# SHIING-SHEN CHERN <br> Phillip A. Griffiths 

## An inequality for the rank of a web and webs of maximum rank

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze $4^{e}$ série, tome 5, n ${ }^{0} 3$ (1978), p. 539-557.

[http://www.numdam.org/item?id=ASNSP_1978_4_5_3_539_0](http://www.numdam.org/item?id=ASNSP_1978_4_5_3_539_0)
© Scuola Normale Superiore, Pisa, 1978, tous droits réservés.
L'accès aux archives de la revue «Annali della Scuola Normale Superiore di Pisa, Classe di Scienze » (http://www.sns.it/it/edizioni/riviste/annaliscienze/) implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/legal.php). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

# An Inequality for the Rank of a Web and Webs of Maximum Rank (*). 

SHIING-SHEN CHERN (**) - PHILLIP A. GRIFFITHS (***)

dedicated to Hans Lewy

## 1, - Statement of results.

A web is given in a neighborhood $U \subset R^{N}$ by a set of codimension $k$ foliations in general position, a notion we shall make precise in a moment. Web geometry is the study of local diffeomorphism invariants of a web ${ }^{(1)}$; for example, we may ask if it is equivalent to a standard web whose foliations consist of parallel linear spaces of dimension $N-k$. An invariant arises from the consideration of the abelian $q$-equations $(1 \leqslant q \leqslant k)\left({ }^{2}\right)$ associated to the web. In this paper we will be concerned with the abelian equations when $q=k$. Specifically, we will find a bound on the rank or maximum number of linearly independent abelian $k$-equations, and will show that webs of maximal rank give a very special $G$-structure in the projectivized tangent spaces $P T_{x}(x \in U)$. In a future paper we hope to use this to show that such webs have a standard local form, generalizing our previous result in the codimension-one case ( ${ }^{3}$ ).

For simplicity of notation we will carry out our study in detail only in the case $k=2$. Therefore we now agree, until specified otherwise, that a
(*) Research partially supported by NSF Grants MCS 74-23180, A01 and 72-07782.
(**) Department of Mathematics, University of California, Berkeley 94720.
(***) Department of Mathematics, Harvard University, Cambridge 02138.
${ }^{(1)}$ The basic reference is Blaschke-Bol [1].
$\left.{ }^{(2}\right)$ These are defined in general in [4]. The definitions relevant to our present discussion will be given below.
$\left.{ }^{(3}\right)$ Cf. [2], and also [3] for an outline of the main result from [2].
Pervenuto alla Redazione il 27 Giugno 1977.
web will be given in an open set $U \subset R^{2 n}\left({ }^{4}\right)$ by $d$ foliations by codimensiontwo submanifolds. The leaves of the $i$-th foliation will be taken as level sets

$$
u_{i}(x)=\text { const }, \quad v_{i}(x)=\text { const } ;
$$

these functions are defined up to a local diffeomorphism in ( $u_{i}, v_{i}$ )-space. The $i$-th web normal is

$$
\Omega_{i}(x)=d u_{i}(x) \wedge d v_{i}(x) .
$$

Under a diffeomorphism of $\left(u_{i}, v_{i}\right), \Omega_{i}$ is multiplied by a non-vanishing factor so that what is intrinsic is the point

$$
\Omega_{i}(x) \in P\left(\Lambda^{2} T^{*}\right)
$$

the latter being the projective space associated to the vector space of 2 -forms at $x \in U$.

We want to say what it means for the web to be in general position. For this some linear algebra preliminaries are required. It will be convenient not to distinguish between a point $u \in P^{N}=P\left(R^{N+1}\right)$ and its homogeneous coordinate vector $u \in R^{N+1}-\{0\}$. A set of points $u_{1}, \ldots, u_{d} \in P^{N}$ is in general position in case any $k \leqslant N+1$ of them span a $P^{k-1}$; i.e. $u_{i_{1}} \wedge \ldots \wedge u_{i_{k}} \neq 0$ for $1 \leqslant i_{1}<\ldots<i_{k} \leqslant d, k \leqslant N+1$.

When we come to the notion of general position of lines some care is necessary. Denote by $G(1,2 n-1)$ the Grassmannian of lines in $P^{2 n-1}=$ $=P\left(R^{2 n}\right)$. We will identify $G(1,2 n-1)$ with its image under the Plücker embedding

$$
G(1,2 n-1) \hookrightarrow P^{\left(2_{2}^{2 n}\right)-1}=P\left(\Lambda^{2} R^{2 n}\right)
$$

given by sending the line spanned by points $u, v \in P^{2 n-1}$ into $u \wedge v \in P\left(\Lambda^{2} R^{2 n}\right)$. A first guess is that a set of lines

$$
\Omega_{1}, \ldots, \Omega_{d} \in G(1,2 n-1)
$$

should be said to be in general position if any $k \leqslant n$ of them span a $P^{2 k-1}$; i.e. if all

$$
\begin{equation*}
\Omega_{i_{1}} \wedge \ldots \wedge \Omega_{i_{k}} \neq 0 \quad 1 \leqslant i_{1}<\ldots<i_{k} \leqslant d, k \leqslant n . \tag{1.1}
\end{equation*}
$$

This condition is certainly necessary, but for some purposes may not be sufficient.
${ }^{(4)}$ The reason for taking the dimension $N=2 n$ will appear in § 4.

For example $\left(^{5}\right)$, consider a set of four lines $\Omega_{1}, \Omega_{2}, \Omega_{3}, \Omega_{4}$ in $P^{3}$. The condition (1.1) is equivalent to those lines being pairwise skew. Now it is well-known that there are «in general» two lines meeting each of four skew lines in $P^{3}$. To see better what «in general» means recall that a non-singular quadric surface $S$ in $P^{3}$ is doubly ruled by two families of lines, called the $A$-lines and $B$-lines. The $A$-lines (resp. $B$-lines) are pairwise skew, and any $A$-line meets any $B$-line exactly once. All of these facts follow easily by representing $S$ as the image under the Segre embedding

$$
P^{1} \times P^{1} \rightarrow P^{3}
$$

given by

$$
(s, t) \rightarrow[1, s, t, s t]
$$

The $A$-lines and $B$-lines are given by $s=$ const and $t=$ const respectively. Now there is a unique non-singular quadric surface $S$ containing $\Omega_{1}, \Omega_{2}, \Omega_{3}$ as $A$-lines.

