Chapter III

INFINITESIMAL VARIATION OF HODGE STRUCTURE

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Giving a polarized Hodge structure $\{H_{\mathbf{Z}}, H^{p,q}, Q\}$ of weight one is equivalent to giving a polarized Abelian variety (A,ω), as follows. We set the complex torus A to be $H^{0,1}/H_{\mathbf{Z}}$. Via the identification

$$\begin{aligned}\mathrm{Hom}(\Lambda^2 H_{\mathbf{Z}}, \mathbf{Z}) &\simeq \mathrm{Hom}(\Lambda^2 H_1(A,\mathbf{Z}),\mathbf{Z})\\ &\simeq \mathrm{Hom}(H_2(A,\mathbf{Z}),\mathbf{Z})\\ &\simeq H^2(A,\mathbf{Z}),\end{aligned}$$

the polarization Q corresponds to a class $[\omega] \in H^2(A,\mathbf{Z})$. It can be checked that as a consequence of the two bilinear relations, $[\omega]$ is a positive (1,1)-class. In case the polarization is unimodular, the class $[\omega]$ defines a divisor Θ, unique up to translation, called the *theta divisor* of A. It is the geometric object Θ that plays the major role in classical Hodge theory.

Much of the formal aspects of classical Hodge theory has been extended to higher weights; for example, the asymptotic behavior of a family of Hodge structures. However, one may argue that the applications to geometry have fallen short of expectations, as evidenced by the lack of progress on higher codimensional cycles. We suggest that one reason for this is the following.

OBSERVATION 1. A general Hodge structure of weight $n \geq 2$ (where $h^{2,0} \geq 2$ if $n = 2$) does not come from geometry, so that there is no "natural" way of assigning a geometric object such as Θ to it.

EXPLANATION. Let $D \subset \check{D}$ be the classifying space for the polarized Hodge structures $\{H_{\mathbf{Z}}, H^{p,q}, Q\}$ and $T_h(\check{D})$ the horizontal subbundle, given by

$$\{\xi \in T(\check{D}) \mid \xi F^p \subset F^{p-1}\}.$$

Set

$$I^{(1)} = \mathcal{O}(T_h(\check{D})^{\perp}) \subset \Omega^1_{\check{D}}.$$

Define $I \subset \Omega^{\cdot}_{\check{D}}$ to be the sheaf of ideals generated by the 1-forms $\theta \in I^{(1)}$ and their exterior derivatives $d\theta$. Then I is a $G_{\mathbf{C}}$-invariant differential system on D and hence induces a differential system on $\Gamma\backslash D$, which we also denote by I. In this context the horizontality condition in the definition of a variation of Hodge structure $\phi: S \to \Gamma\backslash D$ amounts to requiring that ϕ be an integral manifold of the differential system I; that is,

$$\phi^*(\theta) = 0 \quad \text{for all} \quad \theta \quad \text{in} \quad I.$$

The point is that if the weight $n \geq 2$ (assuming $h^{2,0} \geq 2$ if $n = 2$), then $I \neq (0)$. Thus $\phi(S)$ can never contain an open subset of $\Gamma\backslash D$. q.e.d.

Since the differential system I appears to be causing the trouble, we will try and use it to extract some geometry. This leads to the topic of today's talk, *the infinitesimal variation of Hodge structure*, a work which is still in the experimental stage.

DEFINITION 2. An *infinitesimal variation of Hodge structure* $V = \{H_{\mathbf{Z}}, H^{p,q}, Q, T, \delta\}$ of weight n is given by a polarized Hodge structure $\{H_{\mathbf{Z}}, H^{p,q}, Q\}$ of weight n together with a vector space T and a linear map (here $q = n-p$)

$$\delta: T \to \bigoplus_{1 \leq p \leq n} \mathrm{Hom}(H^{p,q}, H^{p-1,q+1})$$

satisfying

(1) $\delta_{p-1}(\xi_1)\delta_p(\xi_2) = \delta_{p-1}(\xi_2)\delta_p(\xi_1)$ for $\xi_1, \xi_2 \in T$,

(2) $Q(\delta(\xi)\psi, \eta) + Q(\psi, \delta(\xi)\eta) = 0$ for $\psi \in H^{p,q}$, $\eta \in H^{q+1,p-1}$.

