Consequences of R.H. (Riemann Hypothesis):

$$\psi(x) = x + O(x^{\frac{1}{2}} \log^2 x)$$

$$\vartheta(x) = x + O(x^{\frac{1}{2}} \log^2 x)$$

$$\pi(x) = \int_2^x \frac{dt}{\log t} + O(x^{\frac{1}{2}} \log x).$$

Also showed that if $\psi(x) = x + R(x)$ then if there are θ and $A > 0$ such that

$$R(x) > -A x^\theta \text{ for all } x > x_0,$$

then $\zeta(s)$ has no zeros in the halfplane $\sigma > \theta$, same conclusion from $R(x) < A x^\theta$ for all $x > x_0$.

If $\zeta(s)$ has no zeros in $\sigma > \theta$, we can show

$$R(x) = O(x^\theta \log^2 x),$$

in a way quite similar to the one used assuming R.H. If $\theta > \frac{1}{2}$ then the stronger conclusion

$$R(x) = O(x^\theta)$$

can be shown to hold, and if $\zeta(s)$ has no zeros in $\sigma \geq \theta > \frac{1}{2}$ then $R(x) = o(x^\theta)$ holds.

We see that if $R(x) = O(x^{\frac{1}{2}+\varepsilon})$ for all $\varepsilon > 0$, or even if for all $\varepsilon > 0$,

$$R(x) > -A(\varepsilon)\, x^{\frac{1}{2}+\varepsilon} \quad \text{for all } x \geq x_0(\varepsilon)$$

where $A(\varepsilon) > 0$, then R.H. must hold and so:

$$R(x) = O(x^{\frac{1}{2}} \log^2 x).$$

We shall now try to see what lower bound we can find for the maximal order of $R(x)$.

On the R.H. we have

$$\int_0^x \psi(t)\,dt = \frac{x^2}{2} - x^{\frac{3}{2}} \sum_\gamma \frac{x^{i\gamma}}{(\frac{1}{2}+i\gamma)(\frac{3}{2}+i\gamma)} + cx + O(1),$$

from this we see that

$$\int_1^x R(t)\,dt = -x^{\frac{3}{2}} \sum_\gamma \frac{x^{i\gamma}}{(\frac{1}{2}+i\gamma)(\frac{3}{2}+i\gamma)} + cx + O(1)$$

From this it is easy to see that we cannot have

$$\int_0^x R(t)\,dt = o(x^{\frac{3}{2}})$$

and so also $R(x) = o(x^{\frac{1}{2}})$ must be false.

Thus the order of $R(x)$ is at least $x^{\frac12}$.

To get further, we write $x = e^u$ and $t = e^v$, and get

$$\int_0^u e^v R(e^v)\,dv = -e^{\frac32 u}\sum_\gamma \frac{e^{i\gamma u}}{(\frac12+i\gamma)(\frac32+i\gamma)} + ce^u + O(1).$$

From this, with $e^{-u} < \delta < 1$, we get

$$\frac{1}{2\delta}\int_{u-\delta}^{u+\delta} e^v R(e^v)\,dv = -\frac{e^{\frac32 u}}{2\delta}\sum_\gamma \frac{e^{i\gamma u}\left(e^{(\frac32+i\gamma)\delta}-e^{-(\frac32+i\gamma)\delta}\right)}{(\frac12+i\gamma)(\frac32+i\gamma)}$$

$$+ce^u\,\frac{e^{\delta}-e^{-\delta}}{2\delta} + O\left(\frac{1}{\delta}\right) =$$

$$-\frac{e^{\frac32 u}}{2\delta}\sum_\gamma \frac{e^{i\gamma u}\left(e^{i\gamma\delta}-e^{-i\gamma\delta}\right)}{(\frac12+i\gamma)(\frac32+i\gamma)} + O(e^{\frac32 u}) =$$

$$= \frac{e^{\frac32 u}}{2\delta}\sum_\gamma \frac{e^{i\gamma u}\left(e^{i\gamma\delta}-e^{-i\gamma\delta}\right)}{\gamma^2} + O(e^{\frac32 u}) =$$

$$= \frac{e^{\frac32 u}}{2\delta}\sum_{\gamma>0} \frac{\left(e^{i\gamma u}-e^{-i\gamma u}\right)\left(e^{i\gamma\delta}-e^{-i\gamma\delta}\right)}{\gamma^2} + O(e^{\frac32 u})$$

$$= -\frac{2e^{\frac32 u}}{\delta}\sum_{\gamma>0} \frac{\sin\gamma u\,\sin\gamma\delta}{\gamma^2} + O(e^{+\frac32 u}).$$

