then studied several geometric and analytic aspects of hyperbolic exterior differential systems of class $s=0$. Recall that such a system is given by the data consisting of a 4 -manifold together with a pair of transverse 2-plane fields. The properties studied included the characteristic variety and initial value problem and systems which are integrable by the method of Darboux. We refer to the introduction of the paper for a more detailed definition and statement of results, together with illustrative examples.

Finally, we analyzed the geometry of hyperbolic systems of class $s=0$. Recall that a geometry means a $G$-structure together with a distinguished class of pseudoconnections (frequently, the pseudo connection is unique). In the second part of the paper we shall use the geometry as a basis for studying the conversation laws of hyperbolic of class $s=0$. The main point is to determine restrictions on the intrinsic invariants (curvatures) associated to the geometry which are equivalent to the condition that the system has a conservation law. Again we refer to the introduction to Part I for further explanations and illustrations of the results given below.

[^0]
## §2. Conservation Laws for Hyperbolic Systems of Class $s=0$

2.1 General form of the conservation laws for hyperbolic systems of class $s=0$.
2.1.1 Conservation laws. In the standard literature on evolution equations, a conservation law for a given evolution equation is a function on the configuration space of the problem which is constant under the specified evolution of states.

For example, when the configuration space is a (finite dimensional) manifold $M$ and the evolution equation is represented by a vector field $X$ on $M$, a conservation law for $X$ is simply a function $f \in C^{\infty}(M)$ which is constant on the flow lines of $X$, i.e., which satisfies $X f=0$. Of course, one wants to ignore "trivial" conservation laws, i.e., functions $f$ which are locally constant, so one might represent the space of conservation laws in this case as a quotient

$$
\mathcal{C}(X, M)=\frac{\left\{f \in C^{\infty}(M) \mid d f(X)=0\right\}}{\left\{f \in C^{\infty}(M) \mid d f=0\right\}} .
$$

If the dimension of $M$ is $n$ and one can find $n-1$ independent conservation laws, say, $f_{1}, \ldots, f_{n-1}$ with independent differentials, then the integral curve of $X$ through a given point $x_{0} \in M$ can be described implicitly by the $n-1$ equations

$$
f_{i}(x)=f_{i}\left(x_{0}\right)
$$

Thus, knowing "enough" independent conservation laws for $X$ describes the integral curves of $X$ completely.

In exterior differential systems, this notion of conservation law generalizes naturally. Given an exterior differential system $\mathcal{I}$ on a manifold $M$, we are interested in studying the ( $p+1$ )-dimensional integral manifolds of $\mathcal{I}$ which contain a given "initial" $p$-dimensional integral manifold $\gamma: L \rightarrow M$. A "conservation law" of degree $p$ should then be a functional on the space of (compact) $p$-dimensional integral manifolds $\gamma: L \rightarrow M$ which gives the same value for any two integral manifolds $\gamma_{0}, \gamma_{1}: L \rightarrow M$ that bound a $(p+1)$-dimensional integral manifold $\Gamma: L \times[0,1] \rightarrow M$ of $\mathcal{I}$.

Now, one source of functionals on the space of compact, oriented $p$-dimensional integral manifolds is the space $\Omega^{p}(M)$ of $p$-forms on $M$. For any $\phi \in \Omega^{p}(M)$ and any immersion $\gamma: L \rightarrow M$ with $L$ compact, we can define the functional

$$
F_{\phi}(\gamma)=\int_{L} \gamma^{*}(\phi)
$$

By Stokes' Theorem, we have

$$
F_{\phi}\left(\gamma_{1}\right)-F_{\phi}\left(\gamma_{0}\right)=\int_{L \times[0,1]} \Gamma^{*}(d \phi)
$$

Thus, in particular, if $\Gamma^{*}(d \phi)=0$ for all integral manifolds $\Gamma: L \times[0,1] \rightarrow M$ of $\mathcal{I}$, then $F_{\phi}$ will be a conservation law in our intended sense. Of course, this will necessarily be true if $d \phi$ lies in $\mathcal{I}$. On the other hand, if $\phi$ itself lies in $\mathcal{I}$, then the functional $F_{\phi}$ will be identically zero. Moreover, if $\phi \equiv d \psi \bmod \mathcal{I}$ for some ( $p-1$ )form $\psi$, then the functional $F_{\phi}$ will, by Stokes' Theorem, take the same value on any two integral manifolds $\gamma_{0}, \gamma_{1}: L \rightarrow M$ of $\mathcal{I}$ which are homologous, whether the homology is via an integral manifold of $\mathcal{I}$ or not. Clearly, we will want to regard such functionals as trivial conservation laws. (Compare this with the case of the trivial conservation laws as described above for vector fields.) This discussion leads us to make a definition of the form

$$
\mathcal{C}_{0}^{p}(M, \mathcal{I})=\frac{\left\{\phi \in \Omega^{p}(M) \mid d \phi \in \mathcal{I}^{p+1}\right\}}{\left\{d \psi \mid \psi \in \Omega^{p-1}(M)\right\}+\mathcal{I}^{p}}
$$

In other words, $\mathcal{C}_{0}^{p}(M, \mathcal{I})$ is the cohomology of the quotient complex $\left(\bar{\Omega}^{*}(M, \mathcal{I}), \bar{d}\right)$ where $\bar{\Omega}^{*}(M, \mathcal{I})=\Omega^{*}(M) / \mathcal{I}$ and $\bar{d}=d \bmod \mathcal{I}$. (The subscript " 0 " is meant to warn the reader that this is our starting point for the definition of conservation laws. The real definition will be given below.)

While this definition of conservation laws properly generalizes the notion of conservation laws for ordinary differential equations, i.e., the case discussed above of a vector field $X$ on a manifold $M$, it has serious shortcomings as a definition for partial differential equations. To see why, consider the exterior differential system associated to the classical wave equation

$$
u_{x x}-u_{y y}=0
$$

As a Monge-Ampère system, this is usually set up as an exterior differential system on $M=\mathbb{R}^{5}$ with coordinates $(x, y, u, p, q)$, an ideal $\mathcal{I}$ generated by $\theta=d u-p d x-$ $q d y$ and the 2-form

$$
\Upsilon=d p \wedge d y+d q \wedge d x
$$

The 1-form

$$
\varphi=\frac{1}{2}\left(p^{2}+q^{2}\right) d x+p q d y
$$

satisfies $d \varphi=-p d \theta+q \Upsilon \in \mathcal{I}$, and hence is closed on all integral surfaces of $\mathcal{I}$. It corresponds to the classical law of conservation of energy for the wave equation:

$$
E(u)=\int_{\mathbb{R}} \frac{1}{2}\left(u_{x}^{2}+u_{y}^{2}\right) d x
$$

However, other expressions which are also known to be conserved, for example

$$
E\left(u_{x}\right)=\int_{\mathbb{R}} \frac{1}{2}\left(u_{x x}^{2}+u_{x y}^{2}\right) d x
$$

cannot be represented by an element of $\mathcal{C}_{0}^{1}(\mathcal{I}, M)$. This is because the manifold $M$ has no coordinates which represent the second derivatives of $u$.

To get around this problem, we pass to the first prolongation. Now, as discussed in Section 1.3 , the underlying manifold $M^{(1)}$ on which the first prolongation is defined is a $\mathbb{P}^{1} \cup \mathbb{P}^{1}$ bundle over $M$. However, the dense open subset $W \subset M^{(1)}$ consisting of those integral elements on which $d x \wedge d y$ is non-vanishing is more easily parametrized: $W$ is simply $M \times \mathbb{R}^{2}=\mathbb{R}^{7}$ with coordinates $(x, y, u, p, q, r, s)$ and the prolonged ideal $\mathcal{I}^{(1)}$ is generated in $W$ by three 1 -forms

$$
\begin{aligned}
& \theta_{0}=d u-p d x-q d y \\
& \theta_{1}=d p-r d x-s d y \\
& \theta_{2}=d q-s d x-r d y
\end{aligned}
$$

On $W$, the conservation law $E\left(u_{x}\right)$ is then represented by the 1 -form

$$
\varphi^{\prime}=\frac{1}{2}\left(r^{2}+s^{2}\right) d x+r s d y
$$

which satisfies $d \varphi^{\prime}=-r d \theta_{1}-s d \theta_{2} \in \mathcal{I}$.
This example points up two problems which must be addressed. First, while this construction captures $E\left(u_{x}\right)$, it clearly cannot capture $E\left(u_{x x}\right)$, which is also a conservation law. In order to capture all of what are classically known as the "local" conservation laws, one must pass to the infinite prolongation $\left(M^{(\infty)}, \mathcal{I}^{(\infty)}\right)$. Second, it can be verified that the class $\left[\varphi^{\prime}\right] \in \mathcal{C}_{0}^{1}\left(W, \mathcal{I}^{(1)}\right)$ is not the restriction of a class in $\mathcal{C}_{0}^{1}\left(M^{(1)}, \mathcal{I}^{(1)}\right)$. This latter problem is not serious if one is only interested in the integral manifolds of $\mathcal{I}$ which represent classical solutions of the wave equation since the lift of such an integral manifold clearly lies in $W$ anyway. However, it does point out that one must be careful about domains of definition when one discusses conservation laws for exterior differential systems.

With these preliminary cautions, we are now ready to define the main object of study for the rest of the paper. Let $(M, \mathcal{I})$ be an exterior differential system and let $\left(M^{(k)}, \mathcal{I}^{(k)}\right)$ be its sequence of prolongations (for integral manifolds of dimension $n$ ). We will assume, as usual, that these are all smooth manifolds and that each of the natural maps $M^{(k+1)} \rightarrow M^{(k)}$ is a smooth, surjective submersion. Also, as usual, we will let $M^{(\infty)}$ denote the inverse limit of this tower of submersions

$$
M=M^{(0)} \longleftarrow M^{(1)} \longleftarrow M^{(2)} \cdots \longleftarrow M^{(k)} \longleftarrow \cdots \longleftarrow M^{(\infty)},
$$

and regard it as a sort of infinite dimensional manifold, whose smooth forms are defined as the direct limit of the inclusions

$$
\begin{aligned}
\Omega^{*}(M)=\Omega^{*}\left(M^{(0)}\right) & \longrightarrow \Omega^{*}\left(M^{(1)}\right) \longrightarrow \Omega^{*}\left(M^{(2)}\right) \cdots \\
& \longrightarrow \Omega^{*}\left(M^{(k)}\right) \longrightarrow \cdots \longrightarrow \Omega^{*}\left(M^{(\infty)}\right)
\end{aligned}
$$

Frequently, it will be necessary to consider forms and other objects defined on "open subsets" $W^{(\infty)} \subset M^{(\infty)}$, where, by definition, an open subset of $M^{(\infty)}$ is a sequence of open subsets $W^{(k)} \subset M^{(k)}$ with the property that the restrictions of the natural submersions $M^{(k+1)} \rightarrow M^{(k)}$ to the various $W^{(k+1)}$ define a sequence of surjective submersions $W^{(k+1)} \rightarrow W^{(k)}$. To save on notation, we will generally just use $\mathcal{I}^{(\infty)}$ to denote the differential ideal generated on $W^{(\infty)}$ by the restriction of the forms in $\mathcal{I}^{(\infty)}$.

The characteristic cohomology of $\mathcal{I}$ on $W^{(\infty)}$ is the cohomology

$$
\bar{H}^{*}\left(W^{(\infty)}, \mathcal{I}\right)=H^{*}\left(\bar{\Omega}^{*}\left(W^{(\infty)}\right), \bar{d}\right)
$$

of the quotient complex

$$
\bar{\Omega}^{*}\left(W^{(\infty)}\right)=\Omega^{*}\left(W^{(\infty)}\right) / \mathcal{I}^{(\infty)}
$$

where $\bar{d}=d \bmod \mathcal{I}^{(\infty)}$.
In $\left[\mathrm{BG}_{1}\right]$ it was proved that, when $\mathcal{I}$ is involutive for integral manifolds of dimension $n$, the characteristic cohomology satisfies a local vanishing result

$$
\bar{H}^{q}\left(W^{(\infty)}, \mathcal{I}\right)=0, \quad 0<q<n-l
$$

Here, $l$ is an easily computed integer measuring what might be called the degree of "over-determination" of $\mathcal{I}$ and should be thought of intuitively as the codimension in an integral $n$-manifold of the appropriate "initial condition" submanifolds. (In most applications, $l=1$.) Thus, in the absence of topological complications, the first (potentially) non-vanishing group is $\bar{I}^{n-l}\left(W^{(\infty)}, \mathcal{I}\right)$, and it is generally defined to be the space $\mathcal{C}\left(W^{(\infty)}, \mathcal{I}\right)$ of conservation laws of the exterior differential system $\mathcal{I} .{ }^{18}$

In cases where the exterior differential system $\mathcal{I}$ arises from a determined PDE system, it is known that this definition (with an appropriate choice of $W^{(\infty)}$ ) recaptures the usual notion of the local conservation laws of the system. An important consequence of this fact and the above definition is that the local conservation laws of a PDE system form a group which is nearly invariant under contact transformations, the only dependence coming from the choice of independence conditions imposed by a choice of independent variables.

Another consequence of the definition is that $\mathcal{C}\left(W^{(\infty)}, \mathcal{I}\right)$, being the first nonvanishing group and therefore occupying a special place in the spectral sequence of the filtered complex $F^{p} \Omega^{*}\left(W^{(\infty)}\right)$, is given canonically as the kernel of a certain linear differential operator

$$
D: \mathcal{E}_{1}\left(W^{(\infty)}\right) \rightarrow \mathcal{E}_{2}\left(W^{(\infty)}\right)
$$

18) Although we will not use any of the attendant machinery or ideas, it would be remiss of us not to point out that what is actually going on is that the quotient complex ( $\bar{\Omega}^{*}, \bar{d}$ ) is a sheaf of differential complexes and that our cohomology theory is actually a sheaf cohomology.
where $\mathcal{E}_{i}\left(W^{(\infty)}\right)$ is the space of sections of a certain vector bundle $E_{i}$ over $W^{(\infty)}$ defined in terms of the crude structure equations of the exterior differential system.

In fact, what happens is this: When $W^{(\infty)}$ has trivial topology, the short exact sequence of complexes

$$
0 \longrightarrow \mathcal{I}^{(\infty)} \longrightarrow \Omega^{*}\left(W^{(\infty)}\right) \longrightarrow \bar{\Omega}^{*}\left(W^{(\infty)}\right) \longrightarrow 0
$$

gives rise to an isomorphism

$$
\bar{H}^{n-l}\left(W^{(\infty)}, \mathcal{I}\right) \xrightarrow{d} H^{n-l+1}\left(W^{(\infty)}, \mathcal{I}^{(\infty)}\right)
$$

which sends the class $[\phi] \in \bar{H}^{n-l}\left(W^{(\infty)}, \mathcal{I}\right)$ to the class $[d \phi] \in H^{n-l+1}\left(W^{(\infty)}\right.$, $\left.\mathcal{I}^{(\infty)}\right)$. Now the vanishing theorem described above actually allows us to find a subbundle $E_{1} \subset\left(I^{(\infty)}\right)^{n-l+1}$ with the properties that, first, $\mathcal{E}_{1}\left(W^{(\infty)}\right) \cap d\left(\left(I^{(\infty)}\right)^{n-l}\right)=$ 0 and, second, there is a containment

$$
Z^{n-l+1}\left(\mathcal{I}^{(\infty)}\right) \subset \mathcal{E}_{1}\left(W^{(\infty)}\right) \oplus d\left(\left(\mathcal{I}^{(\infty)}\right)^{n-l}\right)
$$

It then follows directly that

$$
\bar{H}^{n-l}\left(W^{(\infty)}, \mathcal{I}\right) \simeq H^{n-l+1}\left(W^{(\infty)}, \mathcal{I}^{(\infty)}\right) \simeq\left\{\Phi \in \mathcal{E}_{1}\left(W^{(\infty)}\right) \mid d \Phi=0\right\}
$$

In specific examples, this gives a method of avoiding the traditional problem of factoring out trivial conservation laws, yielding a canonical representative (albeit a differential form of degree $n-l+1$ ) for each conservation law of degree $n-l .{ }^{19}$

Thus, this approach suggests a systematic approach to computing the space of conservation laws, beginning with a normal form which depends only on the symbol of the system. The first step in this computation is to determine this normal form for conservation laws, i.e., the bundle $E_{1}$, and that is what we shall turn to now in the case of hyperbolic systems of class $s=0$.
2.1.2 A normal form for conservation laws. Let $(M, \mathcal{I})$ be a hyperbolic exterior differential system of class $s=0$ which is non-degenerate in the sense of Section 1.5. Let $\left(M^{(\infty)}, \mathcal{I}^{(\infty)}\right)$ be the infinite prolongation of $(M, \mathcal{I})$. Our calculations will take place in the domain $U \subset M$ of a 1 -adapted coframing $\eta=\left(\eta^{1}, \eta^{2}, \eta^{3}, \eta^{4}\right)$. For simplicity, we shall assume that $U$ is connected and that its deRham cohomology vanishes in positive degrees. Let us fix such a coframing $\eta$ and let $U^{(k)} \subset M^{(k)}$
19) In this paper, we will refer to this canonical representative ( $n-l+1$ )-form as the differentiated form of the conservation law that it represents and refer to a representing ( $n-l$ )-form as the undifferentiated form of the conservation law.
denote the (open) subset of integral elements of $\mathcal{I}^{(k-1)}$ on which $\Omega=\eta^{1} \wedge \eta^{3}$ is nonzero and whose projection to $M$ lies in $U .{ }^{20}$ Recall that the assumption that the coframing $\eta$ be 1-adapted implies

$$
\begin{array}{llll}
d \eta^{1} \equiv 0 & \bmod \eta^{1}, \eta^{2}, & d \eta^{3} \equiv 0 & \bmod \eta^{3}, \eta^{4} \\
d \eta^{2} \equiv \eta^{3} \wedge \eta^{4} & \bmod \eta^{1}, \eta^{2}, & d \eta^{4} \equiv \eta^{1} \wedge \eta^{2} & \bmod \eta^{3}, \eta^{4}
\end{array}
$$

Recall now that, in Section 1.3, we showed that there exist (unique) functions $h_{20}$ and $h_{02}$ on $U^{(1)}$ so that the ideal $\mathcal{I}^{(1)}$ on $U^{(1)}$ was generated by $\left\{\eta^{2}-h_{20} \eta^{1}, \eta^{4}-\right.$ $\left.h_{02} \eta^{3}\right\}$. Conforming to the notation established in that section, let us set

$$
\omega_{10}=\eta^{1}, \quad \theta_{10}=\eta^{2}-h_{20} \eta^{1}, \quad \omega_{01}=\eta^{3}, \quad \theta_{01}=\eta^{4}-h_{02} \eta^{3}
$$

Of course, this immediately gives equations of the form

$$
\begin{align*}
d \omega_{10} & \equiv 0 \bmod \left\{\omega_{10}, \theta_{10}\right\}  \tag{1a}\\
d \omega_{01} & \equiv 0 \bmod \left\{\omega_{01}, \theta_{01}\right\}
\end{align*}
$$

Moreover, there clearly exist 1 -forms $\pi_{20}$ and $\pi_{02}$ satisfying

$$
\left.\begin{array}{l}
\pi_{20} \equiv d h_{20} \\
\pi_{02} \equiv d h_{02}
\end{array}\right\} \bmod \omega_{10}, \theta_{10}, \omega_{01}, \theta_{01}
$$

so that

$$
\begin{aligned}
& d \theta_{10} \equiv-\pi_{20} \wedge \omega_{10}+\omega_{01} \wedge \theta_{01} \bmod \theta_{10} \\
& d \theta_{01} \equiv-\pi_{02} \wedge \omega_{01}+\omega_{10} \wedge \theta_{10} \bmod \theta_{01}
\end{aligned}
$$

Note that these equations determine $\pi_{20}$ uniquely modulo $\left\{\omega_{10}, \theta_{10}\right\}$ and $\pi_{02}$ uniquely modulo $\left\{\omega_{01}, \theta_{01}\right\}$. Let us fix choices for these two 1 -forms. Now, again by the argument in Section 1.3, we know that there exist unique functions $h_{30}$ and $h_{03}$ on $U^{(2)}$ so that $\mathcal{I}^{(2)}$ on $U^{(2)}$ is generated by $\left\{\theta_{10}, \theta_{01}, \theta_{20}, \theta_{02}\right\}$ where

$$
\theta_{20}=\pi_{20}-h_{30} \omega_{10}, \quad \theta_{02}=\pi_{02}-h_{03} \omega_{01}
$$

Moreover, there are structure equations of the form

$$
\begin{aligned}
& d \theta_{20} \equiv-\pi_{30} \wedge \omega_{10}+T_{20} \omega_{01} \wedge \theta_{01} \bmod \left\{\theta_{10}, \theta_{20}\right\} \\
& d \theta_{02} \equiv-\pi_{03} \wedge \omega_{01}+T_{02} \omega_{10} \wedge \theta_{01} \bmod \left\{\theta_{01}, \theta_{02}\right\}
\end{aligned}
$$

[^1]for some 1-forms $\pi_{30}$ and $\pi_{30}$, congruent to $d h_{30}$ and $d h_{03}$, respectively, modulo terms semi-basic for $U^{(1)}$. Using $\theta_{20}-T_{20} \theta_{10}$ and $\theta_{02}-T_{02} \theta_{02}$, respectively, instead of our original $\theta_{20}$ and $\theta_{02}$, we can arrange, as we shall, that $T_{20}=T_{02}=0$.