In particular, an infinitesimal variation of Hodge structure is an *integral element* of the horizontal differential system I on D.

EXAMPLE 3. If $\mathfrak{X} \to S$ is a family of polarized algebraic varieties and $\phi: S \to \Gamma\backslash D$ is its period map, then the differential of the period map

$$\phi_*: T_{s_0}(S) \to \bigoplus_{1 \leq p \leq n} \mathrm{Hom}(H^{p,q}, H^{p-1,q+1})$$

gives rise to an infinitesimal variation of Hodge structure in which $T = T_{s_0}(S)$ and $\delta = \phi_*$.

Whereas a polarized variation of Hodge structure has no algebraic invariants (because $G_{\mathbf{R}}$ acts transitively on the classifying space D), an infinitesimal variation of Hodge structure has too many. Of these, five have thus proved useful in geometric problems.

CONSTRUCTION #1. *The kernel of the* n-*th coiterate of the differential.* For $\xi_1, \cdots, \xi_n$ in T consider the map

$$\delta(\xi_1) \cdots \delta(\xi_n): H^{n,0} \to H^{0,n}$$

It follows from (1), (2), and the symmetry of Q that

$$Q(\delta(\xi_1) \cdots \delta(\xi_n)\psi, \eta) = (-1)^n Q(\psi, \delta(\xi_n) \cdots \delta(\xi_1)\eta)$$
$$= Q(\delta(\xi_1) \cdots \delta(\xi_n)\eta, \psi).$$

So in fact $\delta(\xi_1) \cdots \delta(\xi_n)$ is in $\mathrm{Hom}^{(s)}(H^{n,0}, H^{0,n})$, the symmetric maps from $H^{n,0}$ to $(H^{n,0})^*$. Define

$$\delta^n(\xi_1, \cdots, \xi_n) = \delta(\xi_1) \cdots \delta(\xi_n).$$

By (1), δ^n is symmetric in its arguments and so induces a map

$$\delta^n: \mathrm{Sym}^n T \to \mathrm{Hom}^{(s)}(H^{n,0}, H^{0,n}),$$

called the n-th *iterate of the differential.* Note that

$$\mathrm{Hom}^{(s)}(H^{n,0},H^{0,n}) = \mathrm{Sym}^2(H^{n,0})^* .$$

The dual of δ^n is then

$$(\delta^n)^* : \mathrm{Sym}^2 H^{n,0} \to \mathrm{Sym}^n T^* .$$

Our first invariant is

$$\mathcal{Q}(V) = \ker(\delta^n)^* ;$$

it may be viewed as a linear system of quadrics on $PH^{0,n}$.

EXAMPLE 4. Let C be a curve of genus g and $T = H^1(C,\Theta)$. It is well known that this T effectively functions as the tangent space to the moduli space $\mathfrak{M}_g$ at C. Let

$$\delta : T \to \mathrm{Hom}^{(s)}(H^{1,0},H^{0,1})$$

be the differential of the period map at C. This gives the *universal infinitesimal variation of Hodge structure* of C. Then

$$\delta^* : \mathrm{Sym}^2 H^{1,0} \to T^* .$$

Note that $H^{1,0} = H^0(C,K)$ and $T^* = H^0(C,K^2)$. Therefore,

$$\delta^* : \mathrm{Sym}^2 H^0(C,K) \to H^0(C,K^2)$$

is the obvious map and

$$\mathcal{Q}(V) = \text{quadrics through the canonical curve } \phi_K(C) .$$

COROLLARY 5. *A general curve of genus* $g \geq 5$ *can be reconstructed from its universal infinitesimal variation of Hodge structure.*

Proof. A general curve of genus $g \geq 5$ is not hyperelliptic, trigonal, or a plane quintic. By a theorem of Babbage, Enriques, and Petri the canonical image of such a curve is the intersection of all the quadrics through it. Hence, the base locus of $\mathcal{Q}(V)$ is the canonical curve of C. q.e.d.