From this we get, dividing by $e^{\frac{3}{2}u}$ and writing $u+\theta$ for v, that

$$\frac{1}{2\delta}\int_{-\delta}^{\delta} e^{\frac{3}{2}\theta}\frac{R(e^{u+\theta})}{e^{\frac{u+\theta}{2}}}\,d\theta =$$

$$= -\frac{2}{\delta}\sum_{\gamma>0}\frac{\sin\gamma u \sin\gamma\delta}{\gamma^2} + O(1).$$

We shall now try to find values of u that make the expression on the right hand side large positive or large negative. Let $\{x\}$ denote the difference between x and the nearest integer (and in case of ambiguity define $\{x\} = \frac{1}{2}$), given L and T, we can then find an integer $L \le u_0 \le L^{N(T)+1}$, such that for all $0<\gamma<T$ we have $|\{\frac{u_0\gamma}{2\pi}\}| \le \frac{1}{L}$.

Follows from the socalled Dirichlet principle or pigeonhole principle.

We then have for $\gamma<T$

$|\sin\gamma u_0| < \frac{2\pi}{L}$ and $\cos\gamma u_0 > 1-\frac{10}{L^2}$,

so for $u = u_0+\delta$, we have for $\gamma<T$ that $\sin\gamma u = \sin\gamma\delta + O(\frac{1}{L})$, and for $\gamma \ge T$ that $\sin\gamma u = \sin\gamma\delta + O(1)$.

For $u = u_0 + \delta$ we then get

$$\frac{1}{2\delta}\int_{-\delta}^{\delta} e^{\frac{3}{2}\theta}\,\frac{R(e^{u+\theta})}{e^{\frac{1}{2}(u+\theta)}}\,d\theta = -\frac{2}{\delta}\sum_{\gamma}\frac{\sin^2\gamma\delta}{\gamma^2} +$$

$$+ O\Big(\frac{1}{L}\sum_{\gamma<T}\frac{1}{\gamma} + \frac{1}{\delta}\sum_{\gamma\geq T}\frac{1}{\gamma^2} + 1\Big)$$

$$= -\frac{2}{\delta}\sum_{\gamma}\frac{\sin^2\gamma\delta}{\gamma^2} + O\Big(\frac{\log^2 T}{L} + \frac{\log T}{\delta T} + 1\Big)$$

$$= -\frac{2}{\delta}\sum_{\gamma}\frac{\sin^2\gamma\delta}{\gamma^2} + O(1),$$

if we choose $L = [\log^2 T]$ and $\delta = \frac{\log T}{T}$.

Furthermore

$$\frac{2}{\delta}\sum_{\gamma}\frac{\sin^2\gamma\delta}{\gamma^2} = \frac{2}{\delta}\int_{10}^{\infty}\frac{\sin^2 t\delta}{t^2}\,dN(t),$$

where $N(t) = \frac{t}{2\pi}\left(\log\frac{t}{2\pi} - 1\right) + r(t)$

and $r(t) = O(\log t)$. This gives

$$\frac{2}{\delta}\int_{10}^{\infty}\frac{\sin^2 t\delta}{t^2}\,dN(t) = \frac{1}{\delta\pi}\int_{10}^{\infty}\frac{\sin^2 t\delta}{t^2}\log\frac{t}{2\pi}\,dt$$

$$+ \frac{2}{\delta}\int_{10}^{\infty}\frac{\sin^2 t\delta}{t^2}\,dr(t).$$

Here integration by parts gives

$$\frac{2}{\delta}\int_{10}^{\infty}\frac{\sin^2\delta t}{t^2}\,dr(t) = \frac{2}{\delta}\left[\frac{\sin^2\delta t}{t^2}r(t)\right]_{10}^{\infty} +$$

$$+\frac{4}{\delta}\int_{10}^{\infty} r(t)\,\frac{\sin^2\delta t}{t^3}\,dt - 4\int_{10}^{\infty} r(t)\,\frac{\sin\delta t\cos\delta t}{t^2}\,dt$$

$$= O(\delta^{\frac{1}{2}}),$$

Also writing $t\delta = v$

$$\frac{1}{\delta\pi}\int_{10}^{\infty}\frac{\sin^2 t\delta}{t^2}\log\frac{t}{2\pi}\,dt =$$

$$\frac{1}{\pi}\int_{10\delta}^{\infty}\frac{\sin^2 v}{v^2}\log\frac{v}{2\pi\delta}\,dv =$$