Continuing on in this way, we can show that, for every $k \geq 1$, there exist functions $h_{k+1,0}$ and $h_{0, k+1}$ defined on $U^{(k)}$ which restrict to each fiber of $U^{(k)} \rightarrow$ $U^{(k-1)}$ to give a coordinate system and 1-forms $\theta_{k 0}$ and $\theta_{0 k}$ well-defined on $U^{(k)}$ with the following properties:

First, the ideal $\mathcal{I}^{(k)}$ on $U^{(k)}$ is generated by the 1 -forms $\left\{\theta_{10}, \ldots, \theta_{k 0}, \theta_{01}, \ldots\right.$, $\left.\theta_{0 k}\right\}$; second, the following structure equations hold

$$
\begin{align*}
d \theta_{k 0} & \equiv-\theta_{k+1,0} \wedge \omega_{10}+T_{k 0} \omega_{01} \wedge \theta_{01} \bmod \left\{\theta_{10}, \ldots, \theta_{k 0}\right\} \\
d \theta_{0 k} & \equiv-\theta_{0, k+1} \wedge \omega_{01}+T_{0 k} \omega_{10} \wedge \theta_{10} \bmod \left\{\theta_{01}, \ldots, \theta_{0 k}\right\} \tag{1b}
\end{align*}
$$

with $T_{10}=T_{01}=1$ and $T_{k 0}=T_{0 k}=0$ for $k>1$; and, third, for all $k>1$,

$$
\left.\begin{array}{l}
\theta_{k 0} \equiv d h_{k 0} \\
\theta_{0 k} \equiv d h_{0 k}
\end{array}\right\} \text { mod terms semi-basic to } U^{(k-1)}
$$

Note that any 1-form well-defined on $U^{(k)}$ is a linear combination of the forms

$$
\left\{\omega_{10}, \theta_{10}, \ldots, \theta_{k+1,0}, \omega_{01}, \theta_{01}, \ldots, \theta_{0, k+1}\right\}
$$

as these are a basis for the 1 -forms semi-basic to $U^{(k)}$.
In this section, we shall use these equations to deduce general properties about the space $\mathcal{C}\left(U^{(\infty)}, \mathcal{I}\right)$, which we will henceforth denote simply by $\mathcal{C}$ when there is no danger of confusion. We begin with the following

Proposition: The space $\mathcal{C}$ is isomorphic to the space of closed 2-forms on $U^{(\infty)}$ of the form

$$
\begin{equation*}
\Phi=A_{10} \omega_{10} \wedge \theta_{10}+A_{01} \omega_{01} \wedge \theta_{01}+\mathbf{B}_{10}+\mathbf{C}+\mathbf{B}_{01} \tag{2}
\end{equation*}
$$

where

$$
\mathbf{B}_{10}=\sum_{i<j} B_{10}^{i j} \theta_{i 0} \wedge \theta_{j 0}, \quad \mathbf{C}=\sum_{i, j} C^{i j} \theta_{i 0 \wedge} \theta_{0 j}, \quad \mathbf{B}_{01}=\sum_{i<j} B_{01}^{i j} \theta_{0 i} \wedge \theta_{0 j}
$$

Proof: The proof is similar to - but much simpler than - the analogous Proposition in $\left[\mathrm{BG}_{2}\right]$. The relevant Spencer-type complex is

$$
\Lambda^{*}\left[\bar{\omega}_{10}, \bar{\theta}_{10}, \bar{\theta}_{20}, \ldots, \bar{\omega}_{01}, \bar{\theta}_{01}, \bar{\theta}_{02}, \ldots\right]
$$

with boundary operators

$$
\delta \bar{\omega}_{10}=\delta \bar{\omega}_{01}=0, \quad \delta \bar{\theta}_{k, 0}=-\bar{\theta}_{k+1,0} \wedge \bar{\omega}_{10}, \quad \delta \bar{\theta}_{0, k}=-\bar{\theta}_{0, k+1} \wedge \bar{\omega}_{01}
$$

A simple computation shows that $H^{2}$ of this complex has dimension 2 with basis

$$
\bar{\Omega}_{10}=\bar{\omega}_{10} \wedge \bar{\theta}_{10}, \quad \bar{\Omega}_{01}=\bar{\omega}_{01} \wedge \bar{\theta}_{01}
$$

Now, from the general theory, $\mathcal{C}=\bar{H}^{1}$ is isomorphic to

$$
\operatorname{ker} d_{1}: E_{1}^{1,1} \rightarrow E_{1}^{2,1}
$$

where $\left(E_{r}^{p, q}, d_{r}\right)$ is the spectral sequence of the filtered complex $F^{p} \Omega^{*}\left(M^{(\infty)}\right)$. Moreover, again from the general theory $E_{1}^{1,0}=0$ and $E_{1}^{1,1}$ is itself the abutment of the weight spectral sequence $\bar{E}_{r}^{k, q}$ whose $\bar{E}_{1}$-term is given by sections of a vector bundle whose fibers are cohomology groups of the above complex. Unwinding the definitions leads to (2).

Note that, regarding the general description of normal forms we gave earlier, the above proposition serves to identify the vector bundle $E_{1}$ over $U^{(\infty)}$ as

$$
E_{1}=I \oplus \Lambda^{2}\left(I^{(\infty)}\right)
$$

where $I \subset \Lambda^{2}\left(T^{*} U\right)$ is the rank 2 subbundle whose sections are the 2 -forms in $\mathcal{I}$ and $I^{(\infty)} \subset T^{*} U^{(\infty)}$ is the subbundle (of infinite rank) whose sections generate $\mathcal{I}^{(\infty)}$. Thus the above proposition is just the statement that $\mathcal{C}$ consists of the closed 2 -forms in $\mathcal{E}_{i}\left(U^{(\infty)}\right)$.

We shall refer to the expression (2) as the normal form for a conservation law. Recalling that $\mathcal{C}$ was originally defined as a cohomology group (and hence as a quotient space), the import of the above proposition is that each cohomology class has a unique representative 2 -form in the above normal form. In fact, without any danger of confusion, we may clearly identify $\mathcal{C}$ with the space of closed 2 -forms in normal form and we shall do this from now on. We shall write $\Phi \in \mathcal{C}$ as

$$
\Phi=A_{10} \omega_{10} \wedge \theta_{10}+A_{01} \omega_{01} \wedge \theta_{01}+\mathbf{B}_{10}+\mathbf{C}+\mathbf{B}_{01}
$$

and shall refer to the sum

$$
\mathbf{Q}=\mathbf{B}_{10}+\mathbf{C}+\mathbf{B}_{01}
$$

as the quadratic terms in $\Phi$ (quadratic refers to quadratic in the ideal $\mathcal{I}^{(\infty)}$ ). It is not difficult to see that the condition

$$
d \Phi=0
$$

implies, first of all, that the coefficients $B_{10}^{i j}, C^{i j}, B_{01}^{i j}$ are expressible as certain linear differential expressions in $A_{10}$ and $A_{01}$ and, secondly, that $d \Phi=0$ then becomes
a linear system of equations on $A_{10}$ and $A_{01}$. This is, in fact, an overdetermined linear PDE system, and our goal is to relate the analysis of this system to the invariants of the hyperbolic system $\mathcal{I}$.

In order to begin to carry out this program, we shall show that the quadratic terms of a closed 2 -form in the normal form (2) satisfy further restrictions in form. To explain these restrictions, we need to define a certain filtration on the group $\mathcal{C}$.

Definition: A conservation law $\Phi \in \mathcal{C}$ has level $k$ if its normal representative is well-defined as a 2 -form on $U^{(k)}$. We denote by $\mathcal{C}_{k}$ the vector space of conservation laws of level $k$.

Since a 2 -form $\Phi$ in normal form must be closed to represent a conservation law, we see that $\mathcal{C}_{k}$ consists of the 2 -forms $\Phi \in \mathcal{C}$ which are quadratic expressions in

$$
\omega_{10}, \theta_{10}, \ldots, \theta_{k+1,0} ; \omega_{01}, \theta_{01}, \ldots, \theta_{0, k+1}
$$

It is clear that $\mathcal{C}_{k} \subseteq \mathcal{C}_{k+1}$ and that

$$
\bigcup_{k \geq 0} \mathcal{C}_{k}=\mathcal{C}
$$

A somewhat subtle point is that a conservation law $\Phi$ may have level exactly equal to $k$ and yet have another representative 2-form - albeit not in normal form which is defined on $U^{(l)}$ for some $l \leq k .^{21}$ We shall return to this point later on.

Here we want to indicate the proofs of three basic facts:

FACT 1: We have

$$
\begin{equation*}
\mathcal{C}_{2 k}=\mathcal{C}_{2 k-1}, \quad k \geqq 1 \tag{3}
\end{equation*}
$$

That is, when we increase the level we can add new conservation laws in normal form only at odd levels.
21) For example, a hyperbolic system of class $s=0$ may become Darboux integrable exactly at level $k$ for $k=0,1,2, \ldots$ At this point the system gets (at least) "two functions of two variables worth" of new conservation laws. But these new conservation laws will not in general be in normal form, and we may have to pass to a higher level to achieve this. On the other hand, if we pick up two functions of two variables worth of new conservation laws in algebraic normal form at some level $2 k+1$, then the system is Darboux integrable at this level, and perhaps even at a lower one.

FACT 2: If $\Phi \in \mathcal{C}_{2 k-1}$ then the quadratic terms have the form

$$
\begin{align*}
\mathbf{B}_{10} & =B_{10}^{k}\left(\sum_{j=0}^{2 k-1}(-1)^{j} \theta_{2 k-j, 0} \wedge \theta_{j+1,0}\right)+\sum_{\substack{j+i \leq 2 k \\
i<j}} B_{10}^{i j} \theta_{i 0} \wedge \theta_{j 0} \\
\mathbf{C} & =0  \tag{4}\\
\mathbf{B}_{01} & =B_{01}^{k}\left(\sum_{j=0}^{2 k-1}(-1)^{j} \theta_{0,2 k-j} \wedge \theta_{0, j+1}\right)+\sum_{\left\{\begin{array}{c}
i+j \leq 2 k \\
i<j \\
\hline
\end{array}\right.} B_{01}^{i j} \theta_{0 i} \wedge \theta_{0 j}
\end{align*}
$$

Since $\mathbf{C}=0$, the quadratic terms in the conservation law are "unmixed", i.e., if we think of the $\theta_{k 0}$ and $\theta_{0 k}$ as "belonging" to the two characteristic systems, then no cross-terms $\theta_{k 0} \wedge \theta_{0 l}$ occur in $\Phi$. This vanishing is a reflection of the fact that in the structure equations the coupling between the two characteristic systems occurs only at the lowest level and only in the form of the terms $\omega_{10} \wedge \theta_{10}$ and $\omega_{01} \wedge \theta_{01}$.

For a conservation law $\Phi$ of level $k$, we shall refer to the coefficients $B_{10}^{k}$ and $B_{01}^{k}$ as the highest order terms in $\mathbf{Q}$. It follows from the form of $\mathbf{B}_{10}$ and $\mathbf{B}_{01}$ that if we have two conservation laws of level $2 k-1$, say, $\Phi$ and $\bar{\Phi}$, such that (in the obvious notation)

$$
B_{10}^{k}=\bar{B}_{10}^{k}, \quad B_{01}^{k}=\bar{B}_{01}^{k}
$$

then $\Phi-\bar{\Phi}$ is a conservation law of level 0 when $k=1$ and is of level at most $2 k-3$ when $k \geq 2$.

Before stating our final property, we recall that the characteristic systems are given by the formulae

$$
\begin{aligned}
& \Xi_{10}^{(\infty)}=\left\{\omega_{10}, \theta_{10}, \theta_{20}, \ldots\right\} \\
& \Xi_{01}^{(\infty)}=\left\{\omega_{01}, \theta_{01}, \theta_{02}, \ldots\right\}
\end{aligned}
$$

FACT 3: For two conservation laws $\Phi$ and $\bar{\Phi} \in \mathcal{C}_{2 k-1}$, the ratios $\left[B_{10}^{k}: \bar{B}_{10}^{k}\right]=$ $B_{10}^{k} / \bar{B}_{10}^{k}$ and $\left[B_{01}^{k}: \bar{B}_{01}^{k}\right]=B_{01}^{k} / \bar{B}_{01}^{k}$ are functions that satisfy

$$
\begin{align*}
& d\left[B_{10}^{k}: \bar{B}_{10}^{k}\right] \in \Xi_{10}^{(k)} \\
& d\left[B_{01}^{k}: \bar{B}_{01}^{k}\right] \in \Xi_{01}^{(k)} \tag{5}
\end{align*}
$$

To see how useful these facts are, let us note that they imply the following result:

Corollary: If neither of the characteristic systems has an integrable subsystem, then for $k \geqq 1$,

$$
\operatorname{dim} \mathcal{C}_{2 k-1} / \mathcal{C}_{2 k-2} \leqq 2
$$

Discussion: Fact 3 clearly links the issue of adding new conservation laws at a certain (odd) level to integrable subsystems of the characteristic systems. This corollary should be contrasted with the discussion of Darboux's method in Section 1.4, which depended on there being integrable subsystems in the characteristic systems. In fact, if there is a rank 2 integrable subsystem in one of the characteristic systems, then we always have an infinite dimensional space of conservation laws at some finite level. Actually, this and other evidence suggests a sort of converse to this result might be true. Roughly stated, this converse should be: "If the new conservation laws that appear at a certain level "depend on" at least two arbitrary functions of two variables, then the system is integrable by the method of Darboux at this level."

Before turning to the proof of these results, we would like to make some general observations on the equations

$$
d\left(A_{10} \omega_{10} \wedge \theta_{10}+A_{01} \omega_{01} \wedge \theta_{01}+\mathbf{B}_{10}+\mathbf{B}_{01}\right)=0
$$

for a conservation law of level $k$. Denote by $\Xi_{10}^{(k) \perp}$ and $\Xi_{01}^{(k) \perp}$ the distributions dual to the characteristic systems (recall that $\operatorname{dim} M^{(k)}=2 k+4$ and each of these distributions has dimension $k+2$ ), and denote by $\nabla_{10}$ and $\nabla_{01}$ the operation of total differentiation relative to these distributions. Then a consequence of $d \Phi=0$ will be equations

$$
\begin{aligned}
& B_{10}^{i j}=\text { linear function of } A_{10}, A_{01} \text { and their derivatives } \\
& B_{01}^{i j}=\text { linear function of } A_{10}, A_{01} \text { and their derivatives }
\end{aligned}
$$

and

$$
\left\{\begin{array}{l}
\nabla_{01} A_{10}=L_{10}\left(A_{10}, A_{01}, B_{10}^{i j}, B_{01}^{i j}\right)  \tag{6}\\
\nabla_{10} A_{01}=L_{01}\left(A_{10}, A_{01}, B_{10}^{i j}, B_{01}^{i j}\right)
\end{array}\right.
$$

where $L_{10}$ and $L_{01}$ are linear in the $A$ 's and $B$ 's.
We will also find linear equations

$$
\left\{\begin{array}{l}
\nabla_{01} B_{10}^{k}=K_{10} \cdot B_{10}^{k}  \tag{7}\\
\nabla_{10} B_{01}^{k}=K_{01} \cdot B_{01}^{k}
\end{array}\right.
$$

for the highest order terms; these equations obviously imply (5). Of course, the relation $d \Phi=0$ will contain much more information than (6) and (7); we simply wanted to point out once again the role of the characteristic systems.

These considerations - especially equation (6) - perhaps help to explain the following consequence of the corollary to Fact 3:

Suppose that the characteristic distributions $\Xi_{10}^{(k) \perp}$ and $\Xi_{01}^{(k) \perp}$ are bracket generating. Then $\mathcal{C}_{k} / \mathcal{C}_{0}$ is finite dimensional.

We will only prove (3) and (4) in the cases of levels $k=1,2,3$. In this way we believe the general pattern should be clear and the argument more easily understood than with a notationally complicated inductive procedure.

We will, however, first list some general principles that would apply to make the calculations at any level. We will use the notation $\Omega^{(\infty)}$ for the differential forms on $M^{(\infty)}$ and $F^{p}$ for the image

$$
\Lambda^{p} \mathcal{I}^{(\infty)} \otimes \Omega^{(\infty)} \rightarrow \Omega^{(\infty)}
$$

The calculations will use only the structure equations (1). For a 2 -form in normal form giving a conservation law we have

$$
\Phi \equiv A_{10} \omega_{10} \wedge \theta_{10}+A_{01} \omega_{01} \wedge \theta_{01} \quad \bmod F^{2}
$$

and

$$
d \Phi \equiv 0 \quad \bmod F^{2}
$$

Thus $d \Phi \in F^{2}$ and the calculation will only use the assumption that

$$
d \Phi \equiv 0 \quad \bmod F^{3}
$$

which appears to be weaker than assuming that $d \Phi=0$. However, because the ideal $I^{(\infty)}$ contains no integrable subsystems, it is easy to see that any closed 3 -form in $\Lambda^{3}\left(\mathcal{I}^{(\infty)}\right)$ must be identically zero anyway.

For the general principles, we will use the notation

$$
\begin{aligned}
\Theta_{k 0} & =\theta_{k 0} \wedge \ldots \wedge \theta_{10} \\
\Theta_{0 k} & =\theta_{0 k \wedge} \wedge \ldots \wedge \theta_{01} .
\end{aligned}
$$

Then we have from (1)

$$
\begin{align*}
& d\left(\omega_{10} \wedge \theta_{10}\right) \wedge \Lambda^{2}\left[\theta_{10}, \omega_{10}, \theta_{01}, \omega_{01}\right]=0 \\
& d\left(\omega_{01} \wedge \theta_{01}\right) \wedge \Lambda^{2}\left[\theta_{10}, \omega_{10}, \theta_{01}, \omega_{01}\right]=0 \tag{i}
\end{align*}
$$

We note that (i) implies

$$
\begin{equation*}
d\left(A_{10} \omega_{10} \wedge \theta_{10}+A_{01} \omega_{01} \wedge \theta_{01}\right) \wedge \Lambda^{3}\left[\theta_{10}, \omega_{10}, \theta_{01}, \omega_{01}\right]=0 \tag{ii}
\end{equation*}
$$

Next, for any function $F$ and 1-forms $\alpha$ and $\beta$, we have the identity

$$
\begin{equation*}
d(F \alpha \wedge \beta) \wedge \alpha \wedge \beta=0 \tag{iii}
\end{equation*}
$$

Finally, again from (i)

$$
\begin{align*}
& d \theta_{k 0} \wedge \Theta_{k 0} \wedge \omega_{10} \wedge \omega_{01}=d \theta_{k 0} \wedge \Theta_{k 0 \wedge} \wedge \omega_{10} \wedge \theta_{01}=0 \\
& d \theta_{0 k} \wedge \Theta_{0 k} \wedge \omega_{01} \wedge \omega_{10}=d \theta_{0 k} \wedge \Theta_{0 k} \wedge \omega_{01} \wedge \theta_{10}=0 \tag{iv}
\end{align*}
$$