For the remaining line $\Omega_{4}$ there are the three possibilities:

$$
\begin{aligned}
& \Omega_{4} \text { meets } S \text { in distinct points } u_{1}, u_{2} \\
& \Omega_{4} \text { is tangent to } S \text { at } u \\
& \Omega_{4} \text { is an } A \text {-line lying in } S .
\end{aligned}
$$

In the first case each of the $B$-lines through $u_{1}, u_{2}$ meets all four $\Omega_{i}$ once, and the second possibility is the limiting case of the first when $u_{1}=u_{2}$. But in the third case there are infinitely many lines meeting the four skew lines $\Omega_{i}$.

In our study we will give a definition of general position motivated by webs arising from non-degenerate algebraic surfaces in $P^{n+1}$. For this we assume first the condition (1.1). Given any $n-1$ of the $\Omega_{i}$, say $\Omega_{1}, \ldots, \Omega_{n-1}$, spanning a $P^{2 n-3}$ we consider any $P^{2 n-5}$ contained in this $P^{2 n-3}$ and the linear projection

$$
\pi: P^{2 n-1}-P^{2 n-5} \rightarrow P^{3}
$$

Our second requirement is:
(1.2) the lines $\pi\left(\Omega_{n}\right), \ldots, \pi\left(\Omega_{d}\right)$ do not all pass through a common point.
${ }^{(5)}$ This observation is due to Ran Donagi.

[^0]A set of lines satisfying (1.1) and (1.2) will be said to be in general position. It is not the case that lines in general position have as Plücker images points in general position in $P^{\binom{2 n}{2}-1}$. The linear algebra subtlety here is crucial in our study.

A set of foliations is in general position in case the normals $\Omega_{i}(x)$ are lines in general position in $P T_{x}^{*}$.

An abelian equation is a relation

$$
\begin{equation*}
\sum_{i} f_{i}\left(u_{i}(x), v_{i}(x)\right) \Omega_{i}(x)=0 \tag{1.3}
\end{equation*}
$$

and the rank $r$ of the web is the maximum number of linearly independent abelian equations. Our first result is a bound on this rank. Namely, define the integer $t$ uniquely by the conditions

$$
\begin{equation*}
t \equiv-d+1 \bmod n-1, \quad 0 \leqslant t \leqslant n-2 \tag{1.4}
\end{equation*}
$$

and set

$$
\begin{equation*}
p_{g}(d, n)=\frac{1}{6(n-1)^{2}}(d-2 n+1+t)(d-n+t)(d-1-2 t) \tag{1.5}
\end{equation*}
$$

Theorem I. The rank of a d-web in $U \subset R^{2 n}$ satisfies

$$
\begin{equation*}
r \leqslant p_{g}(d, n) \tag{1.6}
\end{equation*}
$$

In particular, $r=0$ when $d<2 n$.
This bound may be seen to be sharp. Webs for which equality holds in (1.6) will be said to be of maximal rank. Our remaining results will, in this case, give a particular type of $G$-structure in the projectivized cotangent spaces $P T_{x}^{*}$.

Before stating the next theorem we recall a little algebraic geometry. A ruled surface $S$ in $P^{N}$ may be constructed by taking two skew subspaces $\boldsymbol{P}^{m}, \boldsymbol{P}^{m^{\prime}}\left(m+m^{\prime}=N-1\right)$ spanning $\boldsymbol{P}^{N}$ together with rational curves $\boldsymbol{C} \subset \boldsymbol{P}^{m}$, $C^{\prime} \subset P^{m^{\prime}}$ in projective correspondence and letting $S$ be the surface of $\infty^{1}$ lines obtained by joining corresponding points. In case $N=2 n-1, m=$ $=m^{\prime}=n-1$, and $C$ together with $C^{\prime}$ are rational normal curves we obtain what will be termed a special ruled surface. In suitable coordinates it is the image of $P^{1} \times P^{1}$ under the map

$$
\begin{equation*}
(t, s) \rightarrow\left[t, \ldots, t^{n} ; s t, \ldots, s t^{n}\right] \tag{1.7}
\end{equation*}
$$

The lines $\Omega(t)$ on $S$ are obtained by holding $t$ fixed. They all have a common linear parameter $s$, and are therefore all in projective correspondence with corresponding points spanning a $P^{n-1}(s)$ (where $P^{n-1}(0)=P^{n-1}, P^{n-1}$. $\cdot(\infty)=P^{(n-1}$.

Theorem II. Assume given a d-web of maximal rank in $U \subset R^{2 n}$ where

$$
d \geqslant 2 n+1, \quad n \geqslant 3\left(^{6}\right)
$$

Then there is defined a field of special ruled surfaces

$$
S(x) \subset P T_{x}^{*}
$$

such that the web normals are lines belonging to this surface. In particular the web normals are all in projective correspondence, written

$$
\begin{equation*}
\Omega_{i}(x) \bar{\wedge} \Omega_{j}(x) \quad 1 \leqslant i, j \leqslant d \tag{1.9}
\end{equation*}
$$

with corresponding points spanning a $P^{n-1}$ in $P T_{x}^{*}$.