$$\frac{\log\frac{1}{\delta}}{\pi}\int_{10\delta}^{\infty}\frac{\sin^2 v}{v^2}\,dv + \frac{1}{\pi}\int_{10\delta}^{\infty}\frac{\sin^2 v}{v^2}\log\frac{v}{2\pi}\,dv$$

$$= \frac{\log\frac{1}{\delta}}{\pi}\left(\frac{\pi}{2} - O(\delta)\right) + O(1)$$

$$= \frac{1}{2}\log\frac{1}{\delta} + O(1).$$

Thus finally for $u = u_0 + \delta$

$$\frac{1}{2\delta}\int_{-\delta}^{\delta} e^{\frac{3}{2}\theta}\,\frac{R(e^{u+\theta})}{e^{\frac{1}{2}(u+\theta)}}\,d\theta = -\frac{1}{2}\log\frac{1}{\delta} + O(1).$$

In the same way we find for $u = u_0 - \delta$, that

$$\frac{1}{2\delta}\int_{-\delta}^{\delta} e^{\frac{3}{2}\theta}\,\frac{R(e^{u+\theta})}{e^{\frac{1}{2}(u+\theta)}}\,d\theta = \frac{1}{2}\log\frac{1}{\delta} + O(1).$$

This shows that there is a u' in the interval $u_0 \leq u' \leq u_0 + 2\delta$ for which

$$e^{\frac{3}{2}\delta}\,\frac{R(e^{u'})}{e^{\frac{1}{2}u'}} \leq -\frac{1}{2}\log\frac{1}{\delta} + O(1)$$

or

$$\frac{R(e^{u'})}{e^{\frac{1}{2}u'}} \leq -\frac{1}{2}\log\frac{1}{\delta} + O(1).$$

Also there is a u'' in the interval $u_0 - 2\delta \leq u'' \leq u_0$ for which

$$e^{\frac{3}{2}\delta}\,\frac{R(e^{u''})}{e^{\frac{1}{2}u''}} \geq \frac{1}{2}\log\frac{1}{\delta} - O(1)$$

or

$$\frac{R(e^{u''})}{e^{\frac{1}{2}u''}} \geq \frac{1}{2}\log\frac{1}{\delta} - O(1).$$

We have $L < u_0 \leq L^{N(T)+1}$,

and $L = [\log^2 T]$, $\delta = \frac{\log T}{T}$.

Since $N(T) < \frac{T}{6} \log T$, we get

$$\mu_0 + 2\delta < e^{\frac{T}{3} \log T \log\log T}$$

and

$$\log \frac{1}{\delta} = \log T - \log\log T.$$

From this

$$\log \frac{1}{\delta} \geq \log\log(\mu_0 + 2\delta) - \frac{3}{2} \log\log\log(\mu_0 + 2\delta).$$

Thus if we write $x_1 = e^{u'}$, $x_2 = e^{u''}$,

we get for T sufficiently large,

$$\frac{R(x_1)}{\sqrt{x_1}} \leq -\frac{1}{2}\log\log\log x_1 + \log\log\log\log x_1,$$

and

$$\frac{R(x_2)}{\sqrt{x_2}} \geq \frac{1}{2}\log\log\log x_2 - \log\log\log\log x_2.$$

Thus there is a sequence of $x \to \infty$ for which

$$\psi(x) - x \geq \frac{x^{\frac{1}{2}}}{2}\log\log\log x - x^{\frac{1}{2}}\log\log\log\log x$$

and similarly

a sequence of $x \to \infty$ for which

$$\psi(x) - x \leq -\frac{x^{\frac{1}{2}}}{2} \log\log\log x + x^{\frac{1}{2}} \log\log\log\log x.$$

This result is essentially due to J. E. Littlewood.

From this we can easily derive corresponding results for $\vartheta(x)$ and finally for $\pi(x)$, since if $\vartheta(x) = x + R'(x)$, then $R'(x) = R(x) - \sqrt{x} + O(x^{\frac{1}{3}})$

and

$$\pi(x) - \int_2^x \frac{dt}{\log t} = \frac{R'(x)}{\log x} + \int_2^x \frac{R'(t)}{t \log^2 t}\, dt =$$

$$= \frac{R'(x)}{\log x} + O\left(\frac{\sqrt{x}}{\log^2 x}\right).$$

Thus there exists a sequence of $x \to \infty$, such that

$$\pi(x) - \int_2^x \frac{dt}{\log t} > \frac{1}{2}\frac{\sqrt{x}}{\log x} \log\log\log x - 2\frac{\sqrt{x}}{\log x} \log\log\log\log x$$

and a sequence $x \to \infty$ such

that

$$\pi(x) - \int_2^x \frac{dt}{\log t} < -\frac{1}{2}\frac{\sqrt{x}}{\log x}\log\log\log x + $$
$$+ 2\frac{\sqrt{x}}{\log x}\log\log\log\log x .$$

In particular

$$\pi(x) - \int_2^x \frac{dt}{\log t}$$

changes sign infinitely often. However no signchange has ever been numerically verified. The first such is extremely far out.

