$\psi(X) = X + o(X)$, $\psi(x) = \sum_{n \le x} \Lambda(n)$

and in general

$\psi_{q,l}(x) = \frac{1}{\varphi(q)} X + o(X)$

for $(l, q) = 1$; $\quad \psi_{q,l}(x) = \sum_{\substack{n \equiv l \,(q) \\ n \le X}} \Lambda(n)$

Define $\vartheta(x) = \sum_{p \le x} \log p$, $\vartheta_{q,l}(x) =$

$= \sum_{\substack{p \equiv l(q) \\ p \le x}} \log p$, and $\pi(x) = \sum_{p \le X} 1$,

$\pi_{q,l}(X) = \sum_{\substack{p \equiv l\,(q) \\ p \le X}} 1.$

We have

$$\vartheta(X) = \psi(X) + O(\sqrt{X}) = X + o(X),$$

and generally

$$\vartheta_{q,l}(x) = \psi_{q,l}(x) + O(\sqrt{x}) = \frac{1}{\varphi(q)} X + o(X).$$

Now we write

$$\pi(X) = \int_{\frac{3}{2}}^{x} \frac{1}{\log t}\, d\vartheta(t) =$$

$$= \frac{\vartheta(X)}{\log X} + \int_{2}^{x} \frac{\vartheta(t)}{t \log^2 t}\, dt,$$

Here
$$\int_2^x \frac{\vartheta(t)}{t\log^2 t}\,dt = O\left(\int_2^x \frac{dt}{\log^2 t}\right) =$$
$$O\left(\frac{x}{\log^2 x}\right),$$

So
$$\pi(x) = \frac{\vartheta(x)}{\log x} + O\left(\frac{x}{\log^2 x}\right)$$
$$= \frac{x}{\log x} + o\left(\frac{x}{\log x}\right),$$

The Prime Number Theorem. In the same way we obtain from the relation
$$\pi_{q,l}(x) = \int_{\frac{3}{2}}^x \frac{1}{\log t}\,d\vartheta_{q,l}(t),$$

that
$$\pi_{q,l}(x) = \frac{1}{\varphi(q)}\frac{x}{\log x} + o\left(\frac{x}{\log x}\right),$$

The P.N.T. for the arithmetic progression.

For later use, we also note that if we write: $\vartheta(x) = x + R(x)$,

then
$$\pi(x) = \int_2^x \frac{dt}{\log t} + \frac{R(x)}{\log x} + \int_2^x \frac{R(t)}{t\log^2 t}\,dt.$$

To obtain sharper estimations we need to have more information about how close the zeros ρ may come to the line $\sigma = 1$

We had for $\zeta(s)$

$$-\frac{\zeta'}{\zeta}(s) = c + \frac{1}{z-1} - \sum_{\rho}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right) - \sum_{n=1}^{\infty}\left(\frac{1}{s+2n} - \frac{1}{2n}\right).$$

We may assume $s = \sigma + it$ with $1 < \sigma < 5$, and $|t| \geq 1$ (since it is easy to show that in this case all $|\gamma| > 1$ if $\rho = \beta + i\gamma$ denotes the zeros. Taking real parts in the above equation, we get

$$R\left(-\frac{\zeta'}{\zeta}(s)\right) = -\sum_{\rho}\frac{\sigma-\beta}{(\sigma-\beta)^2+(t-\gamma)^2} + O(\log(2+|t|)),$$

We consider a zero $\rho_0 = \beta_0 + i\gamma_0$,

and consider the expression:

$$R\left\{-3\frac{\zeta'}{\zeta}(\sigma) - 4\frac{\zeta'}{\zeta}(\sigma+i\gamma_0) - \frac{\zeta'}{\zeta}(\sigma+2i\gamma_0)\right\} \geq 0,$$

since $-\frac{\zeta'}{\zeta}(s) = \sum\frac{\Lambda(n)}{n^s}$, and $3 + 4\cos\theta + \cos 2\theta \geq 0$.