COROLLARY 6 (Generic Global Torelli). *For* $g \geq 5$ *the extended period map*

$$\phi : \mathfrak{M}_g \to \phi(\mathfrak{M}_g) \subset \Gamma \backslash \mathcal{H}_g$$

has degree one.

Proof. Suppose ϕ has degree $d \geq 1$. Let $Z \in \phi(\mathfrak{M}_g)$ be a regular value. Then ϕ is a local isomorphism around $\phi^{-1}(Z) = \{C_1, \cdots, C_d\}$. It follows that $C_1, \cdots, C_d$ all have the same universal infinitesimal variation of Hodge structure. By Corollary 5 all the C_i's are equal. q.e.d.

Of course, this theorem is a consequence of the well-known global Torelli theorem for curves, but the virtue of this proof is that it does not use the theta divisor Θ and so has a chance to generalize. In general, speaking philosophically, if a variety can be reconstructed from its infinitesimal variation of Hodge structure, then the generic global Torelli theorem holds (cf. Chapters XII and XIII).

CONSTRUCTION #2. *Degeneracy loci of iterates of the differential.*

In this construction we consider the (n–2p)-th iterate δ^{n-2p} of the differential. As in the previous construction

$$\delta^{n-2p} : \mathrm{Sym}^{n-2p} T \to \mathrm{Hom}^{(s)}(H^{n-p,p},H^{p,n-p}) .$$

This induces a map

$$f : T \to \mathrm{Hom}^{(s)}(H^{n-p,p},H^{p,n-p})$$

$$f(\xi) = \delta^{n-2p}(\xi, \cdots, \xi) = \delta(\xi) \cdots \delta(\xi) .$$

Define

$$\Sigma_{p,k} = \{\xi \in PT \mid \mathrm{rank}\ f(\xi) \leq k\} .$$

NOTATION.

$\Sigma_k = \Sigma_{0,k}$ = locus where the n-th iterate has rank $\leq k$.

$\Sigma = \Sigma_{h^{n,0}-1}$ = locus where the n-th iterate drops rank.

EXAMPLE 7. We keep the notations of Example 4. Fix $n = 1$ and $p=0$. Then

$$f(\xi) = \delta(\xi): H^0(C,K) \to H^1(C,\mathcal{O}),$$

and

$$\Sigma_{0,1} = \{\xi \in PH^1(C,\Theta) \mid \operatorname{rank} \delta(\xi) \leq 1\}.$$

PROPOSITION 8. *For a general curve* C *of genus* $g \geq 5$ *the rank one degeneracy locus is the bicanonical image*:

$$\Sigma_{0,1} = \phi_{2K}(C).$$

EXPLANATION. For x in C we have $\xi_x = \phi_{2K}(x)$ in $PH^1(C,\Theta)$. It is an easy lemma that

$$\ker \delta(\xi_*) \supseteq H^0(C,K(-x)).$$

Since $H^0(C,K(-x))$ is a hyperplane in $H^0(C,K)$,

$$\operatorname{rank} \delta(\xi_*) \leq 1.$$

This gives a map

$$\phi_{2K}(C) \to \Sigma_{0,1}.$$

The remainder of the proof of the proposition may be found in [4, §Vc)]. From this result we get another proof of the generic global Torelli theorem for curves.

CONSTRUCTION #3. *Annihilator of a Hodge class.*

Given an infinitesimal variation of Hodge structure $V=\{H_{\mathbf{Z}},H^{p,q},Q,T,\delta\}$ of even weight $n = 2m$ and a Hodge class $\gamma \in H^{m,m}_{\mathbf{Z}}$, we set $H^{m+k,m-k}(-\gamma) = \{\psi \in H^{m+k,m-k} \mid Q(\delta^k(\xi)\psi,\gamma) = 0$ for all ξ in $T\}$.