If one looks at the case of arithmetic progressions one finds that only for the case of the progression $qn+1$ can one prove the analogous result, not for the general $qn+l$ with $(l,q)=1$. For the case of $\psi_\chi(x)$ and $\pi_\chi(x)$ one can prove corresponding inequalities for the real part, but not for the imaginary part.

We return to the possibility of an exceptional zero for $L(s,\chi)$ if χ is a primitive quadratic character modulo q.

In addition to the bounds we have established earlier for $\zeta(s)$ and $L(s,\chi)$ for $\sigma \geq 0$

$$\zeta(s) = \frac{s}{s-1} + O((1+|t|)^{1-\sigma} \log(2+|t|))$$

and

$$L(s,\chi) = O(((1+|t|)q)^{1-\sigma} \log q(1+|t|)) + O(1),$$

which for $s = 1+it$

gives $\zeta(s) = \frac{s}{s-1} + O(\log 2+|t|)$,

and $L(s,\chi) = O(\log q(1+|t|))$,

We shall need the better bounds on $\sigma = 0$; $s = it$, that follow from the functional equations:

Namely

$$\zeta(it) = O\left((1+|t|)^{\frac{1}{2}} \log 2+|t|\right)$$

and

$$L(it, \chi) = O\left((q(1+it))^{\frac{1}{2}} \log q(1+|t|)\right).$$

We have for $x > 0$

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \frac{x^{s+2}}{s(s+1)(s+2)}\, ds = \begin{cases} 0 & \text{for } 0 < x \leq 1 \\ \frac{1}{2}(x-1)^2 & \text{for } x \geq 1. \end{cases}$$

Let $f(s) = \zeta(s) L(s, \chi) = \sum \frac{c_n}{n^s}$,

here all $c_n \geq 0$ and $c_{n^2} \geq 1$.

This gives for $x \geq 2$,

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \frac{x^{s+2} f(s-\frac{1}{2})}{s(s+1)(s+2)}\, ds =$$

$$= \sum_{n<x} (x-n)^2 \sqrt{n}\, c_n \geq \sum_{n^2<x} (x-n^2)^2 n \geq \frac{x^3}{10}.$$

But we have also

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \frac{x^{s+2} f(s-\frac{1}{2})}{s(s+1)(s+2)}\, ds$$

$$= \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \frac{x^{s+2}\zeta(s-\frac{1}{2})L(s-\frac{1}{2},\chi)}{s(s+1)(s+2)}\, ds$$

$$= \frac{8}{105}\, x^{3\frac{1}{2}} L(1,\chi)$$

$$+ \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{x^{s+2}\zeta(s-\frac{1}{2})L(s-\frac{1}{2},\chi)}{s(s+1)(s+2)}\, ds$$

$$= \frac{8}{105}\, x^{3\frac{1}{2}} L(1,\chi) + O(x^{2\frac{1}{2}} q^{\frac{1}{2}} \log q).$$

Thus with some constant $A > 0$

$$\frac{x^3}{10} < \frac{8}{105} x^{3\frac{1}{2}} L(1,\chi) + A\, x^{2\frac{1}{2}} q^{\frac{1}{2}} \log q.$$

Choosing

$$\sqrt{x} = 20\, A\, q^{\frac{1}{2}} \log q,$$

we get

$$\frac{x^3}{20} < \frac{8}{105} x^{3\frac{1}{2}} L(1, \chi),$$

and so

$$L(1,\chi) > \frac{1}{2\sqrt{x}} = \frac{1}{40 A q^{\frac{1}{2}} \log q}$$

$$= \frac{\alpha}{q^{\frac{1}{2}} \log q}.$$

From this, and the estimation $L'(\sigma, \chi) = O(\log^2 q)$, which holds for $\sigma > 1 - \frac{1}{\log q}$, we obtain

$$1 - \beta \geqslant \frac{\alpha'}{q^{\frac{1}{2}} \log^3 q},$$

where α' is a computable absolute constant.