Level $k=1$ : A priori we have

$$
\begin{aligned}
\Phi=A_{10} \omega_{10} \wedge \theta_{10} & +A_{01} \omega_{01} \wedge \theta_{01}+B_{10} \theta_{20} \wedge \theta_{10}+B_{01} \theta_{01} \wedge \theta_{02} \\
& +C_{22} \theta_{20} \wedge \theta_{02}+C_{21} \theta_{20} \wedge \theta_{01}+C_{12} \theta_{10} \wedge \theta_{02}+C_{11} \theta_{10} \wedge \theta_{01}
\end{aligned}
$$

and we want to show that all $C_{i j}=0$. By (1), (ii), (iii), (iv)

$$
0=d \Phi \wedge\left(\Theta_{20} \wedge \theta_{01} \wedge \omega_{01}\right)=-C_{22} \theta_{30} \wedge \omega_{10} \wedge\left(\Theta_{20} \wedge \theta_{01} \wedge \omega_{01}\right)
$$

which gives $C_{22}=0$. Once $C_{22}=0$ we have

$$
0=d \Phi \wedge\left(\Theta_{20} \wedge \theta_{02} \wedge \theta_{03} \wedge \omega_{01}\right)=-C_{21} \theta_{30} \wedge \omega_{10} \wedge \theta_{01} \wedge\left(\Theta_{20} \wedge \theta_{02} \wedge \theta_{03} \wedge \omega_{01}\right)
$$

which gives $C_{21}=0$, and then by symmetry $C_{12}=0$. Finally

$$
0=d \Phi \wedge\left(\theta_{10} \wedge \theta_{02} \wedge \omega_{10}\right)=-C_{11} \theta_{20 \wedge \omega_{10} \wedge \theta_{01} \wedge\left(\theta_{10} \wedge \theta_{02} \wedge \omega_{10}\right)}
$$

which gives $C_{11}=0$.
Level $k=2$ : We are given a closed 2-form

$$
\begin{equation*}
\Phi=\mathbf{A}+\mathbf{B}_{10}+\mathbf{C}+\mathbf{B}_{01} \tag{8}
\end{equation*}
$$

where

$$
\mathbf{A}=A_{10} \omega_{10} \wedge \theta_{10}+A_{01} \omega_{01} \wedge \theta_{01}
$$

and

$$
\begin{aligned}
\mathbf{B}_{10} & =\sum_{j<i \leq 3} B_{10}^{i j} \theta_{i 0 \wedge} \theta_{j 0} \\
\mathbf{C} & =\sum_{i, j \leq 3} C^{i j} \theta_{i 0 \wedge} \theta_{0 j} \\
\mathbf{B}_{01} & =\sum_{i<j \leq 3} B_{01}^{i j} \theta_{0 i} \wedge \theta_{0 j}
\end{aligned}
$$

and we want to show that $\Phi$ must already of level 1, i.e. all terms with a $\theta_{30}$ or $\theta_{03}$ should drop out. By (1) and our general principles

$$
0=d \Phi \wedge\left(\theta_{30} \wedge \theta_{10} \wedge \Theta_{03} \wedge \omega_{01}\right)=-B_{10}^{32} \theta_{40 \wedge} \omega_{10} \wedge \theta_{20} \wedge\left(\theta_{30} \wedge \theta_{10} \wedge \Theta_{03} \wedge \omega_{01}\right)
$$

so that $B_{10}^{32}=0$. Next

$$
0=d \Phi \wedge\left(\theta_{30} \wedge \theta_{20 \wedge} \Theta_{03} \wedge \omega_{01}\right)=-B_{10}^{31} \theta_{40 \wedge} \omega_{10} \wedge \theta_{10} \wedge\left(\theta_{30} \wedge \theta_{20 \wedge} \Theta_{03} \wedge \omega_{01}\right)
$$

gives $B_{10}^{31}=0$. By symmetry $B_{01}^{32}=B_{01}^{31}=0$. Turning to the cross-terms we have successively

$$
\begin{aligned}
& 0=d \Phi \wedge\left(\Theta_{30} \wedge \Theta_{02} \wedge \omega_{01}\right)=-C^{33} \theta_{40 \wedge \omega_{10} \wedge \theta_{03} \wedge\left(\Theta_{30} \wedge \Theta_{02} \wedge \omega_{01}\right)}^{0=d \Phi \wedge\left(\Theta_{30} \wedge \theta_{01} \wedge \theta_{03} \wedge \omega_{01}\right)=-C^{32} \theta_{40 \wedge \omega_{10} \wedge \theta_{02} \wedge}\left(\Theta_{30} \wedge \theta_{01} \wedge \theta_{03} \wedge \omega_{01}\right)} \\
& 0=d \Phi \wedge\left(\Theta_{30} \wedge \theta_{02} \wedge \theta_{03} \wedge \omega_{01}\right)=-C^{31} \theta_{40} \wedge \omega_{10} \wedge \theta_{01} \wedge\left(\Theta_{30} \wedge \theta_{02} \wedge \theta_{03} \wedge \omega_{01}\right)
\end{aligned}
$$

which gives $C^{33}=C^{32}=C^{31}=0$, and by symmetry all $C^{i j}=0$.
Level $k=3$ : We again have a closed 2-form (8) where

$$
\begin{aligned}
\mathbf{B}_{10} & =\sum_{j<i \leq 4} B_{10}^{i j} \theta_{i 0} \wedge \theta_{j 0} \\
\mathbf{C} & =\sum_{i, j \leq 4} C^{i j} \theta_{i 0} \wedge \theta_{0 j} \\
\mathbf{B}_{01} & =\sum_{i<j \leq 4} B_{01}^{i j} \theta_{0 i} \wedge \theta_{0 j} .
\end{aligned}
$$

We have

$$
\begin{aligned}
& 0=d \Phi \wedge\left(\theta_{40} \wedge \theta_{20} \wedge \theta_{10} \wedge \Theta_{04} \wedge \omega_{01}\right)=-B_{10}^{43} \theta_{50} \wedge \omega_{10} \wedge \theta_{30} \wedge\left(\theta_{40 \wedge} \wedge \theta_{20} \wedge \theta_{10} \wedge \Theta_{04} \wedge \omega_{01}\right) \\
& 0=d \Phi \wedge\left(\theta_{40} \wedge \theta_{30} \wedge \theta_{10} \wedge \Theta_{04} \wedge \omega_{01}\right)=-B_{10}^{42} \theta_{50} \wedge \omega_{10} \wedge \theta_{20 \wedge}\left(\theta_{40} \wedge \theta_{30} \wedge \theta_{10} \wedge \Theta_{04} \wedge \omega_{01}\right)
\end{aligned}
$$

which gives $B^{43}=B^{42}=0$. More interestingly

$$
0=d \Phi \wedge\left(\theta_{30} \wedge \theta_{10} \wedge \Theta_{04} \wedge \omega_{01}\right)=\left(B_{10}^{41}+B_{10}^{32}\right) \theta_{40 \wedge} \theta_{20 \wedge \omega_{10} \wedge}\left(\theta_{30 \wedge} \theta_{10} \wedge \Theta_{04} \wedge \omega_{01}\right)
$$

which gives

$$
\mathbf{B}_{10}=B_{10}^{41}\left(\theta_{40} \wedge \theta_{10}-\theta_{30} \wedge \theta_{20}\right)+(\text { lower order terms })
$$

and similarly for $\mathbf{B}_{01}$.
An argument similar to those for levels $k=1,2$ shows that the cross terms $\mathbf{C}$ are identically zero.

This completes our discussions of (3) and (4), and we now turn to (5) which again we shall do only for levels 1 and 3 . We consider a conservation law

$$
\Phi=A_{10} \omega_{10} \wedge \theta_{10}+A_{01} \omega_{01} \wedge \theta_{01}+B_{10} \theta_{20 \wedge} \theta_{10}+B_{01} \theta_{01} \wedge \theta_{02}
$$

of level 1 . We will derive a relation for $d B_{10}$, and for the calculation it is notationally convenient to set $\theta_{00}=\omega_{01}$.

From the structure equations we will have a relation

$$
0=d \Phi \wedge \omega_{10}=\left(d B_{10}+\chi B_{10}\right) \wedge \theta_{20} \wedge \theta_{10} \wedge \omega_{10}+\sum_{\nu \geq 0} \rho^{\nu} \wedge \theta_{0 \nu}+\sum_{0 \leq \mu<\nu} \rho^{\mu \nu} \wedge \theta_{0 \mu} \wedge \theta_{0 \nu}
$$

where $\chi$ does not depend on $\Phi$ and where

$$
\rho^{\nu} \equiv 0 \bmod \Lambda^{3}\left[\theta_{10}, \omega_{10}, \omega_{01}, \theta_{01}, \theta_{02}\right]
$$

In particular $\rho^{\nu}$ does not contain a $\theta_{20}$ term, and it follows that

$$
d B_{10}+\chi B_{10} \equiv 0 \bmod \Xi_{10}^{(1)}
$$

If $\bar{\Phi}$ is another conservation law, then this equation implies that

$$
\bar{B}_{10} d B_{10}-B_{10} d \bar{B}_{10} \equiv 0 \bmod \Xi_{10}^{(1)}
$$

which was to be proved.
Turning now to a level 3 conservation law, which we now write as

$$
\Phi=\Phi_{10}+\Phi_{01}
$$

where

$$
\begin{aligned}
& \Phi_{10}=A_{10} \omega_{10} \wedge \theta_{10}+B_{10}^{21} \theta_{20} \wedge \theta_{10}+B_{10}^{31} \theta_{30} \wedge \theta_{10}+B_{10}\left(\theta_{40} \wedge \theta_{10}-\theta_{30} \wedge \theta_{20}\right) \\
& \Phi_{01}=A_{01} \omega_{01} \wedge \theta_{01}+B_{01}^{12} \theta_{01} \wedge \theta_{02}+B_{01}^{13} \theta_{01} \wedge \theta_{03}+B_{01}\left(\theta_{01} \wedge \theta_{04}-\theta_{02} \wedge \theta_{03}\right)
\end{aligned}
$$

we have as before

$$
\begin{aligned}
0 & =d \Phi \wedge \theta_{30} \wedge \theta_{20} \wedge \omega_{10} \\
& =\left(d B_{10}+\chi B_{10}\right) \wedge \theta_{40 \wedge} \theta_{30} \wedge \theta_{20} \wedge \theta_{10} \wedge \omega_{10}+\sum_{0 \leqq \nu} \rho^{\nu} \wedge \theta_{0 \nu}+\sum_{0 \leqq \mu<\nu} \rho^{\mu \nu} \wedge \theta_{0 \mu} \wedge \theta_{0 \nu}
\end{aligned}
$$

Inspection of the structure equations shows that

$$
\rho^{\nu} \equiv 0 \bmod \Lambda^{5}\left[\theta_{30}, \theta_{20}, \theta_{10}, \omega_{01}, \omega_{01}, \theta_{01}, \theta_{02}, \ldots\right]
$$

i.e., $\rho^{\nu}$ does not contain a $\theta_{40}$-term. It follows that

$$
d B_{10}+\chi B_{10} \equiv 0 \bmod \Xi_{10}^{(3)}
$$

and the argument proceeds as before.
2.1.3 An example of higher level conservation laws. In the remaining sections of this paper we shall be concerned with level 0 or, as we shall call them, "classical" conservation laws. Before going on, however, we would like to conclude this section with a subsection in which we give one calculation of higher level conservation laws in an interesting example.

ExAmple: We consider the EDS associated to the (slightly modified) $s=0$ sineGordon system

$$
\begin{aligned}
u_{y} & =\sin v \\
v_{x} & =\sin u
\end{aligned}
$$

As coframing we take

$$
\omega^{1}=d x, \quad \omega^{2}=d u-\sin v d y, \quad \omega^{3}=d y, \quad \omega^{4}=d v-\sin u d x
$$

with structure equations

$$
d \omega^{1}=d \omega^{3}=0
$$

and

$$
\begin{aligned}
d \omega^{2} & =\cos v\left(-\sin u \omega^{1} \wedge \omega^{3}+\omega^{3} \wedge \omega^{4}\right) \\
d \omega^{4} & =\cos u\left(-\sin v \omega^{3} \wedge \omega^{1}+\omega^{1} \wedge \omega^{2}\right)
\end{aligned}
$$

We will show that for this system

$$
\operatorname{dim} \mathcal{C}_{0}=1 \quad \text { and } \quad \operatorname{dim} \mathcal{C}_{1} / \mathcal{C}_{0}=3
$$

Remark: For the $s=0$ sine-Gordon system we will see that both $\Xi_{10}^{(1)}$ and $\Xi_{01}^{(1)}$ have rank one, spanned by $d x$ and $d y$ respectively. Thus, the corollary following Fact 3 does not apply.

First, we compute the level 0 conservation laws. For $\Phi \in \mathcal{\mathcal { C } _ { 0 }}$, we have

$$
\Phi=A \omega^{1} \wedge \omega^{2}+B \omega^{3} \wedge \omega^{4}
$$

the condition $d \Phi=0$ gives

$$
\begin{array}{ll}
A_{4}=0 & B_{2}=0 \\
A_{3}=B \cos u & B_{1}=A \cos v \tag{9}
\end{array}
$$

where we have set $d f=f_{i} \omega^{i}$ for any function $f$. Explicitly, in terms of the classical partials notation,

$$
\begin{array}{ll}
f_{2}=f_{u} & f_{4}=f_{v} \\
f_{1}=f_{v} \sin u+f_{x} & f_{3}=f_{u} \sin v+f_{y} .
\end{array}
$$

Thus, (9) expands to

$$
\begin{aligned}
A_{v} & =0 & B_{u} & =0 \\
A_{u} \sin v+A_{y} & =B \cos u & B_{v} \sin u+B_{x} & =A \cos v
\end{aligned}
$$

The top two equations imply that $A$ does not depend on $v$ and that $B$ does not depend on $u$. Freezing the value of $u$ in the lower left hand equation at some value $u_{0}$ for which $\cos u_{0}$ is non-zero then shows that $B$ must be of the form $B=b_{0}+b_{1} \sin v$ for some functions $b_{0}$ and $b_{1}$ of $x$ and $y$. Similarly, from the lower right hand equation, we see that $A=a_{0}+a_{1} \sin u$ for some functions $a_{0}$ and $a_{1}$ of $x$ and $y$. Substituting these expressions into the lower two equations yields

$$
\begin{aligned}
& a_{1} \cos u \sin v+\left(a_{0}\right)_{y}+\left(a_{1}\right)_{y} \sin u=b_{0} \cos u+b_{1} \sin v \cos u \\
& b_{1} \cos v \sin u+\left(b_{0}\right)_{x}+\left(b_{1}\right)_{x} \sin v=a_{0} \cos v+a_{1} \sin u \cos v
\end{aligned}
$$

Now comparing coefficients in these equations shows that we must have $a_{0}=b_{0}=0$ while $a_{1}=b_{1}$ and $\left(a_{1}\right)_{y}=\left(b_{1}\right)_{x}=0$. These latter equations clearly imply that $a_{1}=b_{1}=c$ for some constant $c$. Thus, $\operatorname{dim} \mathcal{C}_{0}=1$ and the space of classical (i.e., level 0 ) conservation laws of the $s=0$ sine-Gordon system is spanned by

$$
\Phi=\sin u d x \wedge d u+\sin v d y \wedge d v
$$

The corresponding undifferentiated conservation law has the form

$$
\varphi=\cos u d x+\cos v d y
$$

We now turn to the analysis of level 1 conservation laws. On $M^{(1)}=M \times \mathbb{R}^{2}$, where $\mathbb{R}^{2}$ has coordinates ( $h_{20}, h_{02}$ ), we take as coframing

$$
\begin{array}{ll}
\omega_{10}=\omega^{1} & \theta_{10}=\omega^{2}-h_{20} \omega^{1} \\
\omega_{01}=\omega^{3} & \theta_{01}=\omega^{4}-h_{02} \omega^{3}
\end{array}
$$

and

$$
\begin{aligned}
& \pi_{20}=d h_{20}-\cos v \sin u \omega_{01} \\
& \pi_{02}=d h_{02}-\cos u \sin v \omega_{10}
\end{aligned}
$$

With this coframing, we have

$$
\begin{aligned}
& d \theta_{10}=-\pi_{20} \wedge \omega_{10}-\cos v \theta_{01} \wedge \omega_{01} \\
& d \theta_{01}=-\pi_{02} \wedge \omega_{01}-\cos u \theta_{10} \wedge \omega_{10}
\end{aligned}
$$

conforming to the general form of the structure equations as discussed above.
According to Fact 2 of the general theory, any $\Phi \in \mathcal{C}_{1}$ has the general form

$$
\Phi=A^{\prime} \theta_{10} \wedge \omega_{10}+B^{\prime} \theta_{01} \wedge \omega_{01}+C \theta_{10} \wedge \theta_{20}+D \theta_{01} \wedge \theta_{02}
$$

By modifying $A^{\prime}$ and $B^{\prime}$ in the obvious way, we may rewrite $\Phi$ as

$$
\Phi=A \theta_{10} \wedge \omega_{10}+B \theta_{01} \wedge \omega_{01}+C \theta_{10} \wedge \pi_{20}+D \theta_{01} \wedge \pi_{02}
$$

For notational convenience, we will rename the 1 -forms in the coframing of $M^{(1)}$ as follows:

$$
\begin{array}{lll}
\psi_{1}=\omega_{10} & \psi_{3}=\theta_{10} & \psi_{5}=\pi_{20} \\
\psi_{2}=\omega_{01} & \psi_{4}=\theta_{01} & \psi_{6}=\pi_{02}
\end{array}
$$

Thus, $\Xi_{10}^{(1)}=\left\{\psi_{1}, \psi_{3}, \psi_{5}\right\}$ and $\Xi_{01}^{(1)}=\left\{\psi_{2}, \psi_{4}, \psi_{6}\right\}$. For any function $F$ on $M^{(1)}$, we define the functions $F_{i}$ by the rule

$$
d F=F_{1} \psi_{1}+F_{2} \psi_{2}+\cdots+F_{6} \psi_{6}
$$

Now, expanding out the equation $d \Phi=0$ and collecting coefficients yields the following equations

$$
C_{2}=C_{4}=C_{6}=0, \quad D_{1}=D_{3}=D_{5}=0
$$

as well as the equations

$$
\begin{aligned}
& A_{2}=B \cos u+C\left(\sin ^{2} u \sin v-h_{20} \cos v \cos u\right) \\
& A_{4}=D \sin u \sin v \\
& A_{6}=D \cos u \\
& A_{5}=C_{1} \\
& B_{1}=A \cos v+D\left(\sin ^{2} v \sin u-h_{02} \cos u \cos v\right) \\
& B_{3}=C \sin v \sin u \\
& B_{5}=C \cos v \text { and } \\
& B_{6}=D_{2} .
\end{aligned}
$$

Now the first set of equations imply that

$$
d C \in \Xi_{10}^{(1)} \quad d D \in \Xi_{01}^{(1)}
$$

However, it may be verified that the largest integrable subsystem of $\Xi_{10}^{(1)}$ (respectively, $\Xi_{01}^{(1)}$ ) is of rank one and is spanned by $\omega_{10}$ (respectively, $\omega_{01}$ ). This implies the equations

$$
C_{3}=C_{5}=D_{4}=D_{6}=0
$$

In particular, since $\omega_{10}=d x$, it follows that $C$ must be a function of $x$ alone and, for a similar reason, $D$ must be a function of $y$ alone. Of course, this implies that $C_{1}=C_{x}$ and $D_{2}=D_{y}$.