Here $-\frac{\zeta'}{\zeta}(\sigma) = \frac{1}{\sigma-1} + O(1)$,

$$\mathcal{R}\left(-\frac{\zeta'}{\zeta}(\sigma+2i\gamma_0)\right) \leq O(\log(2+|\gamma_0|)),$$

and

$$\mathcal{R}\left(-\frac{\zeta'}{\zeta}(\sigma+i\gamma_0)\right) \leq \frac{-1}{\sigma-\beta_0} + O(\log(2+|\gamma_0|)).$$

Thus

$$\frac{3}{\sigma-1} - \frac{4}{\sigma-\beta_0} + O(\log(2+|\gamma_0|)) \geq 0,$$

or

$$\frac{4}{\sigma-\beta_0} - \frac{3}{\sigma-1} \leq a\log(2+|\gamma_0|),$$

with some absolute constant $a>0$. If we put $\sigma = 1+6(1-\beta_0)$, we get

$$\frac{1}{14}\,\frac{1}{1-\beta_0} \leq a\log(2+|\gamma_0|),$$

or

$$1-\beta_0 \geq \frac{1}{14a\log(2+|\gamma_0|)},$$

$$\text{or } \beta_0 \leq 1 - \frac{\alpha}{\log(2+|\gamma_0|)}.$$

with some absolute positive constant α. Thus $\zeta(s)$ has no zeros in the region

$$\sigma > 1 - \frac{\alpha}{\log(2+|t|)}.$$

We showed last time that

$$\int_1^x \psi(t)\,dt = \frac{x^2}{2} - \sum_\rho \frac{x^{\rho+1}}{\rho(\rho+1)} + O(x).$$

We now see that

$$\left|\sum_\rho \frac{x^{\rho+1}}{\rho(\rho+1)}\right| \leq x^2 \sum_\gamma \frac{e^{-\frac{\alpha \log x}{\log(2+|\gamma|)}}}{|\gamma|^2}$$

$$< 9x^2 \sum_\gamma \frac{e^{-\frac{\alpha \log x}{\log 2+|\gamma|}}}{(2+|\gamma|)^2} =$$

$$9x^2 \sum_\gamma \frac{1}{(2+|\gamma|)^{\frac{3}{2}}} e^{-\frac{\alpha \log x}{\log(2+|\gamma|)} - \frac{1}{2}\log(2+|\gamma|)} <$$

$$< 9x^2 e^{-\sqrt{2\alpha \log x}} \cdot \sum_\gamma \frac{1}{(2+|\gamma|)^{\frac{3}{2}}}$$

$$= O\left(x^2 e^{-\sqrt{2\alpha \log x}}\right).$$

Thus

$$\int_1^x \psi(t)\,dt = \frac{x^2}{2} + O\left(x^2 e^{-\sqrt{2\alpha \log x}}\right).$$

From this

$$\psi(x) \leq \frac{1}{h}\int_x^{x+h} \psi(t)\,dt = x + \frac{h}{2} +$$

$$+ O\left(\frac{x^2}{h} e^{-\sqrt{2\alpha \log x}}\right) = x + O\left(x e^{-\frac{1}{2}\sqrt{2\alpha \log x}}\right),$$

where we have chosen $h = x e^{-\frac{1}{2}\sqrt{2\alpha \log x}}$.
Considering in the same way

$$\psi(x) \geq \frac{1}{h}\int_{x-h}^{x} \psi(t)\,dt \quad , \text{ we get}$$

$$\psi(x) = x + O(x e^{-\alpha' \sqrt{\log x}}),$$

with $\alpha' = \frac{1}{2}\sqrt{2\alpha}$.

This gives of course

$$\vartheta(x) = x + O(x e^{-\alpha' \sqrt{\log x}})$$

and finally

$$\pi(x) = \int_{2}^{x} \frac{dt}{\log t} + O(x e^{-\alpha' \sqrt{\log x}}).$$

This is the Prime Number Theorem with the remainder term first proved by de la Vallée Poussin in 1898. He was the first to obtain a zero free region near $\sigma = 1$, and so an explicit remainder term.

When we use the same approach for the $L(s, \chi)$, there are some new difficulties in the case of the quadratic characters χ.

