

Appendix

It is enough to prove the relation

$$\xi_{\hat{\mu}}(\epsilon^2) = (-1)^{\langle 2\rho, \hat{\mu} \rangle}$$

when \tilde{G} , and therefore also \hat{G}_0 is simple. ϵ^2 certainly lies in the Cartan subgroup of \hat{G}_0 corresponding to $\hat{\mathfrak{h}}$. \hat{G} acts as a group of automorphisms of $\hat{\mathfrak{g}}$ (of course not faithfully). Let (\cdot, \cdot) be the Killing form on $\hat{\mathfrak{g}}$. \hat{G} preserves the Killing form. Let $e_{\hat{\alpha}}$ be a root vector. Then

$$(e_{\hat{\alpha}}, \epsilon e_{\hat{\alpha}}) \neq 0$$

and

$$(e_{\hat{\alpha}}, \epsilon e_{\hat{\alpha}}) = (\epsilon e_{\hat{\alpha}}, \epsilon^2 e_{\hat{\alpha}}) = \xi_{\hat{\alpha}}(\epsilon^2)(\epsilon e_{\hat{\alpha}}, e_{\hat{\alpha}}) = \xi_{\hat{\alpha}}(\epsilon^2)(e_{\hat{\alpha}}, \epsilon e_{\hat{\alpha}})$$

because the Killing form is symmetric. Thus $\xi_{\hat{\alpha}} = 1$. Since $\langle \rho, \hat{\alpha} \rangle$ is integral and in fact equal to 1 for simple $\hat{\alpha}$,

$$\xi_{\hat{\alpha}} = (-1)^{\langle 2\rho, \hat{\alpha} \rangle}.$$

In particular the assertion is true in those cases that $\hat{L}_- = \hat{L}_+$. Thus when $\hat{\mathfrak{g}}$ is of type E_8 , F_4 , or G_2 .

In the other cases let $\hat{\beta}$ be the top root (as defined in Freudenthal-de Vries). Then $\omega e_{\hat{\beta}} = \eta e_{\hat{\beta}}$, with $\eta = \pm 1$, and $\omega e_{-\hat{\beta}} = \eta e_{-\hat{\beta}}$. Thus, taking the standard isomorphism of the group corresponding to

$$\text{Span}\{e_{\hat{\beta}}, e_{-\hat{\beta}}, [e_{\hat{\beta}}, e_{-\hat{\beta}}]\}$$

with $SL(2, \mathbb{C})$, ω corresponds to the trivial automorphism if $\eta = 1$ and to the inner automorphism determined by

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

if $\eta = -1$. The reflection corresponding to $\hat{\beta}$ is determined by

$$\delta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Thus $\omega \delta \omega^{-1} = \delta$ if $\eta = +1$ and

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

if $\eta = -1$. We regard δ as an element of \hat{G}_0 . It takes every root which is not orthogonal to $\hat{\beta}$ to a negative root. The roots orthogonal to $\hat{\beta}$ are linear combinations of the simple roots orthogonal to $\hat{\beta}$.

We consider the various types separately. \hat{H}_0 will be the group corresponding to the algebra generated by root vectors belonging to roots orthogonal to $\hat{\beta}$. \hat{H}_0 is invariant under

ω . \hat{H} will be $\hat{H}_0 \cup \omega \hat{H}_0$. If \hat{H}_0 is simple we may suppose our assertion is proved for \hat{H} . We use the tables in Fr.-De Vries.

(i) A_l : We identify L_- and \hat{L}_- and L_+ and \hat{L}_+ as in the previous letter.

$$2\rho = (n-1, n-3, \dots, -(n-1)), \quad n = l+1.$$

Here ω , acting on \hat{G}_0 which we may take as simply connected, is

$$A \rightarrow \begin{pmatrix} & & & & 1 \\ & & & -1 & \\ & & 1 & & \\ & -1 & & & \\ \pm 1 & \cdots & & & \end{pmatrix} {}^t A^{-1} \begin{pmatrix} & & & & \pm 1 \\ & & & \cdots & \\ & & 1 & & \\ & -1 & & & \\ 1 & & & & \end{pmatrix} = \gamma {}^t A^{-1} \gamma^{-1}$$

and ϵ is $\gamma\omega$. It is clear that $\omega(\gamma) = \gamma$ so that $\epsilon^2 = \gamma\omega\gamma\omega = \gamma^2$ which is $(-1)^{n-1}I$. It is enough to check the relation

$$\xi_{\hat{\mu}}(\epsilon^2) = (-1)^{\langle \hat{\mu}, 2\rho \rangle}$$

for $\hat{\mu}$ fundamental. Let

$$\hat{\mu} = \left(\overbrace{\frac{n-k}{n}, \dots, \frac{n-k}{n}}^{k \text{ times}}, \overbrace{-\frac{k}{n}, \dots, -\frac{k}{n}}^{n-k \text{ times}} \right)$$

Then

$$\xi_{\hat{\mu}}(\epsilon^2) = (-1)^{k(n-1)}$$

and

$$\langle 2\rho, \hat{\mu} \rangle \equiv k(n-1) \pmod{2}.$$

The result follows.