## 2. - Proof of the bound on the rank.

Suppose that

$$
\begin{equation*}
\sum_{i=1}^{d} f_{i}^{\lambda}\left(u_{i}(x), v_{i}(x)\right) d u_{i}(x) \wedge d v_{i}(x)=0 \quad \lambda=1, \ldots, r \tag{2.1}
\end{equation*}
$$

are $r$ linearly independent abelian equations.
Set $\Omega_{i}=d u_{i} \wedge d v_{i}$ and

$$
Z_{i}(x)=\left[\ldots, f_{i}^{\lambda}\left(u_{i}(x), v_{i}(x)\right), \ldots\right] \in P^{r-1}
$$

The abelian equations (2.1) become

$$
\begin{equation*}
\sum_{i} Z_{i}(x) \otimes \Omega_{i}(x)=0 \tag{2.2}
\end{equation*}
$$

As $x$ varies over $U \subset R^{2 n}, Z_{i}(x)$ traces out a piece of two-dimensional surface $S_{i}$ in $P^{r-1}$. We may take ( $u_{i}, v_{i}$ ) as local coordinates on $S_{i}$. There
$\left.{ }^{( }{ }^{6}\right)$ The rank is zero when $d<2 n$ and is $\leqslant 1$ when $d=2 n$. These cases are excluded.
is a 1 -to- $d$ correspondence

$$
x \rightarrow Z_{1}(x), \ldots, Z_{d}(x)
$$

but because of (2.2) corresponding points $Z_{i}(x)$ are not in general position. At first glance it might appear that there are in fact $\binom{2 n}{2}$ independent linear relations among the $Z_{i}$, but this is not so. Letting $\left\{Z_{1}, \ldots, Z_{d}\right\}$ denote the linear span in $P^{r-1}$ of $Z_{1}(x), \ldots, Z_{d}(x)$ we shall prove that

$$
\begin{equation*}
\operatorname{dim}\left\{Z_{1}, \ldots, Z_{d}\right\} \leqslant d-2 n, \tag{2.3}
\end{equation*}
$$

an estimate which will turn out to be sharp.
To see this we note that since the lines $\Omega_{1}(x), \ldots, \Omega_{d}(x) \in G_{x}(1,2 n-1)$ are in general position we may choose points

$$
\omega_{1}=d u_{1}+\lambda_{1} d v_{1} \in \Omega_{1}, \ldots, \omega_{2 n-2}=d u_{2 n-2}+\lambda_{2 n-2} d v_{2 n-2} \in \Omega_{2 n-2}
$$

such that

$$
\omega_{1} \wedge \ldots \wedge \omega_{2 n-2} \wedge \Omega_{2 n-1} \neq 0
$$

If we multiply (2.2) by $\omega_{1} \wedge \ldots \wedge \omega_{2 n-2}$ the first $2 n-2$ terms drop out and $\boldsymbol{Z}_{2 n-1}$ appears with a non-zero coefficient, i.e.

$$
\begin{equation*}
Z_{2 n-1} \in\left\{Z_{2 n}, \ldots, Z_{d}\right\} \tag{2.4}
\end{equation*}
$$

By symmetry it follows that at most $d-2 n+1$ of $Z_{i}(x)$ are linearly independent, which proves (2.3).

Now the argument proceeds as in the codimension-one case. By (2.4)

$$
\begin{equation*}
Z_{2 n-1}=\varrho_{2 n} Z_{2 n}+\ldots+\varrho_{d} Z_{d} \tag{2.5}
\end{equation*}
$$

Choose ( $u_{2 n-1}, v_{2 n-1}, u_{2 n}, v_{2 n}, \ldots, u_{3 n-2}, v_{3 n-2}$ ) as coordinates on $U$ and differentiate (2.5) to obtain

$$
\begin{equation*}
\frac{\partial Z_{2 n-1}}{\partial u_{2 n-1}}, \frac{\partial Z_{2 n-1}}{\partial v_{2 n-1}} \equiv \frac{\partial Z_{3 n-1}}{\partial u_{3 n-1}}, \frac{\partial Z_{3 n-1}}{\partial v_{3 n-1}}, \ldots, \frac{\partial Z_{d}}{\partial u_{d}}, \frac{\partial Z_{d}}{\partial v_{d}} \bmod Z_{1}, \ldots, Z_{d} . \tag{2.6}
\end{equation*}
$$

Let

$$
P^{k(0)}=\left\{Z_{1}, \ldots, Z_{a}\right\}, \quad P^{k(1)}=\left\{Z_{1}, \frac{\partial Z_{1}}{\partial u_{1}}, \frac{\partial Z_{1}}{\partial v_{1}}, \ldots, Z_{a}, \frac{\partial Z_{d}}{\partial u_{d}}, \frac{\partial Z_{d}}{\partial v_{d}}\right\} \quad, \ldots,
$$

$P^{k(\mu)}=$ span of the $\mu$-th osculating spaces to the surfaces $S_{i}$ at corresponding points. By (2.4) and (2.6)

$$
\begin{aligned}
& \operatorname{dim} P^{k(0)} \leqslant(d-2 n+1)-1 \\
& \operatorname{dim} P^{k(1)} \leqslant(d-2 n+1)+2(d-3 n+2)-1
\end{aligned}
$$

and in general

$$
\begin{align*}
& \operatorname{dim} P^{k(\mu)} \leqslant(d-2 n+1)+2(d-3 n+2)+\ldots+  \tag{2.7}\\
& \\
& \quad+(\mu+1)(d-(\mu+2) n+\mu+1)-1
\end{align*}
$$

Here we agree that zero is put in whenever one of the first $\mu+1$ terms on the right becomes $\leqslant 0$, which obviously happens for large $\mu$.

The $P^{k(\mu)}(x)$ give an increasing sequence of linear subspaces of $P^{r-1}$, which eventually terminates at say $P^{k(\infty)}(x)$. Since $P^{k(\infty)}(x)$ does not change by differentiation, it must be a constant linear subspace, and hence is all of $P^{r-1}$ since the equations (2.1) were assumed linearly independent. By (2.7)

$$
\begin{equation*}
r \leqslant \sum_{\mu \geqslant 0} \max [(\mu+1)\{d-n(\mu+2)+(\mu+1)\}, 0] . \tag{2.8}
\end{equation*}
$$

It remains to identify this sum with the expression (1.5). Write

$$
\tau=\frac{d-2 n+1}{n-1}+\frac{t}{n-1}
$$

where $t$ is determined by (1.4); $\tau$ is an integer. Put

$$
a=d-2 n+1, \quad s=n-1
$$

Then the R.H.S. of (2.8) is

$$
\begin{aligned}
a+2(a-s) & +3(a-2 s)+\ldots+\tau(a-(\tau-1) s)= \\
& =\frac{1}{2} \tau(\tau+1) a-\frac{1}{3}(\tau-1) \tau(\tau+1) s=\frac{1}{6} \tau(\tau+1)\{3 a-2(\tau-1) s\}
\end{aligned}
$$