Let us consider the case χ even for simplicity, χ odd can be handled in a similar way. We saw last time that

$$\xi(s,\chi) = \varepsilon_\chi \pi^{-\frac{s}{2}} q^{\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) L(s,\chi)$$

$$= c' e^{cs} \prod_\rho \left(1-\frac{s}{\rho}\right) e^{\frac{s}{\rho}},$$

where the ρ runs over the zeros of $L(s,\chi)$ in the strip $0<\sigma<1$.

This gives

$$-\frac{L'}{L}(s,\chi) = c_\chi - \sum_\rho \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right) -$$

$$-\frac{1}{s} - \sum_{n=1}^{\infty}\left(\frac{1}{s+2n} - \frac{1}{2n}\right) =$$

$$c'_\chi - \sum_{|\gamma|\le 1} \frac{1}{s-\rho} - \sum_{|\gamma|>1}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right)$$

$$-\frac{1}{s} - \sum_{n=1}^{\infty}\left(\frac{1}{s+2n} - \frac{1}{2n}\right).$$

Putting $s=2$, we observe that

$$\left|\frac{L'}{L}(2,\chi)\right| \leq -\frac{\zeta'}{\zeta}(2).$$

Also, from the formula

$$N(T,\chi) = \frac{T}{2\pi}\left(\log\frac{q(1+T)}{2\pi} - 1\right) + O(\log q(2+T)),$$

proved last time, we find easily

$$\sum_{|\gamma|\leq 1} \frac{1}{2-\rho} = O(\log q)$$

and

$$\sum_{|\gamma|>1} \frac{2}{(2-\rho)\rho} = O(\log q).$$

From this we find $c'_\chi = O(\log q)$.

We may now consider

$$R\left\{-3\frac{\zeta'}{\zeta}(\sigma) - 4\frac{L'}{L}(\sigma+i\gamma_0,\chi) - \frac{L'}{L}(\sigma+2i\gamma_0,\chi^2)\right\} \geq 0,$$

we assume that $\gamma_0 \neq 0$ if χ is a quadratic character, at first assume that $|\gamma_0| \geq \frac{1}{\log q}$ if χ is quadratic, then for $1 < \sigma < 5$

$$-\frac{\zeta'}{\zeta}(\sigma) = \frac{1}{\sigma-1} + O(1),$$

also

$$-R\frac{L'}{L}(\sigma+i\gamma_0,\chi)\le\frac{-1}{\sigma-\beta_0}+O(\log q(2+|\gamma_0|)),$$

and

$$-R\frac{L'}{L}(\sigma+2i\gamma_0,\chi^2)\le O(\log q(2+|\gamma_0|)).$$

This gives again

$$\frac{3}{\sigma-1}-\frac{4}{\sigma-\beta_0}+O(\log q(2+|\gamma_0|))\ge 0,$$

and we proceed as in the case of $\zeta(s)$, choosing $\sigma=1+6(1-\beta_0)$

and obtain

$$\beta_0<1-\frac{\alpha}{\log q(2+|\gamma_0|)}.$$

If χ is quadratic and $0<|\gamma_0|<\frac{1}{\log q}$, $\frac{L'}{L}(s,\chi^2)$ is essentially $\frac{\zeta'}{\zeta}(s)$, and

so

$$-R\frac{L'}{L}(\sigma+2i\gamma_0,\chi^2)=\frac{\sigma-1}{(\sigma-1)^2+4\gamma_0^2}+O(\log q).$$

Also $L(s,\chi)$ has beside the zero $\beta_0+i\gamma_0$ also the zero $\beta_0-i\gamma_0$, this gives us a term:

$$- \frac{\sigma-\beta_0}{(\sigma-\beta_0)^2+4\gamma_0^2}$$; thus our inequality becomes

$$\frac{3}{\sigma-1} - \frac{4}{\sigma-\beta_0} - \frac{4(\sigma-\beta_0)}{(\sigma-\beta_0)^2+4\gamma_0^2} + \frac{\sigma-1}{(\sigma-1)^2+4\gamma_0^2}$$
$$+ O(\log q) \geq 0.$$