Now, from the definition of the 1 -forms $\psi_{i}$ it follows that, in the coordinate system $\left(x, y, u, v, h_{20}, h_{02}\right)$, for any function $F$ on $M^{(1)}$, the expression $F_{5}$ is the partial of $F$ with respect to $h_{20}$ and $F_{6}$ is the partial of $F$ with respect to $h_{02}$. The above equations for $A_{5}, A_{6}, B_{5}$ and $B_{6}$ then suggest that the functions $\tilde{A}$ and $\tilde{B}$ defined by

$$
\tilde{A}=A-C_{1} h_{20}-D \cos u h_{02} \quad \tilde{B}=B-C \cos v h_{20}-D_{2} h_{02}
$$

should depend only on $x, y, u$, and $v$. In fact, if we redefine our notation so as to rewrite the conservation law in the form

$$
\begin{aligned}
\Phi=\left(A+C_{x} h_{20}+D h_{02} \cos u\right) \theta_{10} \wedge \omega_{10} & +\left(B+C h_{20} \cos v+D_{y} h_{02}\right) \theta_{01} \wedge \omega_{01} \\
& +C \theta_{10 \wedge \pi_{20}+D \theta_{01} \wedge \pi_{02}}
\end{aligned}
$$

where $C$ (respectively, $D$ ) is understood to be a function only of $x$ (respectively, $y$ ), then the closure conditions for $\Phi$ simplify to

$$
A_{5}=B_{5}=A_{6}=B_{6}=0
$$

together with

$$
\begin{aligned}
& A_{2}=B \cos u+C \sin ^{2} u \sin v-C_{x} \sin u \cos v+D h_{20} \sin v \sin u \\
& B_{1}=A \cos v+D \sin ^{2} v \sin u-D_{y} \sin v \cos u-C h_{02} \cos u \cos v \\
& A_{4}=D \sin u \sin v \\
& B_{3}=C \sin u \sin v
\end{aligned}
$$

These last two equations suggest that, in fact, $A+D \sin u \cos v$ is a function of $x$, $y$, and $u$ only and that $B+C \sin v \cos u$ is a function of $x, y$, and $v$ only. Thus, if we redefine our notation once again so that

$$
\begin{aligned}
\Phi= & \left(A-D \sin u \cos v+C_{x} h_{20}+D \cos u h_{02}\right) \theta_{10} \wedge \omega_{10} \\
& +\left(B-C \sin v \cos u+C \cos v h_{20}+D_{y} h_{02}\right) \theta_{01} \wedge \omega_{01} \\
& +C \theta_{10} \wedge \pi_{20}+D \theta_{01} \wedge \pi_{02}
\end{aligned}
$$

we find that the closure conditions simplify yet again. They simply become (assuming, as usual, that $C$ is a function of $x$ alone and $D$ is a function of $y$ alone)

$$
A_{4}=A_{5}=A_{6}=0, \quad B_{3}=B_{5}=B_{6}=0
$$

plus the relations

$$
\begin{aligned}
& A_{2}=B \cos u+\left(D_{y}-C_{x}\right) \sin u \cos v+(D \cos v-C \cos u) \cos u \sin v \\
& B_{1}=A \cos v+\left(C_{x}-D_{y}\right) \sin v \cos u+(C \cos u-D \cos v) \cos v \sin u
\end{aligned}
$$

Moreover, since $A$ is a function of $x, y$, and $u$ alone while $B$ is a function of $x, y$, and $v$ alone, it easily follows from our definitions that

$$
A_{2}=A_{y}+A_{u} \sin v \quad \text { and } \quad B_{1}=B_{x}+B_{v} \sin u
$$

Let us substitute these expressions into the above equations and take note of the fact that $A$ does not depend on $v$ while $B$ does not depend on $u$. In the second equation, freezing $v$ at some value $v_{0}$ for which $\cos v_{0}$ is non-zero, we see that $A$ is a linear combination of $\{1, \cos u, \sin u, \cos u \sin u\}$ with coefficients which are functions of $x$ and $y$. In a similar manner, the first equation implies that $B$ is a linear combination of $\{1, \cos v, \sin v, \cos v \sin v\}$ with coefficients which are functions of $x$ and $y$. Thus, let us write

$$
\begin{aligned}
& A=a_{0}+a_{1} \cos u+a_{2} \sin u+a_{3} \cos u \sin u \\
& B=b_{0}+b_{1} \cos v+b_{2} \sin v+b_{3} \cos v \sin v
\end{aligned}
$$

where the $a_{i}$ and $b_{i}$ are functions of $x$ and $y$ alone. Expanding the above relations out and setting equal the various coefficients of the resulting trigonometric polynomials then yields the relation $C_{x}-D_{y}=0$, the relations

$$
a_{0}=a_{1}=b_{0}=b_{1}=0
$$

the relations $a_{3}=-C$ and $b_{3}=-D$, and finally equations which imply that $a_{2}=$ $b_{2}=c$ where $c$ is a constant. Since $C$ is a function of $x$ alone and $D$ is a function of $y$ alone, it follows that the common value $C_{x}=D_{y}$ must be a constant. Thus, we finally get

$$
\begin{aligned}
& A=c \sin u-\left(E_{0} x+C_{0}\right) \cos u \sin u \\
& B=c \sin v-\left(E_{0} y+D_{0}\right) \cos v \sin v \\
& C=E_{0} x+C_{0} \\
& D=E_{0} y+D_{0}
\end{aligned}
$$

where $c, C_{0}, D_{0}$, and $E_{0}$ are constants. Moreover, these values of $A, B, C$, and $D$ are easily seen to yield a closed 2 -form $\Phi$. Thus, we have shown that $\operatorname{dim} \mathcal{C}_{3}=4$, as we wanted.

In fact, with a little inspection, we can write

$$
\Phi=d\left(c \phi_{0}+C_{0} \phi_{1}+D_{0} \phi_{2}+E_{0} \phi_{3}\right)
$$

where the $\phi_{i}$ are the undifferentiated conservation laws and are given by formulae of the form

$$
\begin{aligned}
\phi_{0}= & \cos u d x+\cos v d y \\
\phi_{1}= & -h_{20} \theta_{10}-\frac{1}{2}\left(h_{20}^{2}+\sin ^{2} u\right) d x+\cos u \cos v d y \\
\phi_{2}= & -h_{02} \theta_{01}-\frac{1}{2}\left(h_{02}^{2}+\sin ^{2} v\right) d y+\cos u \cos v d x \\
\phi_{3}= & -x h_{20} \theta_{10}-y h_{02} \theta_{01}-\frac{1}{2}\left(x h_{20}^{2}-x \cos u-2 y \cos u \cos v\right) d x \\
& -\frac{1}{2}\left(y h_{02}^{2}-y \cos v-2 x \cos u \cos v\right) d y
\end{aligned}
$$

2.2 Determination of hyperbolic systems having the maximum number of classical conservation laws: first steps. Let $(M, \mathcal{I})$ be a hyperbolic exterior differential system of class $s=0$ and denote by $\mathcal{C}_{0}$ the space of conservation laws of level zero or, as we shall call them, classical conservation laws. Assuming symmetric behavior in the two characteristic systems in this section we will prove that:

If $\operatorname{dim} \mathcal{C}_{0} \geqq 7$, then both characteristic systems have an integrable subsystem.
Later on we will see that if $\operatorname{dim} \mathcal{C}_{0} \geqq 7$, then $\operatorname{dim} \mathcal{C}_{0}=\infty$ and there is a local normal form for such hyperbolic systems.

Our technique is to use the equivalence method to express the structure equations of $(M, \mathcal{I})$ in an invariant manner, which has been done in Section 1.5 above, and then to simply compute. The calculations are generally straightforward but a couple of them are somewhat lengthy. For these we have used MAPLE. The main issue of course is to interpret the calculations as we go along, and we shall give these steps without always reproducing the full printout of MAPLE calculations. We now proceed to the calculations.

Recall that a classical conservation law has the form

$$
\begin{equation*}
\Phi=A \Omega_{10}+C \Omega_{01} \tag{1}
\end{equation*}
$$

where

$$
\Omega_{10}=\omega^{1} \wedge \omega^{2} \quad \text { and } \quad \Omega_{01}=\omega^{3} \wedge \omega^{4}
$$

Using the structure equations (5) in Section 1.5, we easily deduce that the closure condition $d \Phi=0$ is equivalent to the following differential relations on the functions $A$ and $C$

$$
\begin{align*}
& d A=A \phi_{44}+A_{i} \omega^{i} \\
& d C=C \phi_{22}+C_{i} \omega^{i} \tag{2}
\end{align*}
$$

where

$$
\begin{cases}A_{3}=C & A_{4}=0  \tag{3}\\ C_{1}=A & C_{2}=0\end{cases}
$$

Since the 2-plane distributions defined by $\Omega_{10}$ and $\Omega_{01}$ are both non-integrable, we know that by taking commutators we may "capture" at least one more derivative of each of $A$ and $C$. In this regard the main initial step in the computation is given by the

Lemma: With the above notations

$$
A_{2}-C_{4}=C q_{3}-A p_{1} .
$$

Proof: This follows directly from the identities

$$
\begin{aligned}
& d(d A) \equiv 0 \bmod \omega^{1}, \omega^{2} \\
& d(d C) \equiv 0 \bmod \omega^{3}, \omega^{4} .
\end{aligned}
$$

Using the lemma, if we define $B$ by

$$
A_{2}+C_{4}=2 B
$$

then we have the relations

$$
\left\{\begin{array}{l}
A_{2}=B-\frac{1}{2}\left(A p_{1}-C q_{3}\right)  \tag{4}\\
C_{4}=B-\frac{1}{2}\left(C q_{3}-A p_{1}\right) .
\end{array}\right.
$$

Lemma: We have

$$
\begin{equation*}
d B=B\left(\phi_{22}+\phi_{44}\right)+\frac{1}{2}\left(C p_{4} \phi_{21}+A q_{2} \phi_{43}\right)+B_{i} \omega^{i} \tag{5}
\end{equation*}
$$

where ${ }^{22}$

$$
\begin{equation*}
B_{i} \equiv 0 \bmod A, C, B, A_{1}, C_{3} \tag{6}
\end{equation*}
$$

Proof: The identities

$$
\begin{aligned}
d(d A) & \equiv 0 \bmod \omega^{1} \\
d(d C) & \equiv 0 \bmod \omega^{3}
\end{aligned}
$$

give, using (3) and (4), relations of the form (6). The exact formula (6) will result from an explicit computation that we shall give below.
22) In this and similar situations to be encountered below, the expression $F \equiv 0 \bmod G_{1}, \ldots, G_{n}$ will mean "there exists a formula or identity expressing $F$ as a linear combination of the functions $G_{1}, \ldots, G_{n}$ where the coefficients are known combinations of coframing invariants and their derivatives". In this specific instance, these formulae for $B_{i}$ are given explicitly below. However, we will not always write out these formulae for it frequently suffices to know just that there exists such formulae in order to complete a proof. Frequently, the explicit formulae are quite ungainly.

Knowing the values of $A$ and $C$ at a point $x$ gives us $A_{3}, A_{4}, C_{1}, C_{2}$ at that point. Knowing $B(x)$ gives us $A_{2}(x)$ and $C_{4}(x)$. By the above lemma, however, we may determine all the $A_{2 i}(x)$ and $C_{4 i}(x)$ in terms of $A, C, B, A_{1}, C_{3}$ at that point. Iterating this gives us all higher derivatives at $x$ of $A_{2}, A_{3}, A_{4}$ and $C_{1}, C_{2}, C_{3}$ in terms of $A, C, B, A_{1}, C_{3}$ at the point. Thus in the formal Taylor's series of $A$ and $C$ we are only free to specify the derivatives $A_{1}$ and $C_{3}$. This suggests that

The space of classical conservation laws can at most depend on two arbitrary functions of one variable.

One way of thinking about this "function count" is this: Solutions to a determined set of two equations for two unknowns are uniquely determined by initial values along a 3 -dimensional hypersurface. Adding one more equation will in general cut the initial value manifold down to 2 dimensions, and adding a fourth equation will in general cut the initial value set down to a curve. Since (3) is a set of 4 equations for 2 unknowns we expect that the solutions will at most be given by arbitrary initial data along a curve. We say "at most" because this function count will only work if the system is involutive, otherwise there are even fewer solutions. Our main result will be to classify systems for which this upper bound is achieved.

Now, we have already developed linear expressions for

$$
A_{2}, A_{3}, A_{4} ; C_{1}, C_{2}, C_{4}
$$

in terms of $A, C, B, A_{1}, C_{3}$; now we shall turn to $A_{1}$ and $C_{3}$. From the identities $d(d A)=d(d C)=0$ we will show that:

$$
\begin{align*}
& d A_{1}=A_{1}\left(2 \phi_{44}-\phi_{22}\right)+\left(B-\frac{1}{2}\left(A p_{1}-C q_{3}\right)\right) \phi_{21}+A q_{1} \phi_{43}+A_{1 i} \omega^{i} \\
& d C_{3}=C_{3}\left(2 \phi_{22}-\phi_{44}\right)+\left(B-\frac{1}{2}\left(C q_{3}-A p_{1}\right)\right) \phi_{43}+C p_{3} \phi_{21}+C_{3 i} \omega^{i} \tag{7}
\end{align*}
$$

where

$$
\left.\begin{array}{l}
A_{12} \equiv A_{13} \equiv A_{14} \equiv 0 \\
C_{34} \equiv C_{31} \equiv C_{32} \equiv 0
\end{array}\right\} \bmod A, C, B, A_{1}, C_{3}
$$

In particular this implies

$$
\left.\begin{array}{l}
d A_{1} \equiv A_{11} \omega^{1} \\
d C_{3} \equiv C_{33} \omega^{3}
\end{array}\right\} \bmod A, C, B, A_{1}, C_{3}
$$

As expected, there is at most one free derivative of the quantities $A_{1}$ and $C_{3}$ respectively. Iterating this, in the Taylor's series of $A$ and $C$ we find that we are at most free to assign the terms $A_{1 \ldots 1}$ and $C_{3 \ldots 3}$, confirming the statement above.

Proof of (7): The identities $d(d A) \equiv 0 \bmod \omega^{1}$ and $d(d C) \equiv 0 \bmod \omega^{3}$ give

$$
\begin{aligned}
& B_{1}=-\frac{1}{2} A\left(p_{11}-q_{3}\right)+\frac{1}{2} C\left(q_{31}-k_{14}\right)-\frac{1}{2} A_{1} p_{1}+C_{3} q_{1} \\
& B_{3}=-\frac{1}{2} C\left(q_{33}-p_{1}\right)+\frac{1}{2} A\left(p_{13}-k_{32}\right)-\frac{1}{2} C_{3} q_{3}+A_{1} p_{3}
\end{aligned}
$$

and

$$
\begin{aligned}
& B_{2}=-A\left(\frac{1}{2} p_{12}-\frac{1}{4} p_{1}^{2}+p_{4}\right)+C\left(\frac{1}{2} q_{32}-k_{24}-\frac{1}{2} p_{3} q_{1}-\frac{1}{4} p_{1} q_{3}\right)-\frac{1}{2} B p_{1}+C_{3} q_{2} \\
& B_{4}=-C\left(\frac{1}{2} q_{34}-\frac{1}{4} q_{3}^{2}+q_{2}\right)+A\left(\frac{1}{2} p_{14}-k_{42}-\frac{1}{2} q_{1} p_{3}-\frac{1}{4} q_{3} p_{1}\right)-\frac{1}{2} B q_{3}+A_{1} p_{4}
\end{aligned}
$$

From

$$
0=d(d A)=d\left(A \dot{\phi}_{44}+A_{i} \omega^{i}\right)=d A_{1 \wedge} \wedge \omega^{1}+(\text { other terms })
$$

and the formulas for $A_{2}, A_{3}, A_{4}$ and $B_{i}$ which give the "(other terms)", we obtain the expression for $d A_{1}$ where $A_{12}, A_{13}, A_{14} \equiv 0 \bmod A, C, B, A_{1}, C_{3}$. The argument for $d C_{3}$ is similar.

The issue now is: What conditions are imposed on the invariants of the system in order that we may arbitrarily assign the coefficients $A_{1 \ldots 1}$ and $C_{3 \ldots 3}$ ?

To begin developing the answer to this we shall first show that

$$
\left.\begin{array}{rl}
p_{4} A_{11} & \equiv 0  \tag{8}\\
q_{2} C_{33} & \equiv 0
\end{array}\right\} \bmod A, C, B, A_{1}, C_{3}
$$

Proof: Above we have given formulae for $B_{i}$ as explicit linear combinations of $A, C, B, A_{1}, C_{3}$. In turn, except for $A_{11}$ and $C_{33}$ the exterior derivatives of $A, C$, $B, A_{1}, C_{3}$ are again explicit linear combinations of these five quantities. Thus, the condition

$$
d(d B)=0
$$

may impose conditions on $A_{11}$ and $C_{33}$. The explicit formula for $d(d B)$ is lengthy, but when we compute it on MAPLE and then set $A=C=B=A_{1}=C_{3}=0$, the expression reduces to

$$
\begin{equation*}
\left(p_{3} A_{11}-q_{1} C_{33}\right) \omega^{1} \wedge \omega^{3}+p_{4} A_{11} \omega^{1} \wedge \omega^{4}+q_{2} C_{33} \omega^{3} \wedge \omega^{2} \tag{9}
\end{equation*}
$$

The last two terms come from $A_{1} p_{4}$ in $B_{4}$ and $C_{3} q_{2}$ in $B_{2}$ respectively. It is clear that setting this expression equal to zero gives (8).

The following is an immediate consequence of (8).

Proposition: If $p_{4} q_{2} \neq 0$, then $\operatorname{dim} \mathcal{C}_{0} \leqq 5$.

Proof: If $p_{4} q_{2} \neq 0$, then we may inductively determine each term in the formal Taylor's series of $A$ and $C$ as a linear combination of $A, C, B, A_{1}, C_{3}$. Thus, the PDE system for $A$ and $C$ is at best a completely integrable system whose solutions are specified by five constants.

It follows that if $\operatorname{dim} \mathcal{C}_{0} \geqq 6$, then (keeping in mind our assumption of symmetry) we must have $p_{4}=q_{2}=0$, i.e., the rank 2 Pfaffian system $\Theta=\Xi_{10}^{\{1\rangle} \cup \Xi_{01}^{\langle 1\rangle}$ is integrable. Assuming this, it follows from (9) that

$$
p_{3} A_{11}-q_{1} C_{33} \equiv 0 \bmod A, C, B, A_{1}, C_{3}
$$

The exterior derivative of this relation gives

$$
\left.\begin{array}{rl}
p_{3} A_{111} & \equiv 0 \\
q_{1} C_{333} & \equiv 0
\end{array}\right\} \bmod A, C, B, A_{1}, C_{3}, A_{11}, C_{33}
$$

By reasoning similar to that above, this immediately implies the following result.

Proposition: If $p_{4}=q_{2}=0$ but $p_{3} q_{1} \neq 0$, then $\operatorname{dim} \mathcal{C}_{0} \leqq 6$.
It follows that if $\operatorname{dim} \mathcal{C}_{0} \geqq 7$, then we must have $p_{4}=q_{2}=0$ and $p_{3}=q_{1}=0$, i.e., the Pfaffian systems $\Xi_{10}^{\langle 1\rangle}$ and $\Xi_{01}^{\langle 1\rangle}$ are integrable.

For systems with

$$
\begin{equation*}
p_{4}=q_{2}=0 \quad \text { and } \quad p_{3}=q_{1}=0 \tag{10}
\end{equation*}
$$

the formulas for $B_{i}$ simplify to give

$$
\begin{aligned}
& B_{1}=-\frac{1}{2} A\left(p_{11}-q_{3}\right)+\frac{1}{2} C\left(q_{31}-k_{14}\right)-\frac{1}{2} A_{1} p_{1} \\
& B_{3}=-\frac{1}{2} C\left(q_{33}-p_{1}\right)+\frac{1}{2} A\left(p_{13}-k_{32}\right)-\frac{1}{2} C_{3} q_{3}
\end{aligned}
$$

and

$$
\begin{aligned}
& B_{2}=-\frac{1}{2} A\left(p_{12}-\frac{1}{2} p_{1}^{2}\right)+C\left(\frac{1}{2} q_{32}-k_{24}-\frac{1}{4} p_{1} q_{3}\right)-\frac{1}{2} B p_{1} \\
& B_{4}=-\frac{1}{2} C\left(q_{34}-\frac{1}{2} q_{3}^{2}\right)+A\left(\frac{1}{2} p_{14}-k_{42}-\frac{1}{4} q_{3} p_{1}\right)-\frac{1}{2} B q_{3}
\end{aligned}
$$

The condition

$$
d(d B) \equiv 0 \bmod A, C, B
$$

then gives, after a MAPLE calculation using the formulas obtained so far,

$$
\begin{equation*}
0=\left(A_{1} k_{23}-C_{3} k_{41}\right) \omega^{1} \wedge \omega^{3}+A_{1} k_{24} \omega^{1} \wedge \omega^{4}+C_{3} k_{42} \omega^{3} \wedge \omega^{2} . \tag{11}
\end{equation*}
$$

Thus, if $k_{24} k_{42} \neq 0$ this implies as before that $\operatorname{dim} \mathcal{C}_{0} \leqq 3$.
We may summarize our results as the following

## Proposition:

(i) If $\Theta=\Xi_{10}^{\{1\}} \cup \Xi_{01}^{\{1\}}$ is not integrable, then

$$
\operatorname{dim} \mathcal{C}_{0} \leqq 5 ;
$$

(ii) Assuming that $\Theta$ is integrable, if $\Xi_{10}^{\langle 1\rangle}$ and $\Xi_{01}^{\langle 1\rangle}$ are not separately integrable, then

$$
\operatorname{dim} \mathcal{C}_{0} \leqq 6
$$

(iii) Assuming that $\Xi_{10}^{\{1\rangle}$ and $\Xi_{01}^{\{1\rangle}$ are separately integrable, if $k_{24} k_{42} \neq 0$ then

$$
\operatorname{dim} \mathcal{C}_{0} \leqq 3
$$

To compute the conservation laws for a given example we may either compute the invariants of the given system and see how these fit with the general analysis given in this and the following section, or else we may compute directly using the general theory to guide the calculation. At the end of the preceding section, we gave an example of the latter.

### 2.3 The two classes of hyperbolic systems having the maximum number of classical conservation laws.

2.3.1 Symmetric systems for which $\operatorname{dim} \mathcal{C}_{0} \geqq 7$. In this section, our main goal is to prove the following theorem

Theorem: If $(M, \mathcal{I})$ is a non-degenerate, symmetric hyperbolic system with $s=0$ on a connected manifold $M$ for which $\operatorname{dim} \mathcal{C}_{0}(U) \geqq 7$ for all open sets $U \subset M$, then, either $(M, \mathcal{I})$ is locally linearizable, or else $(M, \mathcal{I})$ is locally isomorphic to the $s=0$ Liouville system.

As we shall show in the next subsection, systems of this type actually have $\operatorname{dim} \mathcal{C}_{0}(U)=\infty$ for all open sets $U \subset M$.

Proof: By the results of the previous section, we know that we can now restrict attention to hyperbolic systems whose characteristic systems each have an integrable subsystem; i.e., whose invariants satisfy

$$
\begin{equation*}
p_{4}=q_{2}=0 \quad \text { and } \quad p_{3}=q_{1}=0 \tag{1}
\end{equation*}
$$

and which in addition have

$$
\begin{equation*}
k_{24}=k_{42}=0 . \tag{2}
\end{equation*}
$$

By the discussion in Section 2.2, these conditions hold for any symmetric hyperbolic system which has $\operatorname{dim} \mathcal{C}_{0} \geqq 7$.