EXAMPLE 9. Let S be a smooth surface of degree $d \geq 5$ in $\mathbf{P}^3$ and $\omega \in H^2(S,\mathbf{Z})$ its hyperplane class. A *Hodge line* is by definition a cohomology class $\lambda \in H^{1,1}(S) \cap H^2(S,\mathbf{Z})$ satisfying

$$\begin{cases} \lambda^2 = 2-d \\ \lambda \cdot \omega = 1 . \end{cases}$$

We will view $H^{2,0}(-\lambda)$ as a subspace of $H^0(S,\Omega^2)$, the holomorphic 2-forms on S. It is a theorem (cf. [4]) that the base locus of the forms in $H^{2,0}(-\lambda)$ is a line Λ with fundamental class λ:

$$\bigcap_{\psi \in H^{2,0}(-\lambda)} (\psi) = \Lambda . \tag{9.1}$$

REMARK 10. This result generalizes to the following (*loc. cit.*). Let C be a smooth curve of degree d and genus g in $\mathbf{P}^3$ such that $h^1(C,N_{C/\mathbf{P}^3}) = 0$ and let S be a surface of sufficiently large degree $\geq m(d,g)$ containing C. If V is the universal infinitesimal variation of Hodge structure of S and γ is the class of C, then

$$C = \bigcap_{\psi \in H^{2,0}(-\lambda)} (\psi);$$

that is, the curve C may be reconstructed from V and γ. A similar statement holds for a curve in $\mathbf{P}^r$.

As an application of (9.1) we will sketch in Chapter V a proof of the following theorem.

THEOREM 11. *Any smooth surface in* $\mathbf{P}^3$ *with the same universal infinitesimal variation of Hodge structure as the Fermat surface* $F_d = \{x_0^d + x_1^d + x_2^d + x_3^d = 0\}$ *of degree* $d \geq 5$ *is projectively equivalent to* F_d. *Furthermore,*

$$\operatorname{Aut}(V(F_d)) = \operatorname{Aut}(F_d).$$

We will motivate and then give a result concerning Construction #1. Given a family of polarized algebraic varieties, which we think of as $\{X_s\}_{s \in S}$, there is the Kodaira-Spencer map at $s = s_0$

$$\rho: T_{s_0}(S) \to H^1(X,\Theta), \qquad X = X_{s_0}.$$

There is also the cup product map

$$\kappa : H^1(X,\Theta) \to \bigoplus_{1\le p\le n} \mathrm{Hom}(H^{p,q}(X), H^{p-1,q+1}(X)) .$$

It can be shown ([2]) that the differential of the period map is

$$\delta = \kappa\circ\rho : T_{s_0}(S) \to \bigoplus_{1\le p\le n} \mathrm{Hom}(H^{p,q}(X), H^{p-1,q+1}(X)) .$$

Because $\mathfrak{X} \to S$ is a projective family, the hyperplane class ω is constant. Therefore,

$$0 = \delta(\xi)(\omega) = \rho(\xi)\wedge\omega \quad \text{for any} \quad \xi \in T_{s_0}(S) .$$

Recalling the definition of the primitive cohomology, it follows that $\delta(\xi)$ carries a primitive class to a primitive class. Setting $T = T_{s_0}(S)$, and $H^{p,q} = H^{p,q}_{\mathrm{prim}}(X)$, we then have

$$\delta = \kappa\circ\rho : T \to \bigoplus_{1\le p\le n} \mathrm{Hom}(H^{p,q}, H^{p-1,q+1}) .$$

This infinitesimal variation of Hodge structure $V = \{H_{\mathbb{Z}}, H^{p,q}, Q, T, \delta\}$ is said to *arise from geometry*.

The motivation for Construction #1 is as follows. Except when X is a curve, $H^1(X,\Theta)$ is in general not particularly geometric. However, there is a map

$$\rho^n : \mathrm{Sym}^n H^1(X,\Theta) \to H^n(X, \Lambda^n\Theta) .$$

The map ρ^n is the composition

$$\otimes^n H^1(X,\Theta) \to H^n(X,\otimes^n\Theta) \to H^n(X,\Lambda^n\Theta)$$

of two alternating maps and so is symmetric. It is straightforward to verify that the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{Sym}^n T & \xrightarrow{\delta^n} & \mathrm{Hom}^{(s)}(H^{n,0}, H^{0,n}) = \mathrm{Hom}^{(s)}(H^0(K), H^n(\mathcal{O})) \\ & {\scriptstyle\rho^n}\searrow \quad \nearrow{\scriptstyle\kappa\, =\, \text{cup product}} & \\ & H^n(X,K^*) & \end{array}$$