If we again choose $\sigma = 1+6(1-\beta_0)$, we see that the third term more than cancels out the fourth, and so again

$$\frac{3}{\sigma-1} - \frac{4}{\sigma-\beta_0} + O(\log q) \geq 0,$$

and
$$\beta_0 < 1 - \frac{\alpha}{\log q}.$$

Finally if χ is quadratic and $\gamma_0 = 0$, we cannot exclude that there may be a real zero very close to 1, but it could be at most one such, since from

$$\mathcal{R}\left\{-\frac{\zeta'}{\zeta}(\sigma) - \frac{L'}{L}(\sigma,\chi)\right\} \geq 0,$$

the assumption of two real zeros

leads the inequality for $\sigma > 1$

$$\frac{1}{\sigma-1} - \frac{1}{\sigma-\beta} - \frac{1}{\sigma-\beta'} + O(\log q) \geq 0,$$

where β and β' are the two zeros. if $\beta' < \beta < 1$, then

$$\frac{1}{\sigma-1} - \frac{2}{\sigma-\beta'} + O(\log q) \geq 0,$$

choosing $\sigma = 1 + 2(1-\beta')$,

we get
$$\beta' < 1 - \frac{\alpha}{\log q},$$

so $L(s,\chi)$ where χ is quadratic has at most one zero (which then has to be real) in the region
$$\sigma > 1 - \frac{\alpha}{\log q(2+|t|)}.$$

Finally, if we have two different primitive quadratic characters χ_1 and χ_2 both belonging to modules q_1 and $q_2 \leq q$, then $\chi_1\chi_2$ is also a quadratic character belonging to the module $[q_1, q_2] < q^2$.

We have then

$$\Re\left\{-\frac{\zeta'}{\zeta}(\sigma) - \frac{L'}{L}(\sigma, \chi_1) - \frac{L'}{L}(\sigma, \chi_2) - \frac{L'}{L}(\sigma, \chi_1\chi_2)\right\} \geq 0.$$

If both $L(s, \chi_1)$ and $L(s, \chi_2)$ have real zeros β_1 and β_2 respectively, we get the inequality for $\sigma > 1$

$$\frac{1}{\sigma - 1} - \frac{1}{\sigma - \beta_1} - \frac{1}{\sigma - \beta_2} + O(\log q) \geq 0,$$

from this we can again conclude that

$$\min(\beta_1, \beta_2) < 1 - \frac{\alpha}{\log q}.$$

Thus of the quadratic characters to modules $\leq q$, at most one $L(s, \chi)$ can have at most one real zero

$$\beta \geq 1 - \frac{\alpha}{\log q}.$$

The possibility of this exceptional real zero is very annoying, and complicates the statement of a

P.N.T with remainder terms for the arithmetic progression. If there is no exceptional real zero β for any of the quadratic characters whose module divides q, then we get

$$\psi_{q,\ell}(x) = \frac{1}{\varphi(q)}x + O(xe^{-\alpha'\sqrt{\log x}})$$

and

$$\pi_{q,\ell}(x) = \frac{1}{\varphi(q)}\int_2^x \frac{dt}{\log t} + O(xe^{-\alpha'\sqrt{\log x}}).$$

However, if we have such an exceptional zero for the character χ we get

$$\psi_{q,\ell}(x) = \frac{1}{\varphi(q)}x - \frac{\chi(\ell)}{\varphi(q)}x^{\beta} + O(xe^{-\alpha'\sqrt{\log x}})$$

and

$$\pi_{q,\ell}(x) = \frac{1}{\varphi(q)}\int_2^x \frac{dt}{\log t} - \frac{\chi(\ell)}{\varphi(q)}\int_2^{x^{\beta}} \frac{dt}{\log t} + O(xe^{-\alpha'\sqrt{\log x}}).$$