Now

$$
\begin{aligned}
& (n-1) \tau=d-2 n+1+t \\
& (n-1)(\tau+1)=d-n+t,(n-1)(\tau-1)=d-3 n+2+t \\
& 3 a-2(\tau-1) s=3 d-6 n+3-2(d-3 n+2+t)=d-1-2 t
\end{aligned}
$$

so that the R.H.S. of (2.8) is

$$
\frac{1}{6(n-1)^{2}}(d-2 n+1+t)(d-n+t)(d-1-2 t)=p_{g}(d, n)
$$

according to (1.5). This completes the proof of Theorem I.
When $n=2$, we have $t=0$ and

$$
\begin{equation*}
p_{g}(d, n)=\frac{1}{6}(d-1)(d-2)(d-3)= \tag{2.9}
\end{equation*}
$$

$=$ geometric genus of a smooth surface of degree $d$ in $P^{3}$.

Exactly the same considerations can be carried out for a $d$-web of codimension $k$ in a neighborhood $U \subset R^{k n}, k \leqslant n$. Let $r_{k}$ be its rank, i.e., the maximum number of linearly independent abelian $k$-equations. Then we have

$$
\begin{equation*}
r_{k} \leqslant \pi(d, n, k) \tag{2.10}
\end{equation*}
$$

where

$$
\begin{equation*}
\pi(d, n, k)=\sum_{\mu \geqslant 0} \max \left(\binom{k+\mu-1}{\mu}\{d-(k+\mu) n+k-1+\mu\}, 0\right) \tag{2.11}
\end{equation*}
$$

The first term in $\pi(d, n, k)$ is

$$
d-k n+k-1
$$

Hence we have

$$
\begin{equation*}
\pi(d, n, k)=0 \text { when and only when } d \leqslant k n-k+1 \tag{2.12}
\end{equation*}
$$

The first two terms in $\pi(d, n, k)$ are
$d-k n+k-1+k\{d-(k+1) n+k\}=$

$$
=(k+1) d-k(k+2) n+k^{2}+k-1 .
$$

Hence we have

$$
\begin{equation*}
\pi(d, n, k)=n+k \tag{2.13a}
\end{equation*}
$$

when and only when

$$
\begin{equation*}
d=(k+1) n-(k-1) \tag{2.13b}
\end{equation*}
$$

## 3. - Proof of Theorem II.

By the assumption of maximal rank

$$
\begin{equation*}
\left\{\boldsymbol{Z}_{1}, \ldots, Z_{d}\right\}=\left\{\boldsymbol{Z}_{2 n}, \ldots, Z_{d}\right\}=P^{d-2 n}(x), \tag{3.1}
\end{equation*}
$$

i.e. there are exactly $(2 n-1)$ independent relations among the $Z_{i}(x)$. On the other hand, (2.2) gives what appears to be $\binom{2 n}{2}$ relations, and consequently some of these must be dependent. We will see that the geometrical consequence of this is the presence in $P\left(T_{x}^{*}\right), x \in U \subset R^{2 n}$, of a field of special ruled surfaces. An intermediate step is the normal form (3.16), which we will derive first.

To carry this out we choose $u_{1}, v_{1}, \ldots, u_{n}, v_{n}$ as coordinate system and write, at a fixed point $x_{0} \in U$,

$$
\left\{\begin{array}{l}
d u_{s}=d u_{1}+\sum_{\lambda=2}^{n} A_{s \lambda} d u_{\lambda}+\sum_{\lambda=2}^{n} B_{s \lambda} d v_{\lambda},  \tag{3.2}\\
d v_{s}=d v_{1}+\sum_{\lambda=2}^{n} C_{s \lambda} d u_{\lambda}+\sum_{\lambda=2}^{n} D_{s \lambda} d v_{\lambda}, \quad s=n+1, \ldots, d .
\end{array}\right.
$$

This is possible since by general position all

$$
\frac{\partial\left(u_{s}, v_{s}\right)}{\partial\left(u_{1}, v_{1}\right)} \neq 0
$$

We set

$$
A_{\lambda}=\left(A_{n+1, \lambda}, \ldots, A_{d, \lambda}\right) \in R^{d-n}
$$

and similarly for $B_{\lambda}, C_{\lambda}, D_{\lambda}$.
(3.3) Lemma. The vectors $A_{\lambda}, B_{\lambda}$ are multiples of a vector $E_{\lambda}$, and the $E_{\lambda}$ are linearly independent (here $\lambda=2, \ldots, n$ ).

Proof. The abelian equations (2.2) are

$$
\begin{equation*}
\sum_{\alpha=1}^{n} Z_{\alpha} d u_{\alpha} \wedge d v_{\alpha}+\sum_{s=n+1}^{d} Z_{s} d u_{s} \wedge d v_{s}=0 \tag{3.4}
\end{equation*}
$$

The coefficient of $d u_{\alpha} \wedge d v_{\alpha}$ gives $Z_{\alpha}, \alpha=1, \ldots, n$, as a linear combination of $Z_{s}, s=n+1, \ldots, d$, so that by (3.1) there are at most $n-1$ inde-
pendent relations among $Z_{s}$. In particular, the coefficient of $d u_{1} \wedge d v_{1}$ gives

$$
\begin{equation*}
Z_{1}+\sum_{s=n+1}^{d} Z_{s}=0 \tag{3.5}
\end{equation*}
$$

and the coefficients of $d v_{1} \wedge d u_{\lambda}$ and $d v_{1} \wedge d v_{\lambda}$ give

$$
\left\{\begin{array}{l}
\sum_{s=n+1}^{d} A_{s \lambda} Z_{s}=0  \tag{3.6}\\
\sum_{s=n+1}^{d} B_{s \lambda} Z_{s}=0, \quad \lambda=2, \ldots, n
\end{array}\right.
$$

By the above remark at most $(n-1)$ of the $2(n-1)$ equations (3.6) $)_{\lambda}$ can be independent. In other words, if $R_{\lambda} \subset R^{d-n}$ is the span of $A_{\lambda}, B_{\lambda}$ then

$$
\operatorname{dim}\left(\sum_{\lambda=2}^{n} R_{\lambda}\right) \leqslant n-1 .
$$