The structure equations (from Section 1.5) of systems satisfying (1) and (2) now simplify to

$$
d\left(\begin{array}{c}
\omega^{1} \\
\omega^{2} \\
\omega^{3} \\
\omega^{4}
\end{array}\right)=-\left(\begin{array}{cccc}
\phi_{44}-\phi_{22} & 0 & 0 & 0 \\
\phi_{21} & \phi_{22} & 0 & 0 \\
0 & 0 & \phi_{22}-\phi_{44} & 0 \\
0 & 0 & \phi_{43} & \phi_{44}
\end{array}\right) \wedge\left(\begin{array}{c}
\omega^{1} \\
\omega^{2} \\
\omega^{3} \\
\omega^{4}
\end{array}\right)+\left(\begin{array}{c}
-p_{1} \omega^{1} \wedge \omega^{2} \\
\omega^{3} \wedge \omega^{4} \\
-q_{3} \omega^{3} \wedge \omega^{4} \\
\omega^{1} \wedge \omega^{2}
\end{array}\right)
$$

with

$$
\begin{aligned}
d \phi_{22} & =q_{3} \omega^{1} \wedge \omega^{2}-\kappa_{1} \wedge \omega^{1}-\kappa_{2} \wedge \omega^{2} \\
d \phi_{44} & =p_{1} \omega^{3} \wedge \omega^{4}-\kappa_{3} \wedge \omega^{3}-\kappa_{4} \wedge \omega^{4}
\end{aligned}
$$

and

$$
\begin{aligned}
d p_{1} & =p_{1} \phi_{22}+\nabla p_{1} \\
d q_{3} & =q_{3} \phi_{44}+\nabla q_{3}
\end{aligned}
$$

With these assumptions, the condition $d(d B)=0$ may then be expanded to give (cf. (11) of the preceding section)

$$
A_{1} k_{23}-C_{3} k_{41} \equiv 0 \bmod A, C, B
$$

Differentiating this gives

$$
\left.\begin{array}{l}
A_{11} k_{23} \equiv 0 \\
C_{33} k_{41} \equiv 0
\end{array}\right\} \bmod A, C, B, A_{1}, C_{3}
$$

It follows that if $k_{23} k_{41} \neq 0$ then we must have $\operatorname{dim} \mathcal{C}_{0} \leqq 4$. Consequently, (since we are only considering symmetric systems) we shall assume that

$$
\begin{equation*}
k_{23}=0 \quad \text { and } \quad k_{41}=0 . \tag{3}
\end{equation*}
$$

Using (1), (2) and (3) in the identities $d\left(d \omega^{i}\right)=0$ we obtain

$$
\left\{\begin{array}{l}
p_{13}=k_{32}  \tag{4}\\
q_{31}=k_{14}
\end{array}\right.
$$

With (1)-(4) satisfied, the formula for $d B$ now simplifies to

$$
\begin{align*}
d B=B\left(\phi_{22}+\phi_{44}\right) & -\frac{1}{2}\left(A\left(p_{11}-q_{3}\right)+A_{1} p_{1}\right) \omega^{1} \\
& -\frac{1}{2}\left(A\left(p_{12}-\frac{1}{2} p_{1}^{2}\right)-\frac{1}{2} C p_{1} q_{3}+B p_{1}\right) \omega^{2} \\
& -\frac{1}{2}\left(C\left(q_{33}-p_{1}\right)+C_{3} q_{3}\right) \omega^{3}  \tag{5}\\
& -\frac{1}{2}\left(C\left(q_{34}-\frac{1}{2} q_{3}^{2}\right)-\frac{1}{2} A q_{3} p_{1}+B q_{3}\right) \omega^{4} .
\end{align*}
$$

From $d(d B)=0$ we obtain

$$
\begin{align*}
& \text { (i) } \quad\left(B+\frac{1}{2}\left(A p_{1}+C q_{3}\right)\right) k_{14}-A q_{34}=0 \\
& \text { (ii) } \quad\left(B+\frac{1}{2}\left(A p_{1}+C q_{3}\right)\right) k_{32}-C p_{12}=0 \tag{6}
\end{align*}
$$

Taking the derivative of (i) in the $\omega^{1}$-direction, we see from (5) that $d B$ has a $-\frac{1}{2} A_{1} p_{1}$ term so that cancellation occurs and we are left with

$$
A_{1} q_{34} \equiv 0 \bmod A, C, B
$$

and similarly

$$
C_{3} p_{12} \equiv 0 \bmod A, C, B
$$

It follows that if $p_{12} q_{34} \neq 0$ then $\operatorname{dim} \mathcal{C}_{0} \leqq 3$. Thus, we can assume that the relations

$$
\begin{equation*}
p_{12}=0 \quad \text { and } \quad q_{34}=0 \tag{7}
\end{equation*}
$$

are satisfied.
From (6) and (7) we obtain

$$
\left\{\begin{array}{rl}
\text { (i) } & \left(B+\frac{1}{2}\left(A p_{1}+C q_{3}\right)\right) k_{14} \tag{8}
\end{array}=0.0 .\right.
$$

We are now going to show that $k_{14}=k_{32}=0$. Suppose not, then (8) implies

$$
\begin{equation*}
B=-\frac{1}{2}\left(A p_{1}+C q_{3}\right) \tag{9}
\end{equation*}
$$

Computing $d B$ in two ways - using (5) and using (9) - we find

$$
A q_{3}=0 \quad \text { and } \quad C p_{1}=0
$$

Now, since $A C \neq 0$, we necessarily have $p_{1}=q_{3}=0$ and so $\nabla p_{1}=\nabla q_{3}=0$. Equation (4) now gives $k_{14}=k_{32}=0$ anyway.

Using $k_{14}=k_{32}=0$, the identity $d(d B)=0$ yields

$$
\begin{equation*}
\left(B+\frac{1}{2}\left(A p_{1}+C q_{3}\right)\right)\left(k_{13}-k_{31}\right)-A\left(q_{33}-p_{1}\right)+C\left(p_{11}-q_{3}\right)=0 \tag{11}
\end{equation*}
$$

The $\omega^{1}$ and $\omega^{3}$-derivatives of this relation give

$$
\left.\begin{array}{l}
A_{1}\left(q_{33}-p_{1}\right) \equiv 0 \\
C_{3}\left(p_{11}-q_{3}\right) \equiv 0
\end{array}\right\} \bmod A, C, B
$$

implying that if $\left(p_{11}-q_{3}\right)\left(q_{33}-p_{1}\right) \neq 0$ then $\operatorname{dim} \mathcal{C}_{0} \leqq 3$. We thus can assume that

$$
\begin{align*}
& p_{11}-q_{3}=0 \\
& q_{33}-p_{1}=0 \tag{12}
\end{align*}
$$

in which case, (11) reduces to

$$
\left(B+\frac{1}{2}\left(A p_{1}+C q_{3}\right)\right)\left(k_{13}-k_{31}\right)=0
$$

At this point the analysis splits into two cases:
(i) $k_{13}-k_{31}=0 \quad$ or
(ii) $B+\frac{1}{2}\left(A p_{1}+C q_{3}\right)=0$.

Suppose (13.ii) holds. This is equation (9) above and we get the same conclusion as in that case, namely that $p_{1}=q_{3}=0$. Moreover, we now have, from the structure equations and identities found so far that $d \phi_{22}=\alpha_{1} \wedge \omega^{1}$ and $d \phi_{44}=\alpha_{3} \wedge \omega^{3}$ for some 1 -forms $\alpha_{i}$. Also, differentiating the equations

$$
d \omega^{1}=-\left(\phi_{44}-\phi_{22}\right) \wedge \omega^{1}, \quad d \omega^{3}=-\left(\phi_{22}-\phi_{44}\right) \wedge \omega^{3}
$$

yields that $d\left(\phi_{22}-\phi_{44}\right) \wedge \omega^{1}=d\left(\phi_{22}-\phi_{44}\right) \wedge \omega^{3}=0$. Of course, this implies that $\alpha_{1} \wedge \omega^{1} \wedge \omega^{3}=\alpha_{3} \wedge \omega^{1} \wedge \omega^{3}=0$, so we finally conclude that

$$
d \phi_{22}=k_{13} \omega^{1} \wedge \omega^{3}, \quad \quad d \phi_{44}=k_{31} \omega^{3} \wedge \omega^{1}
$$

We now see that the structure equations satisfy the conditions given by the Proposition in Section 1.5 .6 which characterize linear systems. It follows that the systems which satisfy (13.ii) (as well as the previously derived conditions) are precisely the linear systems.

We shall next analyze the situation when (13.i) holds, and for this we set

$$
k_{13}=k_{31}=k_{00}
$$

If we now differentiate (12) we obtain the relations

$$
\left\{\begin{array}{l}
p_{1}\left(k_{00}-1\right)=0  \tag{14}\\
q_{3}\left(k_{00}-1\right)=0
\end{array}\right.
$$

Thus, there are two possibilities: First, we could have $k_{00} \neq 1$, in which case we would have to have $p_{1}=q_{3}=0$, which, as we have seen, leads to the linear case. Thus, we may set this aside and assume, as we shall, that $k_{00}=1$, i.e., $k_{13}=k_{31}=1$.

The structure equations of the system are

$$
d\left(\begin{array}{c}
\omega^{1}  \tag{15}\\
\omega^{2} \\
\omega^{3} \\
\omega^{4}
\end{array}\right)=-\left(\begin{array}{cccc}
\phi_{44}-\phi_{22} & 0 & 0 & 0 \\
\phi_{21} & \phi_{22} & 0 & 0 \\
0 & 0 & \phi_{22}-\phi_{44} & 0 \\
0 & 0 & \phi_{43} & \phi_{44}
\end{array}\right) \wedge\left(\begin{array}{c}
\omega^{1} \\
\omega^{2} \\
\omega^{3} \\
\omega^{4}
\end{array}\right)+\left(\begin{array}{c}
-p_{1} \omega^{1} \wedge \omega^{2} \\
\omega^{3} \wedge \omega^{4} \\
-q_{3} \omega^{3} \wedge \omega^{4} \\
\omega^{1} \wedge \omega^{2}
\end{array}\right)
$$

where

$$
\begin{align*}
d \phi_{22} & =q_{3} \omega^{1} \wedge \omega^{2}+\omega^{1} \wedge \omega^{3} \\
d \phi_{44} & =p_{1} \omega^{3} \wedge \omega^{4}+\omega^{3} \wedge \omega^{1} \tag{16}
\end{align*}
$$

and

$$
\begin{align*}
d p_{1} & =p_{1} \phi_{22}+q_{3} \omega^{1} \\
d q_{3} & =q_{3} \phi_{44}+p_{1} \omega^{3} . \tag{17}
\end{align*}
$$

The eidetic reader will recognize these equations as (11-13) of Section 1.5.7, which characterize the systems whose first prolongations are integrable by the method of Darboux. In that subsection, we proved that, on the domain in $M$ where $p_{1}$ and $q_{3}$ were both non-zero (the only case we are concerned with due to our hypothesis of symmetry), the ideal $\mathcal{I}$ was diffeomorphic to that of the $s=0$ Liouville system. This completes our proof.
2.3.2 Explicit conservation laws in the case $\operatorname{dim} \mathcal{C}_{0}=\infty$. We are now going to explicitly describe the conservation laws in the two cases found in the previous section.

The first case is that of linear systems. According to Section 1.5.6, such a system is locally of the form

$$
\mathcal{I}=\{(d u-P v d y) \wedge d x,(d v-Q u d x) \wedge d y\}
$$

where $P$ and $Q$ are positive functions of $x$ and $y$ only. Tracing through all of the above calculations (or just doing it directly, we see that all of the conservation laws are of the form

$$
\Phi=A(d u-P v d y) \wedge d x+C(d v-Q u d x) \wedge d y
$$

where $A$ and $C$ are functions of $x$ and $y$ alone satisfying the equations

$$
\begin{aligned}
A_{y} & =Q C \\
C_{x} & =P A .
\end{aligned}
$$

Thus, the space of local conservation laws is clearly of infinite dimension.

Consider now the $s=0$ Liouville system described in Section 1.5. This is described by the ideal

$$
\mathcal{I}=\left\{\left(d u-e^{v} d y\right) \wedge d x,\left(d v-e^{u} d x\right) \wedge d y\right\}
$$

and a short calculation using the above formulae shows that the conservation laws are of the form
$\Phi=\left[(g(y)+f(x)) e^{u}+f^{\prime}(x)\right]\left(d u-e^{v} d y\right) \wedge d x+\left[(f(x)+g(y)) e^{v}+g^{\prime}(y)\right]\left(d v-e^{u} d x\right) \wedge d y$
where $f$ and $g$ are arbitrary functions of one variable. Thus, again, we see that the space of conservation laws is of infinite dimension. The undifferentiated conservation law corresponding to such a $\Phi$ is given by

$$
\begin{aligned}
\varphi & =(f(x)+g(y))\left(d u-e^{v} d y\right)-g(y) d u+(f(x)+g(y))\left(d v-e^{u} d x\right)-f(x) d v \\
& =f(x)\left(d u-e^{v} d y-e^{u} d x\right)+g(y)\left(d v-e^{u} d x-e^{v} d y\right)
\end{aligned}
$$

(The first form of $\varphi$ enables us to easily check that $d \varphi=-\Phi$ ). On classical solution surfaces

$$
(x, y) \rightarrow(x, y, u(x, y), v(x, y))
$$

to $s=0$ Liouville we have

$$
\varphi=f(x)\left(u_{x}-e^{u}\right) d x+g(y)\left(v_{y}-e^{v}\right) d y
$$

clearly $d \varphi=0$ on any such solution surface.

Our calculations have thus proved the following result:

Proposition: If a symmetric hyperbolic system $(M, \mathcal{I})$ has $\operatorname{dim} \mathcal{C}_{0} \geqq 7$, then it has $\operatorname{dim} \mathcal{C}_{0}=\infty$.

We will now show how the conservation laws of $s=0$ Liouville may be used to analyze the singularities of the solutions. This discussion will be heuristic and is merely intended to illustrate how conservation laws might be used to gain analytical insight into the solutions of an equation. Of course, in the particular case of the $s=0$ Liouville equation, the precise results may be either derived analytically or verified from the explicit form of the solutions to the $s=0$ Liouville system given in Section 1.5 above.

The initial value problem is typically posed by prescribing initial data on the line $x+y=0$ and seeking a solution in the half-plane $x+y \geq 0$. Thus, setting

$$
t=\frac{1}{2}(x+y) \quad \xi=\frac{1}{2}(x-y)
$$

or equivalently

$$
x=t+\xi \quad y=t-\xi
$$

we are given initial data $u(\xi, 0), v(\xi, 0)$ and we seek to find a solution $u(\xi, t), v(\xi, t)$ defined for $t \geq 0$. Since

$$
\left\{\begin{array}{l}
u_{x}-e^{u}=u_{\xi}+e^{v}-e^{u} \\
v_{y}-e^{v}=-v_{\xi}+e^{u}-e^{v}
\end{array}\right.
$$

the conservation laws give that for any functions $f, g$ the integrals

$$
\begin{aligned}
& \int f(t+\xi)\left(u_{\xi}+e^{v}-e^{u}\right) d \xi=C_{1} \\
& \int g(t-\xi)\left(-v_{\xi}+e^{u}-e^{v}\right) d \xi=C_{2}
\end{aligned}
$$

are formally independent of $t$. More precisely, if we assume that the initial data is smooth then $u(\xi, t), v(\xi, t)$ will remain smooth for $0 \leq t<t_{0}$. If we now restrict $\xi, t$ to the rectangle $|\xi| \leq a, 0 \leq t<t_{0}$ then for all smooth compactly supported $f, g$ with uniformly bounded $L^{2}$ norm we will have

$$
\begin{aligned}
& \int f(t+\xi)\left(u_{\xi}+e^{v}-e^{u}\right) d \xi=\mathrm{O}(1) \\
& \int g(t-\xi)\left(-v_{\xi}+e^{u}-e^{v}\right) d \xi=\mathrm{O}(1)
\end{aligned}
$$

By taking $f, g$ to be sequences tending to $\delta$-functions we will thus have

$$
\begin{array}{r}
u_{\xi}+e^{v}-e^{u}=\mathrm{O}(1) \\
-v_{\xi}+e^{u}-e^{v}=\mathrm{O}(1) \tag{ii}
\end{array}
$$

on rectangles as above. We will use these relations to analyze the possible singularities that $u$ and $v$ can develop.

For $t=t_{0}$ we assume that $u\left(\xi, t_{0}\right), v\left(\xi, t_{0}\right)$ are of class $C^{1}$ in a neighborhood $0<\xi<\epsilon$ and seek to determine what sort of singularity may develop as we approach the origin. ${ }^{23}$ For this we assume that $u_{\xi}$ and $v_{\xi}$ have asymptotic expansions

$$
\begin{aligned}
& u_{\xi} \sim-\frac{c}{\xi^{\lambda}}+r, \quad c \neq 0 \\
& v_{\xi} \sim-\frac{c^{\prime}}{\xi^{\lambda^{\lambda}}}+s, \quad c^{\prime} \neq 0
\end{aligned}
$$

23) To be more precise, we should consider $u(\xi, t), v(\xi, t)$ as $t \uparrow t_{0}$ and $\xi \downarrow 0$, and assuming that $u, v$ are of class $C^{1}$ in $\xi$ for $t<t_{0}$ see what type of singularity may develop at $t=t_{0}, \xi=0$.
where $\xi^{\lambda} r \rightarrow 0$ and $\xi^{\lambda^{\prime}} s \rightarrow 0$ as $\xi \downarrow 0$. By adding (i) and (ii) we see that

$$
\lambda=\lambda^{\prime}, \quad c=c^{\prime}
$$

and we shall consider the case $c>0$.
If $\lambda>1$ we then have

$$
u=\left(\frac{c}{\lambda-1}\right) \frac{1}{\xi^{\lambda-1}}+R
$$

where $\xi^{\lambda-1} R \rightarrow 0$ as $\xi \downarrow 0$, and similarly for $v$. By (i) we have up to a bounded term

$$
-\frac{c}{\xi^{\lambda}}+r=e^{\left(\frac{c}{\lambda-1}\right)\left(\frac{1}{\xi^{\lambda-1}}\right)}\left(e^{R}-e^{S}\right)
$$

which is impossible unless $u=v$, a case that may be handled directly. ${ }^{24}$ Thus we must have $\lambda \leq 1$. If $0<\lambda<1$ then both $e^{u}$ and $e^{v}$ are $\mathrm{O}(1)$ as $\xi \downarrow 0$ and this contradicts (i). Thus we must have $\lambda=1$ which gives the expansions

$$
\begin{aligned}
u_{\xi} & \sim-\frac{c}{\xi}+O(1) \\
v_{\xi} & \sim-\frac{c}{\xi}+O(1)
\end{aligned}
$$

as $\xi \downarrow 0$. If the initial data is real-analytic then it may be shown that this heuristic reasoning is justified leading to the result:
There is a real analytic curve $\Gamma$ in the half-plane $x+y \geq 0$ such that $u, v$ are bounded and piecewise real analytic away from $\Gamma$, but where singularities of $u, v$ occur at the points of intersection $\Gamma \cap\left\{t=t_{0}\right\}$. Moreover, $c$ is a positive integer giving the order of contact of $\Gamma$ with the line $t=t_{0}$ at the point of intersection. ${ }^{25}$

We offer the above discussion not as a definitive analysis of the singularity structure of the solutions of the $s=0$ Liouville system but rather to illustrate how conservation laws may be used to infer analytic behavior.
2.4 Moment conditions. Given a differential ideal $\mathcal{I}$ on a manifold $M$ we may speak of the complex of piecewise smooth singular $\mathcal{I}$-chains and the resulting $\mathcal{I}$-homology, denoted by $H_{*, \mathcal{I}}(M)$ (cf. Section $6 \mathrm{in}\left[\mathrm{BG}_{1}\right]$ ). Of particular interest are the first non-vanishing local $\mathcal{I}$-homology groups and their relationship to conservation laws.
24) When $u=v$, the $s=0$ Liouville system reduces to a simple Ricatti $f^{\prime}=f^{2}$ with dependence on a parameter.
25) If $t=t_{0}$ is the first time a singularity is encountered, then for $t$ slightly larger than $t_{0}$ the singularity will split into "simple" singularities for each of which the asymptotic expansion holds with $c=1$.