(ii) $B_l (= \hat{\mathfrak{g}})$. Again the notation is that of my previous letter.

$$2\rho = (2l, 2(l-1), \dots, 2),$$

$\hat{\beta}$ is $\omega_1 + \omega_2$. The automorphism ω is trivial; so $\eta = 1$. \hat{H}_0 corresponds to the diagram of B_l with ρ_3 omitted. Thus if δ_1 is the reflection corresponding to ρ_2 and δ_2 is an automorphism of the group obtained by removing ρ_1 and ρ_3 from the diagram of B_l we may take $\epsilon = \omega\delta\delta_1\delta_2$ and, since δ , δ_1 , and δ_2 commute,

$$\epsilon^2 = \delta^2\delta_1^2\delta_2^2.$$

By induction

$$\xi_{\hat{\mu}}(\epsilon^2) = (-1)^x$$

with

$$\begin{aligned} x &= \langle \hat{\mu}, \omega_1 + \omega_2 \rangle + \langle \hat{\mu}, \omega_1 - \omega_2 \rangle + \langle \hat{\mu}, (0, 0, 2(l-2), \dots, 2) \rangle \\ &= \langle \hat{\mu}, (2, 0, 2(l-2), \dots, 2) \rangle. \end{aligned}$$

The difference between 2ρ and the element of \hat{L}_+ appearing here is

$$(2(l-1), 2(l-1), 0, \dots, 0).$$

The value of this element at any $\hat{\mu}$ is even.

(iii) $C_l(= \hat{\mathfrak{g}})$

$$2\rho = (2l-1, 2l-3, \dots, 1)$$

$\hat{\beta}$ is $2\omega_1$. ω is again the trivial automorphism. \hat{H}_0 is now the group obtained by removing ω_1 from the Dynkin diagram. If δ_1 is the element of \hat{H}_0 taking roots to their negatives we may take $\epsilon = \omega\delta_1$ so that $\epsilon^2 = \delta^2\delta_1^2$. By induction

$$\xi_{\hat{\mu}}(\epsilon^2) = (-1)^x$$

with

$$\begin{aligned} x &= \langle \hat{\mu}, \omega_1 \rangle + \langle \hat{\mu}, (0, 2l-3, \dots, 1) \rangle \\ &= \langle \hat{\mu}, (1, 2l-3, \dots, 1) \rangle \end{aligned}$$

The difference between 2ρ and $(1, 2l-3, \dots, 1)$ is $(2(l-1), 0, \dots, 0)$. The value of this element of L_+ at any $\hat{\mu}$ in \hat{L}_+ is even.

(iv) $D_l(= \hat{\mathfrak{g}})$. Here

$$\rho = (2l-2, 2l-4, \dots, 0).$$

$\hat{\beta}$ is $\omega_1 + \omega_2$. We take $l > 3$. We take \hat{H}_0 by removing ρ_4 from the Dynkin diagram. We may take

$$e_{\hat{\beta}} = [e_{\rho_4} [e_{\rho_5} [\dots [e_{\rho_l} [e_{\rho_3} [\dots [e_{\rho_{l-1}} [e_{\rho_2} [e_{\rho_l}, e_{\rho_1}] \dots] \dots] \dots] \dots] \dots] \dots]$$

Now

$$[e_{\rho_2} [e_{\rho_l}, e_{\rho_1}]] + [e_{\rho_1}, [e_{\rho_2}, e_{\rho_l}]] + [e_{\rho_l}, [e_{\rho_1}, e_{\rho_2}]] = 0.$$

the last term is 0; so

$$[e_{\rho_2} [e_{\rho_l}, e_{\rho_1}]] = [e_{\rho_1}, [e_{\rho_l}, e_{\rho_2}]].$$