The lemma amounts to

$$
\sum_{\lambda=2}^{n} R_{\lambda}=\oplus_{\lambda=2}^{n} R_{\lambda} \cong R^{n-1}
$$

which is implied by

$$
R_{\lambda} \not \subset \sum_{\lambda \neq \gamma} R_{\gamma}
$$

for fixed $\lambda$. If, on the contrary, the equations (3.6) $)_{\lambda}$ are linear combinations of (3.6) $)_{v}$ for $\gamma \neq \lambda$, then taking $\lambda=n$ we will have

$$
\begin{aligned}
& A_{n}=\sum_{\nu=2}^{n-1} a_{\gamma} A_{\nu}+b_{\gamma} B_{\gamma} \\
& B_{n}=\sum_{\gamma=2}^{n-1} c_{\gamma} A_{\nu}+d_{\gamma} B_{\gamma}
\end{aligned}
$$

By (3.2)

$$
d u_{s}=d u_{1}+\sum_{\gamma=2}^{1-n} A_{s \gamma}\left(d u_{\nu}+a_{\nu} d u_{n}+c_{\gamma} d v_{n}\right)+\sum_{\gamma=2}^{n} B_{s \gamma}\left(d v_{\gamma}+b_{\gamma} d u_{n}+d_{\nu} d v_{n}\right)
$$

In the $R^{4}$ defined by

$$
\left\{\begin{array}{l}
d u_{\nu}+a_{\nu} d u_{n}+c_{\gamma} d v_{n}=0,  \tag{3.7}\\
d v_{\gamma}+b_{\gamma} d u_{n}+d_{\nu} d v_{n}=0, \quad \gamma=2, \ldots, n-1,
\end{array}\right.
$$

we have

$$
\begin{equation*}
d u_{s}=d u_{1}, \quad s=n+1, \ldots, d \tag{3.8}
\end{equation*}
$$

contradicting general position. More precisely, the $2(n-2)$ one-forms (3.7) span a $P^{2 n-5}$ in $P^{2 n-1}=P\left(T_{x}^{*}\right)$. This $P^{2 n-5}$, does not meet any of the web normals $\Omega_{1}, \Omega_{n+1}, \ldots, \Omega_{d}$. Under the linear projection

$$
P^{2 n-1}-P^{2 n-5} \xrightarrow{\pi} P^{3}
$$

(3.8) says exactly the lines $\pi\left(\Omega_{1}\right), \pi\left(\Omega_{n+1}\right), \ldots, \pi\left(\Omega_{d}\right)$ all pass through a common point, and this contradicts general position. Thus Lemma (3.3) is proved.

By the lemma we have $A_{\lambda}=\alpha_{\lambda} E_{\lambda}, B_{\lambda}=\beta_{\lambda} E_{\lambda}, \alpha_{\lambda}, \beta_{\lambda}$ not both zero. Replacing $d u_{\lambda}$ by $\alpha_{\lambda} d u_{\lambda}+\beta_{\lambda} d v_{\lambda}$ we obtain

$$
A_{\lambda}=E_{\lambda}, \quad B_{\lambda}=0
$$

in (3.2). After a similar argument applied to $C_{\lambda}, D_{\lambda}$ we may assume

$$
C_{\lambda}=0, \quad D_{\lambda}=F_{\lambda}
$$

so that (3.2) is now

$$
\left\{\begin{array}{l}
d u_{s}=d u_{1}+\sum_{\lambda=2}^{n} E_{s \lambda} d u_{\lambda}  \tag{3.9}\\
d v_{s}=d v_{1}+\sum_{\lambda=2}^{n} F_{s \lambda} d v_{\lambda}
\end{array}\right.
$$

(3.10) Lemma. $E_{\lambda}$ is a non-zero multiple of $F_{\lambda}$.

Proof. By the proof of Lemma (3.3) the $2(n-1)$ vectors $E_{\gamma}, F_{\gamma}$ span an $R^{n-1}$ in $R^{d-n}$. Thus, if $R_{\lambda}$ is the span of $E_{\lambda}, F_{\lambda}$

$$
\operatorname{dim} \sum_{i=2}^{n} R_{\lambda}=n-1
$$

If some $E_{\lambda}$ and $F_{\lambda}$ are linearly independent, i.e.

$$
\operatorname{dim} R_{\lambda}=2,
$$

then for some other $\gamma$ we must have

$$
R_{\nu} \subset \sum_{\lambda \neq \gamma} R_{\lambda}
$$

Taking $\gamma=n$ we obtain a relation

$$
\left\{\begin{array}{l}
E_{n}=\sum_{\gamma=2}^{n-1} a_{\gamma} E_{\gamma}+b_{\gamma} F_{\nu}  \tag{3.11}\\
F_{n}=\sum_{\gamma=2}^{n-1} c_{\nu} E_{\gamma}+d_{\gamma} F_{\gamma}
\end{array}\right.
$$

which we will show leads to a contradiction.
Using (3.9) the coefficients of $d u_{\gamma} \wedge d v_{n}$ and $d u_{n} \wedge d v_{\gamma}$ in (3.4) give

$$
\left\{\begin{array}{l}
\sum_{s=n+1}^{d}\left(E_{s \gamma} F_{s n}\right) Z_{s}=0,  \tag{3.12}\\
\sum_{s=n+1}^{d}\left(E_{s n} F_{s \gamma}\right) Z_{s}=0, \quad 2 \leqslant \gamma \leqslant n-1 .
\end{array}\right.
$$

The coefficient of $d u_{\lambda} \wedge d v_{\gamma}$ gives

$$
\begin{equation*}
\delta_{\gamma}^{\lambda} Z_{\gamma}+\sum_{s=n+1}^{d}\left(E_{s \nu} F_{s \lambda}\right) Z_{s}=0, \quad 2 \leqslant \gamma, \lambda \leqslant n-1 . \tag{3.13}
\end{equation*}
$$