In the situation of a hyperbolic exterior differential system on a 4-manifold, we work locally in a contractible open set and the issue is this:
Let $\gamma \subset M$ be a closed curve. Then $\gamma=\partial S$ is the boundary of a (piecewise smooth) surface, and we ask when $S$ may be taken to be an integral surface of $\mathcal{I}$ ?
Necessary conditions are given by the moment conditions coming from the space $\mathcal{C}_{0}$ of classical conservation laws, as follows: For $\Phi \in \mathcal{C}_{0}$ we have

$$
\Phi=d \varphi
$$

for some 1 -form $\varphi$, and by Stokes' theorem

$$
\int_{\gamma} \varphi=\int_{S} \Phi
$$

The right hand side vanishes in case $S$ is an integral surface of $\mathcal{I}$, and by definition the moment conditions are

$$
\int_{\gamma} \varphi=0, \quad \Phi \in \mathcal{C}_{0}
$$

The question is whether they are sufficient as well as necessary in order to be able to fill in $\gamma$ with an integral surface, or at least with a piecewise smooth integral chain whose simplices are integral surfaces.

We shall discuss this question in the following variational form. Let $\gamma_{0} \subset M$ be a non-characteristic smooth curve given as the image of a smooth immersion

$$
f:[0,1] \rightarrow M
$$

and for some $\delta>0$, let $\gamma_{t}, 0 \leqq t \leqq \delta$ be a variation of $\gamma_{0}$ given by a smooth mapping

$$
F:[0,1] \times[0, \delta] \rightarrow M
$$

where $\gamma_{t}=f_{t}([0,1])$ with $f_{t}=\left.F\right|_{[0,1] \times\{t\}}$.


For $\delta$ sufficiently small $f_{t}$ will be an immersion and $\gamma_{t}$ will be non-characteristic. For $0 \leq T \leq \delta$ we set

$$
S_{T}=\underset{\leqq \leq T}{\cup} \gamma_{t} \text { with } S=S_{\delta}
$$

and we make the assumption
For all $T$ and for all $\Phi \in \mathcal{C}_{0}$ we have

$$
\begin{equation*}
\int_{\partial S_{T}} \varphi=0, \quad 0 \leq T \leq \delta \tag{1}
\end{equation*}
$$

We then have the following

Proposition: If $\operatorname{dim} \mathcal{C}_{0}=\infty$, then the moment conditions (1) are necessary and sufficient that the $S$ be an integral surface of $(M, \mathcal{I})$.

Proof: We must show that for each point $p \in S_{\epsilon}$ the tangent plane $T_{p} S_{\epsilon}$ is an integral element of $\mathcal{I}$. Clearly it will suffice to do this for $p \in \gamma_{0}$. Let $\tau$ be the tangent vector field to $\gamma_{0}$ and $\nu$ the variation vector field for the family of curves $\gamma_{t}$. Note that the vector field $\nu$ is only defined modulo $\tau$. We want to show that for each $p \in \gamma_{0}$ the 2-plane $\nu(p) \wedge \tau(p)$ is an integral element.

Suppose now that $\Phi \in \mathcal{C}_{0}$ is compactly supported along each $\gamma_{t}$ - i.e., $\Phi$ vanishes near the endpoints of the $\gamma_{t}$. Then a standard calculation gives

$$
\begin{equation*}
0=\frac{d}{d t}\left(\int_{S_{t}} \Phi\right)_{t=0}=\int_{\gamma_{0}} \nu-\Phi \tag{2}
\end{equation*}
$$

Now $\Phi \in \mathcal{C}_{0}$ has the form

$$
\Phi=A \Omega_{10}+C \Omega_{01}
$$

and depends upon "two arbitrary functions of one variable". In fact, as is clear from the discussion in Section 2.3.2, we may specify $A$ and $C$ arbitrarily along a non-characteristic curve $\gamma_{0}$. Thus, the only way that we can have (2) for all $\Phi$ is that $\nu ـ \Omega_{10}$ and $\nu \mu \Omega_{01}$ both restrict to zero on $\gamma_{0}$. This is equivalent to

$$
\left\langle\Omega_{10}, \nu \wedge \tau\right\rangle=\left\langle\Omega_{01}, \nu \wedge \tau\right\rangle=0
$$

along $\gamma_{0}$, which is what was to be proved.

## §3. Symplectic Hyperbolic Systems

3.1 Definition and structure equations; characterization of Euler-Lagrange systems. A symplectic hyperbolic exterior differential system is given by the data $(M, \Phi, \mathcal{I})$ where $M$ is a 4 -dimensional manifold, $\Phi$ is a symplectic form on $M$, and $\mathcal{I}$ is a differential ideal that defines a hyperbolic system with $\Phi \in \mathcal{I}$. Integral surfaces of $\mathcal{I}$ are thus necessarily Lagrangian surfaces for $\Phi$. Examples of symplectic hyperbolic systems include the Euler-Lagrange systems introduced in Section 1.6.

The automorphisms of a symplectic hyperbolic system ( $M, \Phi, \mathcal{I}$ ) are diffeomorphisms of $M$ that preserve both $\Phi$ and $\mathcal{I}$. As we shall see, inequivalent symplectic hyperbolic systems may well become equivalent when considered as hyperbolic systems. In particular, a non-linear ( $M, \Phi, \mathcal{I}$ ) may linearize when we consider only $(M, \mathcal{I})$.

We shall quickly work through the equivalence problem for symplectic hyperbolic systems. As before, we shall assume that the hyperbolic system itself is symmetric and non-degenerate. This leads us to consider coframes $\omega^{1}, \omega^{2}, \omega^{3}, \omega^{4}$ such that

$$
\left\{\begin{array}{l}
\Omega_{10}=\omega^{1} \wedge \omega^{2}  \tag{1}\\
\Omega_{01}=\omega^{3} \wedge \omega^{4}
\end{array}\right.
$$

and with the additional condition that the symplectic form is

$$
\begin{equation*}
\Phi=\omega^{1} \wedge \omega^{2}+\omega^{3} \wedge \omega^{4} \tag{2}
\end{equation*}
$$

As in Section 1.5 we may use the assumed non-integrability of $\Omega_{10}$ and $\Omega_{01}$ to assume further that $\omega^{1}$ and $\omega^{3}$ generate the first derived systems of $\Xi_{10}$ and $\Xi_{01}$ and that

$$
\left\{\begin{array}{l}
d \omega^{2} \equiv \omega^{3} \wedge \omega^{4} \bmod \left\{\omega^{1}, \omega^{2}\right\}  \tag{3}\\
d \omega^{4} \equiv \omega^{1} \wedge \omega^{2} \bmod \left\{\omega^{3}, \omega^{4}\right\}
\end{array}\right.
$$

Coframes satisfying (1)-(3) will be called symplectic coframes.
Keeping the notation of Section 1.5, diagonal transformations that preserve the conditions (2), (3) must satisfy

$$
\begin{aligned}
& a c^{-1} \cdot a^{-1}=1 \\
& a^{-1} c \cdot c^{-1}=1
\end{aligned}
$$

and hence must all be 1 . Thus the structure group of the $G$-structure $B_{G} \rightarrow M$ is now 2 -dimensional consisting of matrices of the form

$$
\left(\begin{array}{llll}
1 & 0 & 0 & 0 \\
\mu & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & \nu & 1
\end{array}\right)
$$

Being smaller than our previous group we may expect more invariants, although the condition $d \Phi=0$ will tend to impose relations among them.

The equivalence problem proceeds much as in Section 1.5. The structure equations have the form

$$
d\left(\begin{array}{c}
\omega^{1}  \tag{4}\\
\omega^{2} \\
\omega^{3} \\
\omega^{4}
\end{array}\right)=-\left(\begin{array}{cccc}
0 & 0 & 0 & 0 \\
\varphi_{21} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & \varphi_{43} & 0
\end{array}\right) \wedge\left(\begin{array}{c}
\omega^{1} \\
\omega^{2} \\
\omega^{3} \\
\omega^{4}
\end{array}\right)+\left(\begin{array}{rr}
-\varphi_{11} \wedge \omega^{1} & -\varphi_{12} \wedge \omega^{2}-\omega^{1} \wedge \omega^{3} \\
\varphi_{11} \wedge \omega^{2} & +\omega^{3} \wedge \omega^{4} \\
-\varphi_{33} \wedge \omega^{3} & -\varphi_{34} \wedge \omega^{4}-\omega^{3} \wedge \omega^{1} \\
\varphi_{33} \wedge \omega^{4} & +\omega^{1} \wedge \omega^{2}
\end{array}\right)
$$

By modifying the pseudo-connection terms in the obvious way we may assume that the torsion is

$$
\left(\begin{array}{c}
-k_{0} \omega^{1} \wedge \omega^{2}-\left(a_{3} \omega^{3}+a_{4} \omega^{4}\right) \wedge \omega^{1}-\left(p_{3} \omega^{3}+p_{4} \omega^{4}\right) \wedge \omega^{2}-\omega^{1} \wedge \omega^{3}  \tag{5}\\
\left(a_{3} \omega^{3}+a_{4} \omega^{4}\right) \wedge \omega^{2}+\omega^{3} \wedge \omega^{4} \\
-k_{0} \omega^{3} \wedge \omega^{4}-\left(c_{1} \omega^{1}+c_{2} \omega^{2}\right) \wedge \omega^{3}-\left(q_{1} \omega^{1}+q_{2} \omega^{2}\right) \wedge \omega^{4}-\omega^{3} \wedge \omega^{I} \\
\left(c_{1} \omega^{1}+c_{2} \omega^{2}\right) \wedge \omega^{4}+\omega^{1} \wedge \omega^{2}
\end{array}\right)
$$

The fiber variation of the individual terms in the torsion is given by

$$
\delta k_{0}=\delta p_{4}=\delta q_{2}=0
$$

and

$$
\left\{\begin{array}{l}
\delta p_{3}=p_{4} \varphi_{43} \\
\delta a_{3}=a_{4} \varphi_{43}+p_{3} \varphi_{21} \\
\delta a_{4}=p_{4} \varphi_{21} \\
\delta q_{1}=q_{2} \varphi_{21} \\
\delta c_{1}=c_{2} \varphi_{21}+q_{1} \varphi_{43} \\
\delta c_{2}=q_{2} \varphi_{43} .
\end{array}\right.
$$

Thus $k_{0}, p_{4}, q_{2}$ are (absolute) invariants of the $G$-structure. If $p_{4}=q_{2}=0$ then $p_{3}, q_{1}$ and $a_{4}, c_{2}$ become invariants, and if in addition these vanish then $a_{3}$ and $c_{1}$ become invariants. The interpretations of these quantities may be given in an analogous manner to what was done in Section 1.5.

As an application of these structure equations, we shall answer the following interesting question: When is a symplectic hyperbolic exterior differential system an Euler-Lagrange system? This is a special case of the more general question: When is a hyperbolic exterior differential system an Euler-Lagrange system? We shall not attempt to answer this latter question in this paper, but, in principle, the techniques we use could do so.

Our answer is summarized in the following

Proposition: The necessary and sufficient conditions that a hyperbolic system with torsion given by (5) be Euler-Lagrange is that the invariants satisfy

$$
\begin{equation*}
k_{0}=0, \quad p_{4}=q_{2}, \quad a_{4}=q_{1}, \quad c_{2}=p_{3}, \quad a_{3}=c_{1} . \tag{6}
\end{equation*}
$$

Proof: We seek the conditions that there exist

$$
\Psi=A \omega^{1} \wedge \omega^{2}+C \omega^{3} \wedge \omega^{4}
$$

satisfying

$$
\begin{align*}
\Phi \wedge \Psi & =0  \tag{i}\\
d \Psi & =0 . \tag{ii}
\end{align*}
$$

Clearly (i) gives that

$$
\Psi=B\left(\omega^{1} \wedge \omega^{2}-\omega^{3} \wedge \omega^{4}\right)
$$

for some function $B$. Then (ii) gives

$$
d B=-2 B\left(\omega^{1}+\omega^{3}\right)
$$

and then the identity $d(d B)=0$ implies the conditions (6).
This result has a curious

Corollary: Let $(M, \mathcal{I})$ be a hyperbolic Euler-Lagrange exterior differential system, and assume that each characteristic system $\Xi_{10}, \Xi_{01}$ has an integrable subsystem. ${ }^{26}$ Then $(M, \mathcal{I})$ is linear.

This corollary stands in strong contrast to the exterior differential system associated to $f$-Gordon equations $z_{x y}=f(z)$, all of which are Euler-Lagrange but which are non-linear unless $f$ is linear.

For the proof we note that under the above assumptions we have

$$
k_{0}=p_{4}=q_{2}=p_{3}=q_{1}=a_{4}=c_{2}=0 \quad \text { and } \quad a_{3}=c_{1}=s_{0}
$$

26) This is equivalent to ( $M, \mathcal{I}$ ) being locally the exterior differential system associated to a PDE system

$$
\left\{\begin{array}{l}
u_{y}=f(x, y, u, v) \\
v_{x}=g(x, y, u, v) .
\end{array}\right.
$$

and hence the structure equations reduce to

$$
\begin{aligned}
d \omega^{1} & =\left(s_{0}-1\right) \omega^{1} \wedge \omega^{3} \\
d \omega^{3} & =\left(s_{0}-1\right) \omega^{3} \wedge \omega^{1} \\
d \omega^{2} & \equiv s_{0} \omega^{3} \wedge \omega^{2}+\omega^{3} \wedge \omega^{4} \bmod \omega^{1} \\
d \omega^{4} & \equiv s_{0} \omega^{1} \wedge \omega^{4}+\omega^{1} \wedge \omega^{2} \bmod \omega^{3} \\
d s_{0} & \equiv 0 \bmod \omega^{1}, \omega^{3}
\end{aligned}
$$

The last equation results from the identities $d^{2} \omega^{1}=d^{2} \omega^{3}=0$. These structure equations can now be applied to compute the invariants of the ideal $\mathcal{I}$ as described in Section 1.5. The result of comparing these structure equations to those is that the system satisfies the conditions to be a linear system set forth in Section 1.5.6.

Another interesting consequence is:
Corollary: The $s=0$ Liouville exterior differential system is not an EulerLagrange system.

Proof: We consider the above situation of a symplectic manifold $(M, \Phi)$ and a hyperbolic Euler-Lagrange exterior differential system $\mathcal{I}$ such that $\Phi \in \mathcal{I}$. The above proposition implies that we may write the structure equations in the form

$$
\begin{aligned}
& d\left(\begin{array}{c}
\omega^{1} \\
\omega^{2} \\
\omega^{3} \\
\omega^{4}
\end{array}\right)=-\left(\begin{array}{cccc}
\phi_{44}-\phi_{22} & 0 & 0 & 0 \\
\phi_{21} & \phi_{22} & 0 & 0 \\
0 & 0 & \phi_{22}-\phi_{44} & 0 \\
0 & 0 & \phi_{43} & \phi_{44}
\end{array}\right) \wedge\left(\begin{array}{c}
\omega^{1} \\
\omega^{2} \\
\omega^{3} \\
\omega^{4}
\end{array}\right) \\
&+\left(\begin{array}{c}
\omega^{2} \wedge\left(-p_{3} \omega^{1}+p_{3} \omega^{3}+t_{0} \omega^{4}\right) \\
\omega^{3} \wedge \omega^{4} \\
\omega^{4} \wedge\left(-q_{1} \omega^{3}+q_{1} \omega^{1}+t_{0} \omega^{2}\right)
\end{array}\right) \\
& \omega^{1} \wedge \omega^{2}
\end{aligned}
$$

where

$$
\begin{aligned}
& \phi_{22}=-\left(s_{0} \omega^{3}+q_{1} \omega^{4}\right)-\omega^{1} \\
& \phi_{44}=-\left(s_{0} \omega^{1}+p_{3} \omega^{2}\right)-\omega^{3} \\
& \phi_{21} \equiv \varphi_{21}-\omega^{2} \bmod \omega^{1} \\
& \phi_{43} \equiv \varphi_{43}-\omega^{4} \bmod \omega^{3} .
\end{aligned}
$$

Comparing this with the structure equations (5) in Section 1.5 we deduce that If a hyperbolic EDS has the properties that (i) each characteristic system contains an integrable subsystem and (ii) $p_{1} q_{3} \neq 0$, then the system is not Euler-Lagrange.

By the computation at the end of Section 1.5, the $s=0$ Liouville system has these properties and hence cannot be Euler-Lagrange.
3.2 The three classes of symplectic hyperbolic systems having an infinite number of classical conservation laws. In this section we shall discuss the conditions imposed on a (symmetric) symplectic hyperbolic exterior differential system ( $M, \Phi, \mathcal{I}$ ) to have $\operatorname{dim} \mathcal{C}_{0}=\infty$. Writing

$$
\Psi=A \Omega_{10}+C \Omega_{01}
$$

the condition

$$
d \Psi=0
$$

is equivalent to the overdetermined system

$$
\begin{cases}A_{3}=C-A & A_{4}=0  \tag{1}\\ C_{1}=A-C & C_{2}=0\end{cases}
$$

The analysis of the solutions to these equations proceeds in exactly the same way as that given in sections 2.2 and 2.3 , and we shall only give the conclusions.

It turns out that there are three classes of symplectic hyperbolic systems with $\operatorname{dim} \mathcal{C}_{0}=\infty$ and we shall discuss these.
Class A: The structure equations have the form

$$
d\left(\begin{array}{c}
\omega^{1}  \tag{2}\\
\omega^{2} \\
\omega^{3} \\
\omega^{4}
\end{array}\right)=-\left(\begin{array}{cccc}
0 & 0 & 0 & 0 \\
\varphi_{21} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & \varphi_{43} & 0
\end{array}\right) \wedge\left(\begin{array}{c}
\omega^{1} \\
\omega^{2} \\
\omega^{3} \\
\omega^{4}
\end{array}\right)+\left(\begin{array}{r}
\left(a_{3}-1\right) \omega^{1} \wedge \omega^{3} \\
-a_{3} \omega^{2} \wedge \omega^{3}+\omega^{3} \wedge \omega^{4} \\
\left(c_{1}-1\right) \omega^{3} \wedge \omega^{1} \\
-c_{1} \omega^{4} \wedge \omega^{1}+\omega^{1} \wedge \omega^{2}
\end{array}\right)
$$

where

$$
\left\{\begin{array}{l}
d a_{3}=a_{31} \omega^{1}+a_{33} \omega^{3}  \tag{3}\\
d c_{1}=c_{13} \omega^{3}+c_{11} \omega^{1}
\end{array}\right.
$$

We shall show that any such structure is linear. More precisely, we shall say that $(M, \Phi, \mathcal{I})$ is symplectic linearizable in case there are local coordinates $(x, y, u, v)$ together with a 2 -form $\Psi \in \mathcal{I}$ such that $\Phi$ and $\Psi$ generate $\mathcal{I}$ and each is linear in $u$ and $v$. Such a system is clearly linearizable in our previous sense, but not conversely. We shall prove that symplectic hyperbolic systems satisfying (2) and (3) are symplectic linearizable.

Proof: Since $\omega^{1}$ and $\omega^{3}$ are separately integrable we may find local coordinates $x, y$ such that

$$
\begin{aligned}
\omega^{1} & =A d x \\
\omega^{3} & =C d y
\end{aligned}
$$

These coordinates are determined up to

$$
(x, y)=(X(\bar{x}), Y(\bar{y}))
$$

From the structure equations we infer that $A, C$ and $a_{3}, c_{1}$ are all functions of $x, y$ and furthermore that the hyperbolic PDE system

$$
\left\{\begin{array}{l}
A_{y}=-\left(a_{3}-1\right) A C  \tag{4}\\
C_{x}=-\left(c_{1}-1\right) A C
\end{array}\right.
$$

is satisfied.
Next we may find functions $u, v$ such that

$$
\left.\begin{array}{l}
\omega^{2} \equiv B d u \\
\omega^{4} \equiv D d v
\end{array}\right\} \bmod \omega^{1}, \omega^{3}
$$

From the structure equations we have that

$$
B_{v}=D_{u}=0
$$

Thus we may introduce new coordinates $u, v$ so that

$$
\begin{aligned}
\omega^{2} & \equiv d u+E d y \bmod \omega^{1} \\
\omega^{4} & \equiv d v+F d x \bmod \omega^{3}
\end{aligned}
$$

From the structure equations we obtain

$$
\begin{aligned}
& E_{u}=-a_{3} C \quad E_{v}=-C \\
& F_{v}=-c_{1} A \quad F_{u}=-A .
\end{aligned}
$$

Integrating these gives

$$
\begin{aligned}
& E=-C(x, y)\left(a_{3}(x, y) u+v\right)-E_{0}(x, y) \\
& F=-A(x, y)\left(c_{1}(x, y) v+u\right)-F_{0}(x, y) .
\end{aligned}
$$

Thus the exterior differential system

$$
\begin{align*}
& \omega^{1} \wedge \omega^{2}=A d x \wedge(d u+E d y)=0 \\
& \omega^{3} \wedge \omega^{4}=C d y \wedge(d v+F d x)=0 \tag{5}
\end{align*}
$$

models the affine linear PDE system

$$
\begin{aligned}
& u_{y}=C(x, y)\left(a_{3}(x, y) u+v\right)+E_{0}(x, y) \\
& v_{x}=A(x, y)\left(c_{1}(x, y) v+u\right)+F_{0}(x, y) .
\end{aligned}
$$

By subtracting from $u, v$ a particular solution we may assume that $E_{0}=F_{0}=0$. Clearly (5) is linear in $u, v$ as is $\Phi=\omega^{1} \wedge \omega^{2}+\omega^{3} \wedge \omega^{4}$.