If l is even, ω is trivial and η is 1. If l is odd it follows from these calculations that η is also 1, because $\omega e_{\rho_2} = e_{\rho_1}$ and $\omega e_{\rho_1} = e_{\rho_2}$. Let δ_1 be the reflection corresponding to ρ_3 and let δ_2 be that element of the Weyl group of \hat{H}_0 taking positive roots to negative roots. Take $\epsilon = \omega\delta_1\delta_2$. Then by induction

$$\xi_{\hat{\mu}}(\epsilon^2) = (-1)^x$$

with

$$x = \langle \hat{\mu}, \omega_1 + \omega_2 \rangle + \langle \hat{\mu}, \omega_1 - \omega_2 \rangle + \langle \hat{\mu}, (0, 0, 2l - 6, \dots, 0) \rangle = \langle \hat{\mu}, (2, 0, 2l - 6, \dots, 0) \rangle.$$

The difference between ρ and $(2, 0, 2l - 6, \dots, 0)$ is $(2l - 4, 2l - 4, 0, \dots, 0)$ which takes an even value at every $\hat{\mu}$. If $l = 4$ the induction argument cannot be used. In this case δ_2 is the product of the reflection corresponding to ρ_1 and that corresponding to ρ_2 . Thus

$$\xi_{\hat{\mu}}(\delta_2^2) = (-1)^y$$

with

$$y = \langle \hat{\mu}, \omega_3 - \omega_4 \rangle + \langle \hat{\mu}, \omega_3 + \omega_4 \rangle.$$

This justifies the use of the induction.

E_6 : The toproot is

$$\begin{array}{cccccc} & & & 2 & & \\ & & & | & & \\ 1 & 2 & 3 & 2 & 1 & \end{array}$$

The scheme $\langle \hat{\beta}, \hat{\alpha} \rangle$ is given by

$$\begin{array}{ccccccccc} & & & & & & 4 - 3 = 1 & & & \\ & & & & & & | & & & \\ & & & & & & | & & & \\ & & & & & & | & & & \\ \cdot & & \cdot & & \cdot & & \cdot & & \cdot & \\ 2 - 2 = 0 & 4 - 3 - 1 = 0 & 6 - 2 - 2 - 2 = 0 & 4 - 3 - 1 = 0 & 2 - 2 = 0 & & & & & \end{array}$$

Thus \hat{H}_0 corresponds to A_5 . Using the table of roots in Freudenthal-de Vries, we see that

$$e_{\hat{\beta}} = [e_{\rho_2}, [e_{\rho_6}, [e_{\rho_4}, [e_{\rho_5}, [e_{\rho_6}, [e_{\rho_2}, [e_{\rho_3}, [e_{\rho_1}, [e_{\rho_5}, [e_{\rho_4}, [e_{\rho_6}], \dots]]]]]]]]]$$

Applying ω we get

$$[e_{\rho_2}, [e_{\rho_6}, [e_{\rho_5}, [e_{\rho_4}, [e_{\rho_6}, [e_{\rho_2}, [e_{\rho_1}, [e_{\rho_3}, [e_{\rho_4}, [e_{\rho_5}, [e_{\rho_6}], \dots]]]]]]]]]$$

Now

$$[e_{\rho_4}, [e_{\rho_5}, e_{\rho_6}]] + [e_{\rho_6}, [e_{\rho_4}, e_{\rho_5}]] + [e_{\rho_5}, [e_{\rho_6}, e_{\rho_4}]] = 0,$$

so that

$$[e_{\rho_4}, [e_{\rho_5}, e_{\rho_6}]] = [e_{\rho_5}, [e_{\rho_4}, e_{\rho_6}]].$$

For the same reason the interchange of ρ_1 and ρ_3 has no effect. Neither has the other interchange of ρ_4 and ρ_5 . Thus $\eta = 1$. By induction

$$\xi_{\hat{\mu}(\epsilon^2)} = (-1)^{\langle \hat{\mu}, \beta + 2\rho_0 \rangle},$$

where β is the root dual to $\hat{\beta}$. ρ_0 is one-half the sum of the positive roots for the A_5 obtained by throwing out the root ρ_2 . From the table of fundamental wts. (ρ is the sum of these) we find that

$$\begin{array}{cccccc}
 & & & & \cdot 22 & \\
 2\rho & \longleftrightarrow & \cdot 16 & \cdot 30 & \cdot 42 & \cdot 30 & \cdot 16 \\
 & & & & \cdot 0 & & \\
 2\rho_0 & \longleftrightarrow & \cdot 5 & \cdot 8 & \cdot 9 & \cdot 8 & \cdot 5
 \end{array}$$

Thus $2\rho - \beta - 2\rho_0$ corresponds to

$$\begin{array}{cccccc}
 & & & & \cdot 20 & \\
 & & & & \cdot 10 & \cdot 20 & \cdot 30 & \cdot 20 & \cdot 10
 \end{array}$$

and takes even values on every $\hat{\mu}$.