Finally the coefficient of $d u_{n} \wedge d v_{n}$ gives, after we plug in (3.11),

$$
\begin{equation*}
Z_{n}+\sum_{\lambda_{\lambda}, \gamma=2}^{n-1} \sum_{s=n+1}^{d}\left(a_{\lambda} E_{s} \lambda+b_{\lambda} F_{s \lambda} \lambda\right)\left(c_{\gamma} E_{s \gamma}+d_{\gamma} F_{s \gamma}\right) Z_{s}=0 . \tag{3.14}
\end{equation*}
$$

Substituting (3.11) into (3.12) and using (3.13) gives

$$
\left\{\begin{array}{l}
\sum_{\lambda=2}^{n-1} \sum_{s=n+1}^{d}\left(c_{\lambda} E_{s \gamma} E_{s \lambda}\right) Z_{s}=-d_{\gamma} Z_{\gamma},  \tag{3.15}\\
\sum_{\lambda=2}^{n-1} \sum_{s=n+1}^{d}\left(b_{\lambda} F_{s \lambda} F_{s \gamma}\right) Z_{s}=-a_{\gamma} Z_{\gamma}, \quad 2 \leqslant \gamma \leqslant n-1
\end{array}\right.
$$

Expanding out (3.14) and plugging in (3.15) and (3.13) we obtain

$$
Z_{n}-\sum_{\gamma=2}^{n-1}\left(3 a_{\gamma} d_{\gamma}+b_{\gamma} c_{\gamma}\right) Z_{\gamma}=0,
$$

which contradicts the maximal rank assumption (3.1). This proves Lemma (3.10).

We now arrive at our normal form for the $d u_{i}, d v_{i}$. Namely, we may multiply by a scale factor and assume $E_{\lambda}=F_{\lambda}$. If we relabel and define $A_{i \alpha}$ by

$$
\begin{array}{ll}
A_{s \lambda}=E_{s \lambda}, & n+1 \leqslant s \leqslant d, 2 \leqslant \lambda \leqslant n, \\
A_{s 1}=1, & n+1 \leqslant s \leqslant d, \\
A_{\alpha \beta}=\delta_{\alpha \beta}, & 1 \leqslant \alpha, \beta \leqslant n,
\end{array}
$$

then (3.9) becomes

$$
\left\{\begin{array}{l}
d u_{i}=\sum_{\alpha=1}^{n} A_{i \alpha} d u_{\alpha},  \tag{3.16}\\
d v_{i}=\sum_{\alpha=1}^{n} A_{i \alpha} d v_{\alpha}, \quad 1 \leqslant i \leqslant d .
\end{array}\right.
$$

(3.17) Lemma. The vectors $A_{i}=\left[\ldots, A_{i \alpha}, \ldots\right] \in P^{n-1}$ lie on $\infty^{(n-1)(n-2) / 2}$ linearly independent quadrics.

Proof. The basic abelian equation (3.4) gives upon substituting in (3.16)

$$
\begin{equation*}
\sum_{i=1}^{d} A_{i \alpha} A_{i \beta} Z_{i}=0, \quad 1 \leqslant \alpha, \beta \leqslant n \tag{3.18}
\end{equation*}
$$

These are $n(n+1) / 2$ relations among the $Z_{i}$, and by (3.1) only $2 n-1$ of the equations (3.18) can be independent. In other words we have

$$
\frac{n(n+1)}{2}-(2 n-1)=\frac{(n-1)(n-2)}{2}
$$

linearly independent relations

$$
\sum_{\alpha, \beta=1}^{n} k_{\alpha \beta} A_{i \alpha} A_{i \beta}=0, \quad k_{\alpha \beta}=k_{\beta \alpha},
$$

among the coefficients in (3.18), and this gives the lemma.

Now we can complete the proof of Theorem II. Namely, by (3.17) the $A_{i}$ lie on a rational normal curve $C$ in $P^{n-1}$. After a linear change of coordinates we may assume that $C$ is given parametrically by

$$
\begin{equation*}
t \rightarrow\left[t, t^{2}, \ldots, t^{n}\right] \tag{3.19}
\end{equation*}
$$

According to (3.16) we have now written $P^{2 n-1}=P\left(T_{x}^{*}\right)$ as the span of the $P^{n-1}$ determined by the $d u_{\alpha}$ and $P^{\prime n-1}$ determined by the $d v_{\alpha}$, and in each of $P^{n-1}, P^{\prime n-1}$ we have the rational normal curve (3.19) such that setting $t=t_{i}$ gives $d u_{i} \in P^{n-1}$ and $d v_{i} \in P^{\prime n-1}$ respectively. The $i$-th web normal $\Omega_{i}$ is the line $d u_{i}+s d v_{i}$, which is just the line $t=t_{i}$ on the standard ruled surface given parametrically by

$$
(s, t) \rightarrow\left[t, t^{2}, \ldots, t^{n} ; s t, s t^{2}, \ldots, s t^{n}\right]
$$

## 4. - Webs defined by algebraic varieties.

A projective algebraic variety of dimension $k, V_{k} \subset P^{m}$ is non-degenerate in case it does not lie in a $P^{m-1}$. The degree $d$ is the number of intersections with a generic $P^{m-k}$, written

$$
\begin{equation*}
V \cdot P^{m-k}=p_{1}+\ldots+p_{d} \tag{4.1}
\end{equation*}
$$

(Here and in what follows, we frequently omit the index of $V_{k}$.) For nondegenerate $V$, which we will always assume, the $\boldsymbol{p}_{i} \in \boldsymbol{P}^{m-k}$ are in general position (c.f. Lemma 1.8 in [2]).