The dual of this is our *basic diagram*:

$$(12) \qquad \begin{array}{ccc} \mathrm{Sym}^2 H^0(X,K) & \xrightarrow{(\delta^n)^*} & \mathrm{Sym}^n T^* \\ & {\scriptstyle\mu}\searrow \quad \nearrow{\scriptstyle\lambda} & \\ & H^0(X,K^2) , & \end{array}$$

where μ is the dual of the cup product and is given by multiplication. To get some feeling for what is going on, we assume that μ is onto. Then quite formally we get the exact sequence

$$(*) \qquad 0 \to \ker\mu \to \ker\delta^{n*} \to \ker\lambda \to 0 .$$

Note that

$$\ker\mu = I_{\phi_K(X)}(2) = \{\text{quadrics through the canonical image } \phi_K(X)\} .$$

So (*) may be rewritten as

$$0 \to I_{\phi_K(X)}(2) \to \mathfrak{Q}(V) \to \ker\lambda \to 0 .$$

When $n = 1$ and $\mathfrak{X} \to S$ is the Kuranishi family of curves, $\ker\lambda = (0)$ and we have

$$\mathfrak{Q}(V) \simeq I_{\phi_K(X)}(2)$$

as in Example 4. In the general situation, to be able to interpret geometrically the *infinitesimal Schottky relations* $\mathfrak{Q}(V)$ one must first understand $\ker\lambda$. We give a partial result in this direction.

For $X = X_{s_0} \subset \mathbf{P}^r$, we take $T = H^0(X, N_{X/\mathbf{P}^r})$. Consider the composition of the Gauss map followed by the Plücker embedding:

$$X \xrightarrow{\gamma} G(n+1, r+1) \xrightarrow{P} P(\Lambda^{n+1} C^{r+1}) = \mathbf{P}^N .$$

If X is locally given by $x(u_1, \cdots, u_n) \in \mathbf{C}^{r+1} - \{0\}$, then

$$(P \circ \gamma)(u) = x(u) \wedge \frac{\partial x}{\partial u_1}(u) \wedge \cdots \wedge \frac{\partial x}{\partial u_n}(u) .$$

Since x is a section of $L = \mathcal{O}_X(1)$,

$$(P \circ \gamma)^* \mathcal{O}_{\mathbf{P}^N}(1) = KL^{n+1} .$$

We define Γ to be the pullback of the hyperplane sections

$$\Gamma = (P \circ \gamma)^* H^0(\mathbf{P}^N, \mathcal{O}(1)) \subset H^0(X, KL^{n+1})$$

and the *Gauss linear system* to be

$$\Gamma_{2K} = \text{image of } \Gamma \otimes H^0(X, KL^{-(n+1)}) \to H^0(X, K^2) .$$

THEOREM 13 (see [1]). $\Gamma_{2K} \subseteq \ker \lambda$.

It follows that $\mu^{-1}(\Gamma_{2K}) \subseteq \ker (\delta^n)^* = \mathcal{Q}(V)$; that is to say, *the Gauss linear system always gives infinitesimal Schottky relations.* This is consistent with the experimentally observed phenomenon that *a geometric understanding of Hodge theory frequently involves dual varieties* (e.g., Andreotti's proof of Torelli for curves, the intermediate Jacobian of the cubic and other Fano threefolds, the generic global Torelli theorem for hypersurfaces (cf. Chapters XII and XIII below)).

Since this talk was given (in November 1981) there has been progress in IVHS. Some of this occurred during the year and is reported on in Chapter XIII. Additional applications of the methods of IVHS by F. Catanese (On the period map of surfaces with $K^2 = \chi = 2$, preprint available from Dipartimento di Matematica, Universitá Degli Studi di Pisa), Donagi and Tu (work in progress), and M. Green (also work in progress) are encouraging signs for a stronger interplay between formal Hodge theory and geometry.

Finally, an excellent exposition [5] of IVHS in the setting of general moduli problems will soon appear.

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