E_7 : In this case ω is trivial. The toproot is

$$\begin{array}{cccccc}
 & & & & \cdot 2 & \\
 \cdot 1 & \cdot 2 & \cdot 3 & \cdot 4 & \cdot 3 & \cdot 2
 \end{array}$$

We give $(\hat{\beta}, \hat{\alpha})$, α simple, schematically. Note that in this case \mathfrak{g} is also of type E_7 . The normalization is such that $(\hat{\alpha}, \hat{\alpha}) = 1$.

$$\begin{array}{cccccc}
 & & & & \cdot & \\
 & & & & 4 - 4 = 0 & \\
 \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 2 - 2 = 0 & 4 - 4 = 0 & 6 - 6 = 0 & 8 - 8 = 0 & 6 - 6 = 0 & 4 - 3 = 1
 \end{array}$$

Thus \hat{H}_0 is of type D_6 . Let $2\rho_0$ be the sum of the positive roots for this D_6 . We have to show that $\rho - \beta - 2\rho_0$ is a sum of simple roots in which each root enters an even number of times. Using the tables as before, we obtain

$$\begin{array}{cccccc}
 & & & & \cdot 49 & \\
 2\rho & \longleftrightarrow & \cdot 27 & \cdot 52 & \cdot 75 & \cdot 96 & \cdot 66 & \cdot 34 \\
 & & & & \cdot 15 & & & \\
 2\rho_0 & \longleftrightarrow & \cdot 10 & \cdot 18 & \cdot 24 & \cdot 28 & \cdot 15 & \cdot 0
 \end{array}$$

Everything checks.

As an extra check let's work out G_2 and F_4 along these lines. Again the Lie algebras and their duals are the same.

F_4 : Toproot is

$$2 - 3 \Rightarrow 4 - 2.$$

ω is of course trivial. The diagram is

$$\begin{array}{ccc} 2 & 2 & 1 & 1 \\ \bullet - - - \bullet & \Rightarrow & \bullet - - - \bullet \\ -1 & -1 & -1/2 \end{array}$$

Thus $(\hat{\beta}, \hat{\alpha})$ is given by

$$\begin{array}{ccc} 2 & 2 & 1 & 1 \\ \bullet & - - - - - \bullet & \Rightarrow & \bullet & - - - - - \bullet \\ 4 - 3 = 1 & 6 - 2 - 4 = 0 & 4 - 1 - 3 = 0 & 2 - 4/2 = 0 \end{array}$$

\hat{H}_0 is in this case C_3 and its dual is B_3 .

$$2\rho_0 \quad \longleftrightarrow \quad \bullet - - - - \bullet \Leftarrow \bullet - - - - \bullet \\ 0 \qquad \qquad \qquad 9 \quad 8 \qquad \qquad \qquad 5$$

Note that in passing to the dual the direction of the arrow is reversed.

$$\begin{array}{ccc} 22 & 42 & 30 & 16 \\ 2\rho & \longleftrightarrow & \bullet - - - - \bullet \Leftarrow \bullet - - - - \bullet \\ & & & \\ & & 2 & 3 & 2 & 1 \\ \beta & \longleftrightarrow & \bullet - - - - \bullet \Leftarrow \bullet - - - - \bullet \end{array}$$

because $(\beta, \rho_2) = (\beta, \rho_4) = (\beta, \rho_3) = 0$. Note again that arrows are reversed. Note also that there is only one positive root orthogonal to ρ_2, ρ_3 and ρ_4 . In any case it checks.

G_2 : Toproot is

$$\begin{array}{cc} 2 & 3 \\ \bullet & \Rightarrow \bullet \end{array}$$

The diagram is

$$\begin{array}{ccc} 3 & & 1 \\ \bullet & \Leftarrow & \bullet \\ & & -3/2 \end{array}$$

$(\hat{\beta}, \hat{\alpha})$ is given by

$$6 - \overset{\bullet}{\frac{9}{2}} = \frac{3}{2} \quad \Rightarrow \quad 3 - \overset{\bullet}{\frac{3}{2}} \cdot 2 = 0$$

Thus, in this case,

$$2\rho_0 \longleftrightarrow \begin{array}{c} \bullet \Leftarrow \bullet \\ 0 \quad 1 \end{array}$$

$$2\rho \longleftrightarrow \begin{array}{c} \bullet \Leftarrow \bullet \\ 10 \quad 6 \end{array}$$

and, since β annihilates the root on the right

$$\beta \longleftrightarrow 2 \Leftarrow 1.$$

Everything checks in this case also.