We continue to denote by $G(m-k, m)$ the Grassmannian of $P^{m-k}{ }_{s}^{\prime}$ in $P^{m}$, and for fixed $p \in P^{m}$ we let $\Sigma(p)$ designate the Schubert variety of all $P^{m-k}{ }_{\text {s }}^{\prime}$ which pass through the point $p$. Note that $\Sigma(p) \cong G(m-k-1$, $m-1$ ) and has codimension $k$ in $G(m-k, m)$. The algebraic variety $V$ defines a web in open sets $U \subset G(m-k, m)$ by specifying the $i$-th web leaf through $P^{m-k}$ to be $\Sigma\left(p_{i}\right)$ where the $p_{i}$ are given by (4.1). The basic geometric object here is the incidence correspondence

$$
\begin{equation*}
I_{V} \subset V \times G(m-k, m) \tag{4.2}
\end{equation*}
$$

defined by $V$, where $I_{V}=\{(p, A): p \in V, A \in G(m-k, m), p \in A \cap V\}$. By taking $V$ and the $p_{i}$ to be defined over the real numbers, we have associated to a projective variety $V_{k} \subset P^{m}$ a $d(=$ degree $V$ ) web of codimension $k$ ( $=\operatorname{dim} V$ ) submanifolds in $U \subset R^{k(m-k+1)}$.

We now wish to verify that the web defined by a non-degerate algebraic variety is non-degenerate according to our definition, which we shall do for a surface $S \subset P^{n+1}$. For this consider the linear projection

$$
\begin{equation*}
\pi: P^{n+1}-P^{n-3} \rightarrow P^{3} \tag{4.3}
\end{equation*}
$$

with center $P^{n-3}$ defined by

$$
\pi(p)=\left(p \wedge P^{n-3}\right) \cdot P^{3}
$$

where $p \wedge P^{n-3}$ is the $P^{n-2}$ spanned by $p \in P^{n+1}-P^{n-3}$ and the center. Under such a projection, $\pi(S)=S^{\prime}$ is a non-degenerate surface in $P^{3}$ of degree

$$
d^{\prime}=d-\# \text { of points in } S \cap P^{n-3} .
$$

The projection induces an inclusion

$$
\begin{equation*}
\pi^{-1}: G(1,3) \rightarrow G(n-1, n+1) \tag{4.4}
\end{equation*}
$$

whose image is the Schubert cycle of all $P^{n-1}$ 's containing the center $P^{n-3}$. Our first observation is that the web in $G(1,3)$ defined by $S^{\prime}$ is the intersection of $\pi^{-1} G(1,3)$ with the web in $G(n-1, n+1)$ defined by $S$, even in case there are finitely many points in $S \cap P^{n-3}$.

Now consider the web in $G(1,3)$ defined by a non-degenerate surface $S^{\prime} \subset P^{3}$. For a generic line $P^{1}$ in $P^{3}$ the intersection

$$
P^{1} \cdot S^{\prime}=p_{1}+\ldots+p_{d^{\prime}}
$$

where the $p_{i}$ are distinct. The Schubert cycle $\Sigma(p)$ consists of all lines passing through $p \in P^{3}$, and under the Plücker embedding

$$
G(1,3) \rightarrow P^{5}
$$

$\Sigma(p)$ is a plane. If $\Sigma(p)$ and $\Sigma(q)$ fail to intersect transversely, then they must have in common a line in $P^{5}$. Any line on $G(1,3)$ is the $\Sigma\left(p, P^{2}\right)$ ( $p \in P^{2} \subset P^{3}$ ) of lines in $P^{3}$ passing through $p$ and contained in $P^{2}$. Consequently $\Sigma(p)$ and $\Sigma(q)$ meet transversely unless $p=q$. From this we deduce that the normals to the web defined by $S^{\prime} \subset P^{3}$ are skew lines in the projectivized cotangent spaces to $G(1,3)$, these being $P^{3^{\prime}}$ s.

Finally, for the web defined by a non-degenerate surface $S$ in $P^{n+1}$, the projection

$$
P^{2 n-1}-P^{2 n-5} \rightarrow P^{3} \quad\left(P^{2 n-1}=P\left(T_{x}^{*}\right)\right)
$$

in the definition of web non-degeneracy corresponds to the transposed differential of the inclusion (4.4) induced by the linear projection (4.3) whose center $P^{n-3}$ contains $p_{1}, \ldots, p_{n-1}$. But since $S^{\prime}=\pi(S)$ is still non-degenerate we deduce that the web defined by $S$ in the neighborhood of a generic $P^{n-1} \in G(n-1, n+1)$ is non-degenerate according to the definition used in Theorems I and II.

Given $V_{k} \subset P^{m}$ we consider a meromorphic $k$-form $\omega$ on $V$ and define the trace $\omega$, a meromorphic $k$-form on the Grassmannian $G(m-k, m)$, by

$$
\omega(A)=\sum_{i=1}^{d} \omega\left(p_{i}(A)\right), \quad A \in G(m-k, m)
$$

where the intersection

$$
A \cdot V=p_{1}(A)+\ldots+p_{d}(A)
$$

for a variable $(m-k)$-plane $A$. In terms of the diagram (4.2)

$$
\omega=\left(\pi_{2}\right)_{*} \pi_{1}^{*} \omega
$$

where $\pi_{1}, \pi_{2}$ are respectively the projections $I_{V} \rightarrow V, I_{V} \rightarrow G(m-k, m)$. The form $\omega$ is a differential of the first kind (d.f.k.) if $\omega$ is holomorphic (cf. § II of [4]). The space of d.f.k. will be denoted by $H^{k, 0}(V)$ and its dimension by $h^{k, 0}(V)$. In case $V$ is non-singular $H^{k, 0}(V)$ are just the holomorphic $k$-forms and $h^{k, 0}(V)$ is the usual Hodge number.

Since there are no holomorphic forms on $G(m-k, m)$, for $\omega$ a d.f.k. we have $\omega=0$, which is Abel's theorem

$$
\begin{equation*}
\sum_{i=1}^{d} \omega\left(p_{i}(A)\right)=0 \tag{4.5}
\end{equation*}
$$

Clearly (4.5) gives an abelian $k$-equation on the web defined by $V$. Conversely, it is not difficult to see that every abelian $k$-equation is of this form, and consequently the rank of the web is equal to $h^{k, 0}(V)$. From Theorem I we deduce the bound

$$
\begin{equation*}
h^{k, 0}(V) \leqslant \pi(d, m-k+1, k) \tag{4.6}
\end{equation*}
$$

on the number of linearly independent d.f.k. of a non-degenerate $V_{k} \subset P^{m}$.
In case $k=1$ and $m=n$ we obtain Castelnuovo's bound (cf. [2])

$$
\begin{equation*}
\pi(C) \leqslant \pi(d, n)=\pi(d, n, 1) \tag{4.7}
\end{equation*}
$$

on the genus of a curve of degree $d$ in $P^{n}$. The curves for which equality holds in (4.1) were extensively discussed in our previous paper [2], where in fact we proved that their properties could be deduced by web-theoretic methods.