In these estimations α' and the constants implied by the O's do not depend on q. β if it exists, clearly does, but in a way we know very little about. We proved earlier only that $\beta < 1$. A lower bound for $1-\beta$ is hard to come by. If we have a positive lower bound for $L(1,\chi)$ and an upper bound for the derivative $L'(s,\chi)$ in the neighbourhood of 1, we can obviously get a lower bound for $1-\beta$. It is easy to show that for $\sigma \geq 1 - \frac{1}{\log q}$,

$$L'(\sigma,\chi) = O(\log^2 q).$$

Since $\zeta(s)L(s,\chi)$ is the Dedekind zetafunction of some quadratic field, the connection of $L(1,\chi)$ with the class number (which has to be at least 1) gives that

$$L(1,\chi) \geq \frac{c}{\sqrt{q}}, \text{ from which}$$

we see that $1-\beta \geq \frac{c'}{\sqrt{q}\log^2 q}$, with an effective constant c'. There is a theorem by Siegel which shows that for any $\varepsilon > 0$ there is $c(\varepsilon) > 0$ such that

$$1-\beta > c(\varepsilon)\, q^{-\varepsilon}.$$

But only the existence of $c(\varepsilon)$ is proved so this is not an effective constant we have no way of giving bounds for it. The best effective bound for $1-\beta$ is by Goldfeld - Gross - Zagier, it improves the bound obtained from the connection with the class number by a power of $\log q$.

Riemann conjectured in his paper of 1859 that all the zeros of $\xi(s)$ lie on the line $\sigma = \frac{1}{2}$, this has later been extended to the more general conjecture that for any χ, all the zeros of $\xi(s,\chi)$ lie on the line $\sigma = \frac{1}{2}$. We shall see what consequences this has for the estimation of $\psi(x)$, $\pi(x)$ and $\psi_{q,l}(x)$ and $\pi_{q,l}(x)$.

We go back to the formulas

$$\int_0^x \psi(t)\,dt = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \frac{x^{s+1}}{s(s+1)}\left(-\frac{\zeta'}{\zeta}(s)\right) ds,$$

and

$$-\frac{\xi'}{\xi}(s) = c + \frac{1}{s-1} - \sum_\rho \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right) - \sum_{n=1}^{\infty}\left(\frac{1}{s+2n} - \frac{1}{2n}\right).$$

Inserting the second formula in the first and integrating term by term, we get

$$\int_1^x \psi(t)\,dt = \frac{x^2}{2} - \sum_\rho \frac{x^{\rho+1}}{\rho(\rho+1)} + c'x + c'' - \sum_{n=1}^{\infty} \frac{x^{1-2n}}{2n(2n-1)},$$

where c' and c'' are certain constants.
If the Riemann hypothesis is true we may write $\rho = \frac{1}{2} + i\gamma$.
So we have

$$\int_1^x \psi(t) = \frac{x^2}{2} - \sum_\gamma \frac{x^{\frac{3}{2}+i\gamma}}{(\frac{1}{2}+i\gamma)(\frac{3}{2}+i\gamma)} + c'x + O(1).$$

Forming, with $1 < h < x$

$$\psi(x) \leq \frac{1}{h}\int_x^{x+h} \psi(t)\,dt = x + \frac{h}{2}$$

$$-\frac{1}{h}\sum_\gamma \frac{(x+h)^{\frac{3}{2}+i\gamma} - x^{\frac{3}{2}+i\gamma}}{(\frac{1}{2}+i\gamma)(\frac{3}{2}+i\gamma)} + O(1),$$

and observing that

$$\left|\frac{(x+h)^{s+1} - x^{s+1}}{h}\right| \leq \begin{cases} |s+1|\ |x+h|^{\sigma} \\ \frac{2}{h}\ |x+h|^{\sigma+1} \end{cases},$$

using the upper inequality for $|\gamma| \le \frac{2x}{h}$ and the lower when $|\gamma| > \frac{2x}{h}$, we get

$$\psi(x) \le x + \frac{h}{2} + \sqrt{2x} \sum_{|\gamma| \le \frac{2x}{h}} \frac{1}{|\gamma|}$$

$$+ \frac{2(2x)^{\frac{3}{2}}}{h} \sum_{|\gamma| \ge \frac{2x}{h}} \frac{1}{|\gamma|^2} + O(1)$$

From $N(T) = \frac{T}{2\pi}\left(\log \frac{T}{2\pi} - 1\right) + O(\log T)$.