Of particular interest is the case when $a_{3}=a_{0}$ and $c_{1}=c_{0}$ are constant. Then the group of symmetries of $(M, \Phi, \mathcal{I})$ is an infinite-dimensional transitive pseudogroup whose general element depends on two functions of one variable. Note that the PDE system (4) is then a constant scaling in $x$ and $y$ of the $s=0$ Liouville system for the quantities $\log A, \log C$.

Assuming that $\left(a_{0}-1\right)\left(c_{0}-1\right) \neq 0$, the general solution is given by

$$
\begin{aligned}
& A=\frac{-a^{\prime}(x)}{\left(c_{0}-1\right)(c(y)-a(x))} \\
& C=\frac{-c^{\prime}(y)}{\left(a_{0}-1\right)(a(x)-c(y))}
\end{aligned}
$$

and so (5) models the interesting linear PDE system

$$
\left\{\begin{aligned}
u_{y} & =-\frac{c^{\prime}(y)\left(a_{0} u+v\right)}{\left(a_{0}-1\right)(a(x)-c(y))} \\
v_{x x} & =-\frac{a^{\prime}(x)\left(c_{0} v+u\right)}{\left(c_{0}-1\right)(c(y)-a(x))}
\end{aligned}\right.
$$

In case $a_{0}=c_{0}=1,(4)$ gives

$$
\begin{aligned}
& A_{y}=0 \\
& C_{x}=0
\end{aligned}
$$

and so

$$
\begin{aligned}
& A=A(x) \\
& C=C(y)
\end{aligned}
$$

Thus (5) yields the following linear system

$$
\left\{\begin{aligned}
u_{y} & =\frac{u+v}{C(y)} \\
v_{x} & =\frac{u+v}{A(x)}
\end{aligned}\right.
$$

Class B: The structure equations have the form

$$
\begin{align*}
d\left(\begin{array}{c}
\omega^{1} \\
\omega^{2} \\
\omega^{3} \\
\omega^{4}
\end{array}\right)= & -\left(\begin{array}{cccc}
0 & 0 & 0 & 0 \\
\varphi_{21} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & \varphi_{43} & 0
\end{array}\right) \wedge\left(\begin{array}{c}
\omega^{1} \\
\omega^{2} \\
\omega^{3} \\
\omega^{4}
\end{array}\right) \\
& +\left(\begin{array}{r}
-k_{0} \omega^{1} \wedge \omega^{2}-\left(a_{3} \omega^{3}+a_{4} \omega^{4}\right) \wedge \omega^{1}-\omega^{1} \wedge \omega^{3} \\
\left.-k_{0} \omega^{3} \wedge \omega^{4}-\left(a_{3} \omega^{3}+a_{4} \omega^{4}\right) \wedge \omega^{2}+\omega^{3} \wedge \omega^{4}+c_{2} \omega^{2}\right) \wedge \omega^{3}-\omega^{3} \wedge \omega^{1} \\
\left(c_{1} \omega^{1}+c_{2} \omega^{2}\right) \wedge \omega^{4}+\omega^{1} \wedge \omega^{2}
\end{array}\right) \tag{6}
\end{align*}
$$

where $k_{0} \neq 0$ and

$$
\left\{\begin{array}{l}
d k_{0}=-\left(a_{4}+c_{1} k_{0}\right) \omega^{1}-c_{2} k_{0} \omega^{2}-\left(c_{2}+a_{3} k_{0}\right) \omega^{3}-a_{4} k_{0} \omega^{4}  \tag{7}\\
d a_{3}=a_{4} \varphi_{43}+a_{3}\left(c_{1}-2\right) \omega^{1}+a_{3} c_{2} \omega^{2}+a_{33} \omega^{3}-\left(2 a_{3} k_{0}+a_{4}\right) \omega^{4} \\
d a_{4}=-a_{4}\left(c_{1}+1\right) \omega^{1}-a_{4} c_{2} \omega^{2}-\left(a_{4}+a_{3} k_{0}\right) \omega^{3}-a_{4} k_{0} \omega^{4} \\
d c_{1}=c_{2} \varphi_{21}+c_{1}\left(a_{3}-2\right) \omega^{3}+c_{1} a_{4} \omega^{4}+c_{11} \omega^{1}-\left(2 c_{1} k_{0}+c_{2}\right) \omega^{2} \\
d c_{2}=-c_{2}\left(a_{3}+1\right) \omega^{3}-c_{2} a_{4} \omega^{4}-\left(c_{2}+c_{1} k_{0}\right) \omega^{1}-c_{2} k_{0} \omega^{2}
\end{array}\right.
$$

Comparing (6), (7) with the proposition in Section 1.5 we obtain the following

Proposition: Any system satisfying (6) and (7) is Darboux integrable on the first prolongation.

In order to give a normal form we need to separate into the subcases

$$
\begin{array}{r}
a_{4}=c_{2}=0 \\
a_{4} c_{2} \neq 0 . \tag{9}
\end{array}
$$

We shall deal mainly with the first case and shall show that
Proposition: A hyperbolic exterior differential system satisfying (6), (7) and (8) has $k_{0}=$ constant and is equivalent to the system modeled on the $s=0$ Liouville system

$$
\left\{\begin{array}{l}
u_{y}=e^{k_{0} v} \\
v_{x}=e^{k_{0} u}
\end{array}\right.
$$

Proof: From (7) we have $a_{3}=c_{1}=0$ and so $k_{0}=$ constant. Thus (6) reduces to

$$
d\left(\begin{array}{c}
\omega^{1}  \tag{10}\\
\omega^{2} \\
\omega^{3} \\
\omega^{4}
\end{array}\right)=-\left(\begin{array}{cccc}
0 & 0 & 0 & 0 \\
\varphi_{21} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & \varphi_{43} & 0
\end{array}\right) \wedge\left(\begin{array}{c}
\omega^{1} \\
\omega^{2} \\
\omega^{3} \\
\omega^{4}
\end{array}\right)+\left(\begin{array}{r}
-k_{0} \omega^{1} \wedge \omega^{2}-\omega^{1} \wedge \omega^{3} \\
\omega^{3} \wedge \omega^{4} \\
-k_{0} \omega^{3} \wedge \omega^{4}-\omega^{3} \wedge \omega^{1} \\
\omega^{1} \wedge \omega^{2}
\end{array}\right)
$$

where $k_{0} \neq 0$ parametrizes the equivalence classes of these structures. Since $\omega^{1}$ and $\omega^{3}$ are integrable we may introduce local coordinates ( $x, y, u, v$ ) such that

$$
\left\{\begin{array}{l}
\omega^{1}=A d x \\
\omega^{3}=C d y
\end{array}\right.
$$

and

$$
\left\{\begin{array}{l}
\omega^{2} \equiv B d u+E d y \bmod \omega^{1} \\
\omega^{4} \equiv D d v+F d x \bmod \omega^{3}
\end{array}\right.
$$

From the structure equations for $d \omega^{2}$ and $d \omega^{4}$ we have

$$
B_{v}=D_{u}=0
$$

By changing $u, v$ appropriately we may assume that

$$
\left\{\begin{array}{l}
\omega^{2}=d u+E d y \bmod \omega^{1} \\
\omega^{4}=d v+F d x \bmod \omega^{3}
\end{array}\right.
$$

Again from the structure equations we have

$$
\left\{\begin{array}{rll}
(\mathrm{i}) & E_{u}=0 & E_{v}=-C \\
(\mathrm{ii}) & F_{v}=0 & F_{u}=-A
\end{array}\right.
$$

and

$$
\left\{\begin{array}{lll}
\text { (iii) } & A_{u}=k_{0} A & A_{v}=0 \\
\text { (iv) } & C_{v}=k_{0} C & C_{u}=0
\end{array}\right.
$$

Integrating (iii) and (iv) gives

$$
A=k_{0} A_{0}(x, y) e^{k_{0} u} \quad C=k_{0} C_{0}(x, y) e^{k_{0} v}
$$

while from (i) and (ii) we obtain

$$
\begin{aligned}
& E=-C_{0} e^{k_{0} v}+E_{0}(x, y) \\
& F=-A_{0} e^{k_{0} u}+F_{0}(x, y) .
\end{aligned}
$$

Thus we now have

$$
\left\{\begin{array}{l}
\omega^{2} \equiv d u+\left(-C_{0} e^{k_{0} v}+E_{0}\right) d y \bmod d x \\
\omega^{4} \equiv d v+\left(-A_{0} e^{k_{0} u}+F_{0}\right) d x \bmod d y
\end{array}\right.
$$

Clearly we may make a substitution

$$
\begin{aligned}
& u \rightarrow u+\int E_{0} d y \\
& v \rightarrow v+\int F_{0} d x
\end{aligned}
$$

to set $E_{0}=F_{0}=0$ (with a new $A_{0}$ and $C_{0}$ ). The structure equations for $d \omega^{1}$ and $d \omega^{3}$ then give

$$
\left(A_{0}\right)_{y}=\left(C_{0}\right)_{x}=0
$$

By choosing new coordinates $x$ and $y$ we may assume that

$$
\left\{\begin{array}{l}
\omega^{1}=e^{k_{0} u} d x \\
\omega^{3}=e^{k_{0} v} d y
\end{array}\right.
$$

and

$$
\left\{\begin{array}{l}
\omega^{2}=d u-e^{k_{0} v} d y \bmod d x \\
\omega^{4}=d v-e^{k_{0} u} d x \bmod d y
\end{array}\right.
$$

which gives the $s=0$ Liouville system.
REmark: These systems with parameter $k_{0}$ are inequivalent as symplectic hyperbolic systems but equivalent as hyperbolic systems.

For symplectic hyperbolic systems satisfying (6), (7) and (9) we may restrict to the subbundle of coframes that satisfy

$$
a_{3}=c_{1}=0,
$$

in which case the structure equations reduce to

$$
d\left(\begin{array}{l}
\omega^{1}  \tag{11}\\
\omega^{2} \\
\omega^{3} \\
\omega^{4}
\end{array}\right)=\left(\begin{array}{l}
-k_{0} \omega^{1} \wedge \omega^{2}-a_{4} \omega^{4} \wedge \omega^{1}-\omega^{1} \wedge \omega^{3} \\
-\varphi_{21} \wedge \omega^{1}+a_{4} \omega^{4} \wedge \omega^{2}+\omega^{3} \wedge \omega^{4} \\
-k_{0} \omega^{3} \wedge \omega^{4}-c_{2} \omega^{2} \wedge \omega^{3}-\omega^{3} \wedge \omega^{1} \\
-\varphi_{43} \wedge \omega^{3}+c_{2} \omega^{2} \wedge \omega^{4}+\omega^{1} \wedge \omega^{2}
\end{array}\right)
$$

where

$$
\left\{\begin{array}{l}
d k_{0}=-a_{4} \omega^{1}-c_{2} k_{0} \omega^{2}-c_{2} \omega^{3}-a_{4} k_{0} \omega^{4}  \tag{12}\\
d a_{4}=-a_{4} \omega^{1}-a_{4} c_{2} \omega^{2}-a_{4} \omega^{3}-a_{4} k_{0} \omega^{4} \\
d c_{2}=-c_{2} \omega^{1}-c_{2} k_{0} \omega^{2}-c_{2} \omega^{3}-c_{2} a_{4} \omega^{4}
\end{array}\right.
$$

In addition, from (7) we have

$$
\left\{\begin{array}{l}
\varphi_{21} \equiv \omega^{2} \bmod \omega^{1}  \tag{13}\\
\varphi_{43} \equiv \omega^{4} \bmod \omega^{3}
\end{array}\right.
$$

An example of such a system is provided by the hyperbolic system associated to the $s=0$ Goursat equations

$$
\begin{aligned}
& u_{y}=\frac{2 \sqrt{u v}}{(x+y)}, \\
& v_{x}=\frac{2 \sqrt{u v}}{(x+y)} .
\end{aligned}
$$

This system is Darboux integrable and is in fact equivalent, as hyperbolic systems, to the linear system $\mathcal{I}_{1,0}$ given in Section 1.5.

Since every system of Class B is Darboux integrable, from the discussion in Section 1.5, we see that as hyperbolic exterior differential systems (though not as symplectic hyperbolic EDS) there are exactly two equivalence classes, corresponding to the $s=0$ Liouville system and the $s=0$ Goursat system.
Class C: The structure equations have the form

$$
\begin{aligned}
d\left(\begin{array}{c}
\omega^{1} \\
\omega^{2} \\
\omega^{3} \\
\omega^{4}
\end{array}\right)= & -\left(\begin{array}{cccc}
0 & 0 & 0 & 0 \\
\varphi_{21} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & \varphi_{43} & 0
\end{array}\right) \wedge\left(\begin{array}{c}
\omega^{1} \\
\omega^{2} \\
\omega^{3} \\
\omega^{4}
\end{array}\right) \\
& +\left(\begin{array}{r}
-k_{0} \omega^{1} \wedge \omega^{2}-\left(a_{3} \omega^{3}+k_{0} \omega^{4}\right) \wedge \omega^{1}-\omega^{1} \wedge \omega^{3} \\
-k_{0} \omega^{3} \wedge \omega^{4}-\left(a_{3} \omega^{3}+k_{0} \omega^{4}\right) \wedge \omega^{2}+\omega^{3} \wedge \omega^{4} \\
\left.-k_{0} \omega^{2}\right) \wedge \omega^{3}-\omega^{3} \wedge \omega^{1} \\
\left.c_{1} \omega^{1}+k_{0} \omega^{2}\right) \wedge \omega^{4}+\omega^{1} \wedge \omega^{2}
\end{array}\right)
\end{aligned}
$$

where

$$
d k_{0}=-k_{0}\left(c_{1}+1\right) \omega^{1}-k_{0}^{2} \omega^{2}-k_{0}\left(a_{3}+1\right) \omega^{3}-k_{0}^{2} \omega^{4}
$$

and

$$
\begin{aligned}
d a_{3} & =k_{0} \varphi_{43}+\left(b_{00}+c_{1}-a_{3}\right) \omega^{1}+a_{3} k_{0} \omega^{2}+a_{33} \omega^{3}-k_{0}\left(2 a_{3}+1\right) \omega^{4} \\
d c_{1} & =k_{0} \varphi_{21}+\left(b_{00}+a_{3}-c_{1}\right) \omega^{3}+c_{1} k_{0} \omega^{4}+c_{11} \omega^{1}-k_{0}\left(2 c_{1}+1\right) \omega^{2}
\end{aligned}
$$

In addition $k_{0} \neq 0$ and so from the last two equations we see that we may restrict to a subbundle defined by $a_{3}=c_{1}=0$. After further computation the structure equations then become

$$
d\left(\begin{array}{c}
\omega^{1}  \tag{14}\\
\omega^{2} \\
\omega^{3} \\
\omega^{4}
\end{array}\right)=\left(\begin{array}{r}
-k_{0} \omega^{1} \wedge \omega^{2}+k_{0} \omega^{1} \wedge \omega^{4}-\omega^{1} \wedge \omega^{3} \\
-\left(\omega^{2}+f_{0} \omega^{3}\right) \wedge \omega^{1}-k_{0} \omega^{2} \wedge \omega^{4}+\omega^{3} \wedge \omega^{4} \\
-\quad k_{0} \omega^{3} \wedge \omega^{4}+k_{0} \omega^{3} \wedge \omega^{2}-\omega^{3} \wedge \omega^{1} \\
-\left(\omega^{4}+f_{0} \omega^{1}\right) \wedge \omega^{3}-k_{0} \omega^{4} \wedge \omega^{2}+\omega^{1} \wedge \omega^{2}
\end{array}\right)
$$

where

$$
\begin{equation*}
d k_{0}=-k_{0}\left(\omega^{1}+k_{0} \omega^{2}+\omega^{3}+k_{0} \omega^{4}\right) \tag{15}
\end{equation*}
$$

and

$$
\begin{equation*}
d f_{0} \equiv k_{0} f_{0}\left(\omega^{2}+\omega^{4}\right) \bmod \omega^{1}, \omega^{3} \tag{16}
\end{equation*}
$$

Remark: Since $k_{0} \neq 0$ the structures of Class C cannot be homogeneous as symplectic hyperbolic systems. The structure equations may be proved to be involutive with Cartan characters $s_{1}=s_{2}=1, s_{3}=s_{4}=0$ (the $\omega^{1}$ and $\omega^{3}$ derivatives of $f_{0}$
may be specified arbitrarily); hence structures of Class C depend on one function of two variables.

We will prove that

Proposition: (i) No structure satisfying (14)-(16) is symplectic linearizable. (ii) On the other hand, every such structure is linearizable as a hyperbolic exterior differential system.

Proof: For the proof, since $\omega^{1}$ and $\omega^{3}$ are integrable we may introduce local coordinates $x, y, u, v$ such that

$$
\left\{\begin{array}{l}
\omega^{1}=A d x \\
\omega^{3}=C d y
\end{array}\right.
$$

and

$$
\left\{\begin{array}{l}
\omega^{2}=(B d u+P d x+Q d y) / A \\
\omega^{4}=(D d v+S d x+T d y) / C
\end{array}\right.
$$

From the structure equations we deduce that

$$
B_{v}=D_{u}=0
$$

so we may introduce new coordinates so that

$$
\left\{\begin{array}{l}
\omega^{2}=(d u+P d x+Q d y) / A \\
\omega^{4}=(d v+S d x+T d y) / C
\end{array}\right.
$$

Our exterior differential system thus models the PDE system

$$
\left\{\begin{align*}
u_{y}+Q(x, y, u, v) & =0  \tag{17}\\
v_{x}+S(x, y, u, v) & =0
\end{align*}\right.
$$

The coordinates $x, y$ are determined up to

$$
(x, y)=(X(\bar{x}), Y(\bar{y}))
$$

and $A, C$ then undergo the transformation

$$
(A, C) \rightarrow\left(X^{\prime}(\bar{x}) A, Y^{\prime}(\bar{y}) C\right)
$$

while $u, v$ undergo

$$
(u, v) \rightarrow\left(X^{\prime}(\bar{x}) u, Y^{\prime}(\bar{y}) v\right) .
$$

Noting that

$$
\Phi=(d u+Q d y) \wedge d x+(d v+S d x) \wedge d y
$$

it follows that the condition that $Q(x, y, u, v)$ and $S(x, y, u, v)$ be non-linear in $u, v$ has intrinsic meaning under symplectic equivalence and measures the non-linearity of the original system.

From the $d \omega^{2}$ and $d \omega^{4}$ equations we have

$$
\begin{aligned}
Q_{v}+A & =0 \\
S_{u}+C & =0
\end{aligned}
$$

and

$$
\begin{aligned}
k_{0}(A T-C Q)+C\left(A_{y}+A Q_{u}\right) & =0 \\
-k_{0}(A S-C P)+A\left(C_{x}+C S_{v}\right) & =0
\end{aligned}
$$

while the $d \omega^{1}$ and $d \omega^{3}$ equations give

$$
\begin{aligned}
Q_{u}+C & =0 \\
S_{v}+A & =0 .
\end{aligned}
$$

Thus

$$
\begin{aligned}
& (Q-S)_{u}=0 \\
& (Q-S)_{v}=0
\end{aligned}
$$

and so

$$
\begin{array}{r}
Q=R+f(x, y) \\
S=R-f(x, y)
\end{array}
$$

for some function $f$. We may then choose new coordinates such that

$$
\left\{\begin{array}{l}
\omega^{2}=(d u+P d x+R d y) / A \\
\omega^{4}=(d v+R d x+T d y) / C
\end{array}\right.
$$

The PDE system (17) becomes

$$
\left\{\begin{array}{l}
u_{y}+R(x, y, u, v)=0  \tag{18}\\
v_{x}+R(x, y, u, v)=0
\end{array}\right.
$$

We will show that: $R$ is necessarily non-linear in $u, v$.
From the $d \omega^{2}$ and $d \omega^{4}$ equations we obtain

$$
\begin{aligned}
R_{u} & =-C \\
R_{v} & =-A
\end{aligned}
$$

while from the $d \omega^{1}$ and $d \omega^{3}$ equations we find

$$
\begin{align*}
A_{u}=k_{0} & A_{v}=-\frac{k_{0} A}{C} \\
C_{v}=k_{0} & C_{u}=-\frac{k_{0} C}{A} . \tag{19}
\end{align*}
$$

Thus

$$
R_{u v}=-k_{0} \neq 0
$$

so that $R$ cannot be linear in $u, v$.
To complete the argument we need to show that systems satisfying (14)-(16) linearize as hyperbolic systems. But in Section 2.2 we have seen that there is only one class of non-linear hyperbolic system having an infinite number of classical conservation laws, namely the class of the $s=0$ Liouville system. We have already seen that this system is of Class B.