When $k=2$ we set $m=n+1$ so that our variety is a surface $S \subset P^{n+1}$ corresponding to a codimension-2 web in $U \subset R^{2 n}$. We denote by $p_{g}(S)$ the number $h^{2,0}(S)$ of d.f.k.; for smooth $S$ this is the geometric genus. Theorem I gives the bound

$$
\begin{equation*}
p_{g}(S) \leqslant p_{g}(d, n)=\pi(d, n, 2) \tag{4.8}
\end{equation*}
$$

This inequality has been proved algebro-geometrically by Joe Harris in his Harvard thesis, which contains general methods of estimating the superabundance (= «number of relations among conditions imposed by ") of linear systems with base conditions imposed.

A special case of (4.7) is ( ${ }^{7}$ )

$$
p_{g}(S)=0 \quad \text { for degree } S<2 n
$$

The general statement for a non-degenerate $V_{k} \subset P^{m}$ is, by (2.12) and with $n=m-k+1$,

$$
\begin{equation*}
h^{k, 0}(V)=0 \quad \text { for } d<k(m-k)+2 \tag{4.8}
\end{equation*}
$$

These bounds are sharp. For example, for each $n \geqslant 2$ there are $K 3$ surfaces $S \subset P^{n+1}$ of degree $2 n$, characterized by having as hyperplane sections canonical curves of genus $n+1$. In general

$$
h^{k, 0}(V) \leqslant 1 \quad \text { for } d=k(m-k)+2
$$

and if $V$ is smooth and if $h^{k, 0}(V)=1$, then $V$ is simply-connected (for $k \geqslant 2$ ) with trivial canonical bundle.

To give another application we first observe that, by (2.13a) and (2.13b), there is, for each $k$ a unique function $m \rightarrow d(m)$ satisfying

$$
\pi(d(m), m-k+1, k)=m+1
$$

$\left(^{7}\right)$ After we mentioned this result to R. Hartshorne, he showed us an algebraicgeometric proof, together with the result that $S$ must be a $K 3$-surface, if $\operatorname{deg} S=2 n$, $p_{g}(S)=1$.
(Notice that $n=m-k+1$ ). For example we have

$$
\begin{cases}d(m)=2 m & \text { when } k=1  \tag{4.9}\\ d(m)=3 m-4 & \text { when } k=2\end{cases}
$$

and in general

$$
\begin{equation*}
d(m)=(k+1) m-(k-1)(k+2) \tag{4.10}
\end{equation*}
$$

Next we remark that (4.6) can be inverted to

$$
\begin{equation*}
d \geqslant d\left(h^{k, 0}, m\right) \tag{4.11}
\end{equation*}
$$

bounding from below the degree of a non-degenerate $V_{k} \subset P^{m}$ with fixed $h^{k, 0}$.
In particular we consider canonical algebraic varieties, defined by the property that their canonical linear system $|K|$ gives a birational and biregular mapping of the abstract variety onto its image in $P^{m}\left(m+1=h^{k, 0}\right)$. For such varieties the degree of the canonical image is

$$
(-1)^{k} c_{1}^{k}
$$

where $c_{1}$ is the 1 -st Chern class, and by combining (4.10) and (4.11) we deduce the bound

$$
\begin{equation*}
h^{k .0}(V) \leqslant \frac{1}{k+1}\left[(-1)^{k} c_{1}^{k}+k^{2}+2 k-1\right] \tag{4.12}
\end{equation*}
$$

on the Hodge number of a canonical variety. For $k=1,2$ we may use (4.9) to obtain

$$
\pi(C) \leqslant \frac{-c_{1}}{2}+1, \quad p_{d}(S) \leqslant \frac{c_{1}^{2}}{3}+\frac{7}{3}
$$

The first is an equality due to $c_{1}=2-2 \pi$, but the second is in general an inequality. It may be compared with Max Noether's estimate

$$
\begin{equation*}
p_{g}(S) \leqslant \frac{c_{1}^{2}}{2}+2 \tag{4.13}
\end{equation*}
$$

valid for any surface. We remark that here the factor $\frac{1}{2}$ ultimately comes from the 2 in

## Clifford's Theorem:

$$
\operatorname{dim}|L| \leqslant \frac{\operatorname{deg} L}{2}
$$

for a special linear series $|L|$ on a curve, and consequently the generalization of (4.13) to higher dimension is

$$
h^{k, 0} \leqslant \frac{(-1)^{k} c_{1}^{k}}{2}+\text { const },
$$

which is sharp for suitable double coverings of $P^{k}$.
The estimates (4.6)-(4.8), (4.11)-(4.13) were consequences of Theorem I. It is of course, interesting to ask whether or not these bounds are sharp, and if so to determine the structure of the extremal varieties defined as those for which equality holds. Now Theorem II gives at least the infinitesimal structure of extremal surfaces $S \subset P^{n+1}$ where the degree $d>2 n$. By continuing the reasoning in the proof of that result we may show that an extremal surface lies in a very special way as a divisor on a threefold $V \subset P^{n+1}$ of minimal degree $n-1$, and this leads to an effective determination of all extremal surfaces. These matters will be taken up in a future paper.

## REFERENCES

[1] W. Blaschke - G. Bol, Geometrie der Gewebe, Springer, Berlin (1938).
[2] S. Chern - P. A. Griffiths, Abel's theorem and webs, to appear in Jahresberichte der deutschen Mathematiker Vereinigung (1978).
[3] S. Chern - P. A. Griffiths, Linearization of webs of codimension one and maximum rank, to appear in Proc. of International Symposium on Algebraic Geometry, Kyoto 1977.
[4] P. A. Griffiths, Variations on a theorem of Abel, Inventiones Math., 35 (1976), pp. 321-390.


[^0]:    35 - Annali della Scuola Norm. Sup. di Pisa