we get

$$\sum_{|\gamma| \le T} \frac{1}{|\gamma|} = 2 \sum_{0<\gamma<T} \frac{1}{\gamma} < A \sum_{1 \le n \le T} \frac{\log n}{n}$$

$$= O(\log^2 T),$$

and

$$\sum_{|\gamma| > T} \frac{1}{|\gamma|^2} = 2 \sum_{\gamma > T} \frac{1}{\gamma^2} < A \sum_{n \ge T} \frac{\log n}{n^2}$$

$$= O\left(\frac{\log T}{T}\right).$$

Inserting this, with $T = \frac{2x}{h}$

we get

$$\psi(x) \leq x + \frac{h}{2} + O\left(x^{\frac{1}{2}} \log^2 \frac{2x}{h}\right)$$

$$+ O\left(x^{\frac{1}{2}} \log \frac{2x}{h}\right),$$

choosing, say $h = 2\sqrt{x}$, we get

$$\psi(x) \leq x + O\left(x^{\frac{1}{2}} \log^2 x\right).$$

From

$$\psi(x) \geq \frac{1}{h} \int_{x-h}^{x} \psi(t)\, dt,$$

we get in a similar way

$$\psi(x) \geq x + O\left(x^{\frac{1}{2}} \log^2 x\right).$$

Thus

$$\psi(x) = x + O\left(x^{\frac{1}{2}} \log^2 x\right),$$

and so

$$\vartheta(x) = x + O\left(x^{\frac{1}{2}} \log^2 x\right)$$

and finally

$$\pi(x) = \int_2^x \frac{dt}{\log t} + O\left(x^{\frac{1}{2}} \log x\right).$$

In the same way one gets, assuming the general Riemann hypothesis, that

$$\psi_{q,l}(x) = \frac{1}{\varphi(q)}x + O(x^{\frac{1}{2}}\log^2 x),$$

$$\vartheta_{q,l}(x) = \frac{1}{\varphi(q)}x + O(x^{\frac{1}{2}}\log^2 x)$$

and

$$\pi_{q,l}(x) = \frac{1}{\varphi(q)}\int_2^x \frac{dt}{\log t} + O(x^{\frac{1}{2}}\log x).$$

The Riemann hypothesis is today supported by very massive numerical evidence. A proof seems still far off.

Let $\psi(x) = x + R(x)$,

we shall show that if for all large x

$$R(x) \geq -Ax^{\theta}, \quad \theta > 0$$

that implies that $\zeta(s)$ has no zero in $\sigma > \theta$. The same conclusion holds if $R(x) < Ax^{\theta}$

for all large x.

We consider for $\sigma > 1$

$$\int_1^\infty \frac{R(x) + Ax^{\theta} + B}{x^{s+1}}\,dx,$$

where the constant B is chosen large enough that

$$R(x) + Ax^{\theta} + B > 0$$

for all $x \geq 1$. We have

$$\int_1^\infty \frac{R(x)}{x^{s+1}}\,dx = \int_1^\infty \frac{\psi(x)\,dx}{x^{s+1}} - \int_1^\infty \frac{dx}{x^s} =$$

$$= \sum_n \Lambda(n) \int_n^\infty \frac{dx}{x^{s+1}} - \frac{1}{s-1} =$$

$$= \frac{1}{s} \sum_n \frac{\Lambda(n)}{n^s} - \frac{1}{s-1} = \frac{-1}{s} \frac{\zeta'}{\zeta}(s) - \frac{1}{s-1},$$

also

$$\int_1^\infty \frac{Ax^{\theta}}{x^{s+1}}\,dx = \frac{A}{s-\theta}, \quad \int_1^\infty \frac{B\,dx}{x^{s+1}} = \frac{B}{s}.$$

So

$$\int_1^\infty \frac{R(x) + Ax^{\theta} + B}{x^{s+1}}\,dx$$

$$= -\frac{1}{s} \frac{\zeta'}{\zeta}(s) - \frac{1}{s-1} + \frac{A}{s-\theta} + \frac{B}{s}.$$

This function is regular analytic up to the nearest singularity on the real axis. This is $s = \theta$. So $\frac{\zeta'}{\zeta}(s)$ can have no singularity in $\sigma > \theta$, which proves the statement.