We may summarize the situation by the following schematic

| Symplectic equivalence classes of <br> hyperbolic systems having an infinite <br> number of classical conservation laws |
| :--- |



The notation means that B contains exactly two classes, one of which maps to I and the other of which maps to II.

This illustrates again in a rather dramatic way just how an exterior differential system may linearize when we increase its symmetry group.

To complete the story we shall derive a symplectic normal form for equations of Class C.

Proposition: Any hyperbolic exterior differential system of Class $C$ is symplectically equivalent to the exterior differential system generated $b y^{27}$

$$
\left\{\begin{array}{l}
u_{y}=F(x, y) \sqrt{u v}  \tag{20}\\
v_{x}=F(x, y) \sqrt{u v}
\end{array}\right.
$$

Proof: From the expression for $d k_{0}$ in (15) we obtain

$$
\left(k_{0}\right)_{u}=-\frac{k_{0}^{2}}{A} \quad\left(k_{0}\right)_{v}=-\frac{k_{0}^{2}}{C}
$$

Combining these equations with (19) yields the relations

$$
\begin{equation*}
\left(A A_{u}\right)_{u}=0 \quad\left(C C_{v}\right)_{v}=0 \tag{i}
\end{equation*}
$$

(ii)

$$
(A C)_{u}=0 \quad(A C)_{v}=0
$$

From (i) it follows that

$$
\left(R_{v}^{2}\right)_{u u}=0 \quad\left(R_{u}^{2}\right)_{v v}=0
$$

and from (ii) it follows that

$$
\left(R_{u} R_{v}\right)_{u}=\left(R_{u} R_{v}\right)_{v}=0
$$

Holding $x, y$ fixed for the moment we shall prove the following

Lemma: Let $R(u, v)$ be a function that satisfies
(i) $R_{u}^{2}$ is linear in $v$ and $R_{v}^{2}$ is linear in $u$;
(ii) $R_{v} R_{v}=C$ is constant;
then

$$
R=\gamma \sqrt{u+\beta} \sqrt{v+\alpha}+\delta
$$

27) The symplectic form of the EDS that models (20) is

$$
\begin{aligned}
& \Phi=\omega^{1} \wedge \omega^{2}+\omega^{3} \wedge \omega^{4} \\
& \text { where } \omega^{3}=-\frac{1}{2} F(x, y) \sqrt{\frac{u}{v}} d x \quad \omega^{2}=-\frac{2}{F(x, y)} \sqrt{\frac{v}{u}}(d u-F(x, y) \sqrt{u v} d y) \\
& \omega^{3}=-\frac{1}{2} F(x, y) \sqrt{\frac{u}{v}} d y \quad \omega^{4}=-\frac{2}{F(x, y)} \sqrt{\frac{u}{v}}(d v-F(x, y) \sqrt{u v} d x) .
\end{aligned}
$$

where $\alpha, \beta, \gamma, \delta$ are constants.
Proof: By (i)

$$
\begin{aligned}
& R_{u}=a(u) \sqrt{v+\alpha(u)} \\
& R_{v}=b(v) \sqrt{u+\beta(v)}
\end{aligned}
$$

for suitable functions $a(u), \alpha(u), b(v), \beta(v)$. In the following argument we shall assume that $a(u) \alpha(u) \neq 0$ and $b(v) \beta(v) \neq 0$; the cases where these conditions are not satisfied may be handled by (easier) special arguments. Differentiating

$$
\begin{aligned}
& R_{u}^{2}=a(u)^{2}(v+\alpha(u)) \\
& R_{v}^{2}=b(v)^{2}(u+\beta(v))
\end{aligned}
$$

with respect to $v$ and $u$ respectively gives

$$
\begin{aligned}
& 2 R_{u} R_{u v}=a^{2}(u) \\
& 2 R_{v} R_{v u}=b^{2}(v)
\end{aligned}
$$

Multiplying these equations and using (ii) gives

$$
4 C R_{u v}^{2}=a^{2}(u) b^{2}(v)
$$

Thus

$$
\frac{a^{2}(u) b^{2}(v)}{4 C}=R_{u v}^{2}=\frac{a^{4}(u)}{4 R_{u}^{2}}=\frac{a^{2}(u)}{4(v+\alpha(u))}
$$

which implies that

$$
b^{2}(v)=\frac{C}{v+\alpha(u)}
$$

and therefore

$$
\alpha(u)=\alpha \quad \text { is a constant }
$$

Similarly, $\beta(v)=\beta$ is a constant and

$$
R_{v}=\sqrt{C} \sqrt{\frac{u+\beta}{v+\alpha}}
$$

which integrates to give

$$
R=\gamma \sqrt{(u+\beta)(v+\alpha)}+\delta(u), \quad \gamma=2 \sqrt{C}
$$

But then

$$
R_{u}=\sqrt{C} \sqrt{\frac{v+\alpha}{u+\beta}}+\delta^{\prime}(u)
$$

and so $\delta^{\prime}(u)=0$, i.e., $\delta(u)=\delta$ is a constant.

Applying the lemma to $R(x, y, u, v)$ with $x, y$ variable gives

$$
\begin{equation*}
R=\gamma(x, y) \sqrt{u+\beta(x, y)} \sqrt{v+\alpha(x, y)}+\delta(x, y) \tag{21}
\end{equation*}
$$

Hence we can now introduce new coordinates

$$
\begin{aligned}
& \tilde{u}=u+\beta(x, y) \\
& \tilde{v}=v+\alpha(x, y)
\end{aligned}
$$

such that (dropping the tildes)

$$
\begin{aligned}
& \omega^{2}=\left(d u+P d x+\left(R_{0}+\mu(x, y)\right) d y\right) / A \\
& \omega^{4}=\left(d v+T d y+\left(R_{0}+\nu(x, y)\right) d x\right) / C
\end{aligned}
$$

where $R_{0}=\gamma(x, y) \sqrt{u v}$.
To complete the proof we need to show that $\mu(x, y)=\nu(x, y)=0$. To establish this, we return to the structure equation (14). Expanding out the $d \omega^{2}$-equation and using (19), the coefficient of the $[d y \wedge d v]$-term yields the relation

$$
\begin{equation*}
A=-\frac{1}{2} \gamma(x, y) \sqrt{\frac{u}{v}} \tag{22}
\end{equation*}
$$

while the $[d x \wedge d v]$-term implies

$$
P_{v}=0 .
$$

Now the $[d x \wedge d u]$-term gives

$$
\begin{aligned}
-P_{u} & =A+\frac{A_{x}-A_{u} P}{A}+\frac{k_{0}\left(R_{0}+\nu\right)}{C} \\
& =A+\frac{A_{x}-A_{u} P-A_{v}\left(R_{0}+\nu\right)}{A}
\end{aligned}
$$

Differentiating with respect to $v$, keeping in mind (22), yields

$$
\nu(x, y)=0 .
$$

Similarly, from the $d \omega^{4}$-equation we obtain

$$
\mu(x, y)=0 .
$$

Remark: The explicit linearization of the $F$-Goursat system (20) alluded to above is achieved by the mapping

$$
u=p^{2}, \quad v=q^{2}
$$

which transform the $F$-Goursat system into the following linear PDE system

$$
\begin{aligned}
p_{y} & =\frac{1}{2} F(x, y) q, \\
q_{x} & =\frac{1}{2} F(x, y) p .
\end{aligned}
$$

The above result stands in interesting contrast to the fact that the usual $F$-Goursat equation

$$
z_{x y}=F(x, y) \sqrt{z_{x} z_{y}}
$$

cannot be linearized by a contact transformation in $\left(x, y, z, z_{x}, z_{y}\right)$-space.
3.3 Euler-Lagrange representations of hyperbolic systems. In this final section, we shall discuss the question: What is the maximum number of ways in which a given hyperbolic exterior differential system of class $s=0$ may be realized as an EulerLagrange system? Given $(M, \mathcal{I})$ what we are asking for is the maximum dimension of the space $\Sigma$ of orthogonal pairs ( $\Phi, \Psi$ ) of conservation laws. We shall establish the following

Proposition: For hyperbolic systems $\mathcal{I}$ of class $s=0$, the dimension of the space $\Sigma$ is at most 4 , with equality holding if and only if $\mathcal{I}$ is locally diffeomorphic to one of the linear systems $\mathcal{I}_{K, 0}$ discussed in Section 1.5.6

Proof: What we shall actually prove is that the maximum dimension of $\Sigma$ is achieved exactly when the structure equations of $(M, \mathcal{I})$ are

$$
d\left(\begin{array}{c}
\omega^{1} \\
\omega^{2} \\
\omega^{3} \\
\omega^{4}
\end{array}\right)=-\left(\begin{array}{cccc}
\phi_{44}-\phi_{22} & 0 & 0 & 0 \\
\phi_{21} & \phi_{22} & 0 & 0 \\
0 & 0 & \phi_{22}-\phi_{44} & 0 \\
0 & 0 & \phi_{43} & \phi_{44}
\end{array}\right) \wedge\left(\begin{array}{c}
\omega^{1} \\
\omega^{2} \\
\omega^{3} \\
\omega^{4}
\end{array}\right)+\left(\begin{array}{c}
0 \\
\omega^{3} \wedge \omega^{4} \\
0 \\
\omega^{1} \wedge \omega^{2}
\end{array}\right)
$$

where

$$
\left\{\begin{array}{l}
d \phi_{22}=K \omega^{1} \wedge \omega^{3} \\
d \phi_{44}=K \omega^{3} \wedge \omega^{1}
\end{array}\right.
$$

where $K$ is an arbitrary constant. Of course, comparing this with the structure equations at the end of Section 1.5 .6 will then yield the result.

The proof will proceed by calculations very similar to those used in determining hyperbolic systems with infinitely many classical conservation laws. We shall only write down the main steps in the computation.

A pair of conservation laws may be written as

$$
\begin{aligned}
& \Phi=A \omega^{1} \wedge \omega^{2}+C \omega^{3} \wedge \omega^{4} \\
& \Psi=B \omega^{1} \wedge \omega^{2}+D \omega^{3} \wedge \omega^{4}
\end{aligned}
$$

and the orthogonality condition

$$
\Phi \wedge \Psi=0
$$

is

$$
A D+B C=0
$$

Thus we have

$$
B=\frac{H}{C}, \quad D=-\frac{H}{A}
$$

for some function $H$. The closure conditions

$$
d \Phi=d \Psi=0
$$

give

$$
\begin{aligned}
& d A=A \phi_{44}+A_{i} \omega^{i} \\
& d C=C \phi_{22}+C_{i} \omega^{i}
\end{aligned}
$$

with

$$
\begin{cases}A_{3}=C & A_{4}=0  \tag{1}\\ C_{1}=A & C_{2}=0\end{cases}
$$

and

$$
d H=H\left(\phi_{22}+\phi_{44}\right)+H_{i} \omega^{i}
$$

where

$$
\left\{\begin{array}{l}
H_{1}=H\left(C A_{1}-A^{2}\right) / A C  \tag{2}\\
H_{2}=H A_{2} / A \\
H_{3}=H\left(A C_{3}-C^{2}\right) / A C \\
H_{4}=H C_{4} / C
\end{array}\right.
$$

As in Section 2.2 above, from the identities $d(d A) \equiv 0 \bmod \omega^{1}, \omega^{2}$ and $d(d C) \equiv$ $0 \bmod \omega^{3}, \omega^{4}$ we obtain

$$
\left\{\begin{array}{l}
A_{2}=K_{0}-\frac{1}{2}\left(A p_{1}-C q_{3}\right)  \tag{3}\\
C_{4}=K_{0}-\frac{1}{2}\left(C q_{3}-A p_{1}\right)
\end{array}\right.
$$

Then $d(d A) \equiv 0 \bmod \omega^{1}$ and $d(d C) \equiv 0 \bmod \omega^{3}$ gives

$$
\begin{equation*}
d K_{0}=K_{0}\left(\phi_{22}+\phi_{44}\right)+\frac{1}{2}\left(C p_{4} \phi_{21}+A q_{2} \phi_{43}\right)+K_{0 i} \omega^{i} \tag{4}
\end{equation*}
$$

where

$$
K_{0 i} \equiv 0 \bmod A, C, K_{0}, A_{1}, C_{3}
$$

From the identity $d(d H) \equiv 0 \bmod \omega^{1}, \omega^{3}$ we obtain

$$
H\left(p_{4} A^{2}-q_{2} C^{2}\right)=0
$$

If $p_{4} q_{2} \neq 0$, then we may normalize to have $p_{4}=q_{2}=1$ and therefore $A= \pm C$. This imposes conditions beyond (1)-(3) and cuts down the dimension of the solution space. Thus we assume that

$$
p_{4}=q_{2}=0 .
$$

From the identities $d(d A)=d(d C)=0$ we find that

$$
\begin{aligned}
& d A_{1}=A_{1}\left(\phi_{22}-2 \phi_{44}\right)+\left(K_{0}-\frac{1}{2}\left(A p_{1}-C q_{3}\right)\right) \phi_{21}+A q_{1} \phi_{43}+A_{1 i} \omega^{i} \\
& d C_{3}=C_{3}\left(\phi_{44}-2 \phi_{22}\right)+\left(K_{0}-\frac{1}{2}\left(C q_{3}-A p_{1}\right)\right) \phi_{43}+C p_{3} \phi_{21}+C_{3 i} \omega^{i}
\end{aligned}
$$

where

$$
A_{12} \equiv A_{13} \equiv A_{14} \equiv C_{31} \equiv C_{32} \equiv C_{34} \equiv 0 \bmod A, C, K_{0}, A_{1}, C_{3}
$$

This is just as we found in Section 2.2 above. New conditions arise from $d(d H)=0$, which gives

$$
\begin{equation*}
K_{0}=-\frac{1}{2}\left(A p_{1}+C q_{3}\right) \tag{5}
\end{equation*}
$$

and

$$
\left\{\begin{array}{l}
A_{1}=\frac{L_{0}}{C^{3}}  \tag{6}\\
C_{3}=\frac{L_{0}}{A^{3}}
\end{array}\right.
$$

for some function $L_{0}$. From (4) and $d K_{0}=-d\left(\frac{1}{2}\left(A p_{1}+C q_{3}\right)\right)$ we obtain

$$
k_{24}=k_{42}=0
$$

Plugging back into $d(d H)=0$ then gives

$$
\left\{\begin{array}{l}
A^{2} p_{3}+C^{2} p_{1}=0 \\
C^{2} q_{1}+A^{2} q_{3}=0
\end{array}\right.
$$

Again, if $p_{3} q_{1} \neq 0$ we impose further conditions on the system. Thus we assume that

$$
p_{3}=q_{1}=0
$$

which then gives also that

$$
p_{1}=q_{3}=0
$$

Substituting these conditions back into $d K_{0}$ and comparing with (4) gives

$$
k_{14}=k_{32}=0
$$

Now we use the expressions for $A_{1}$ and $C_{3}$ in (6) to obtain

$$
d L_{0}=2 L_{0}\left(\phi_{22}+\phi_{44}\right)+L_{0 i} \omega^{i}
$$

where

$$
L_{02}=L_{04}=0
$$

and

$$
L_{01} \equiv L_{03} \equiv 0 \bmod A, C .
$$

Moreover, the conditions

$$
A_{11} \equiv C_{33} \equiv 0 \bmod A, C
$$

are also forced. Noting (3), (5) and (6), at this point we have proved that $\Sigma$ is finite dimensional.

From $d\left(d \phi_{22}\right)=d\left(d \phi_{44}\right)=0$ we infer that

$$
\begin{aligned}
& d k_{13}=k_{131} \omega^{1}+k_{133} \omega^{3} \\
& d k_{31}=k_{311} \omega^{1}+k_{313} \omega^{3} .
\end{aligned}
$$

Then $d\left(d L_{0}\right)=0$ gives

$$
\begin{equation*}
3\left(k_{13}-k_{31}\right)\left(L_{0}-A^{2} C^{2}\right)+A C\left(A^{2} k_{133}-C^{2} k_{311}\right)=0 \tag{7}
\end{equation*}
$$

If $k_{13} \neq k_{31}$ then the above relation impose further conditions, cutting down $\Sigma$. Thus we need to assume that

$$
k_{13}=k_{31}=k_{00}
$$

We then have

$$
d k_{00}=k_{00 i} \omega^{i}
$$

where

$$
k_{002}=k_{004}=0
$$

Moreover, (7) gives

$$
A^{2} k_{003}-C^{2} k_{001}=0 .
$$

Again, if $k_{003} k_{001} \neq 0$ we impose additional conditions, so we may assume that

$$
k_{001}=k_{003}=0
$$

In other words, $k_{00}$ is a constant, say $K$. At this point we have

$$
\begin{aligned}
& d A=A \phi_{44}+\frac{L_{0}}{C^{3}} \omega^{1}+C \omega^{3} \\
& d C=C \phi_{22}+\frac{L_{0}}{A^{3}} \omega^{3}+A \omega^{1}
\end{aligned}
$$

and

$$
\begin{aligned}
& d H=H\left(\phi_{22}+\phi_{44}\right)-\frac{H\left(A^{2} C^{2}-L_{0}\right)}{A C}\left(\frac{\omega^{1}}{C^{2}}+\frac{\omega^{3}}{A^{2}}\right) \\
& d L_{0}=L_{0}\left(\phi_{22}+\phi_{44}\right)-\frac{\left(A^{4} C^{4}\left(k_{00}-1\right)-3 L_{0}^{2}\right)}{A C}\left(\frac{\omega^{1}}{C^{2}}+\frac{\omega^{3}}{A^{2}}\right)
\end{aligned}
$$

This is a (non-linear) integrable Frobenius system whose solution depends on four constants. Moreover, the structure equations of the exterior differential system are the same as the structure equations of the ideal $\mathcal{I}_{K, 0}$ described in Section 1.5.6. This proves the desired equivalence.

## References

[AK] I. Anderson and N. Kamran. The variational bicomplex for second order partial differential equations in the plane I: generalized Laplace invariants and their applications. preprint, Fall 1993.
$\left[\mathrm{BCG}^{3}\right]$ R. Bryant, S.-S. Chern, R. Gardner, H. Goldschmidt, and P. Griffiths. Exterior differential systems. Mathematical Sciences Research Institute Publications \#18, Springer-Verlag (1991).
$\left[\mathrm{BG}_{1}\right]$ R. Bryant and P. Griffiths. Characteristic cohomology of differential systems (I): General theory. IAS preprint (1993).
$\left[\mathrm{BG}_{2}\right]$ R. Bryant and P. Griffiths. Characteristic cohomology of differential systems (II): Conservation Laws for a Class of Parabolic Equations. IAS preprint (1993).
[CH] R. Courant and D. Hilbert. Methods of Mathematical Physics, Volume I. Interscience, New York (1953), Volume II, Interscience, New York (1962).
[Da] G. Darboux. Leçons sur la théorie general des surfaces, Tome II. Hermann, Paris (1915).
[FX] Ph. Le Floch and Z. Xin. Uniqueness via the adjoint problems for systems of conservation laws. Comm. Pure and Applied Math., 46 (1993), 1499-1533.
[Go] E. Goursat. Leçons sur l'intégration des équations aux dérivées partielles du second ordre à deux variables indépendantes, Tome 1. Hermann, Paris (1896), Tome 2, Hermann, Paris (1898).
[Ka] K. Kakié. A fundamental property of Monge characteristics in involutive systems of non-linear partial differential equations and its application. Math. Ann., 273 (1985), 89-114.
[La] P. Lax. Development of Singularities of Solutions of Nonlinear Hyperbolic Partial Differential Equations. J. Math. Phys., 5 (1964), 611-613.
[Li] S. Lie. Diskussion der Differential Gleichung $s=F(z)$. Arc. Math., 6 (1881), 112-124.
[Ya] D. Yang. Involutive hyperbolic differential systems. Memoirs of the American Math. Society 68, no. 370, (1987).
R. Bryant

Department of Mathematics
Duke University
Durham, NC 27708, USA
P. Griffiths

Institute for Advanced Study
Princeton, NJ 08540, USA
L. Hsu

School of Mathematics
Institute for Advanced Study
Princeton, NJ 08540, USA


[^0]:    * This research was supported in part by NSF Grant DMS 9205222 (Bryant), an NSERC Postdoctoral Fellowship (Hsu), and the Institute for Advanced Study (Griffiths and Hsu). IAS Preprint: 1/25/94.
    ** Part I appeared in Vol. 1, No. 1 of this journal.

[^1]:    20) In most applications, one can find a 1-adapted coframing $\eta$ whose domain is all of $M$ anyway. Then $U^{(k)}$ is actually open and dense in $M^{(k)}$, consisting exactly of the integral elements tangent to integral manifolds which satisfy the "natural" independence condition $\Omega \neq 0$.
