We shall consider further the possibility of the presence of an exceptional zero, a real zero $\beta > 1 - \frac{1}{\log q}$ for the primitive quadratic character modulo q. We saw last time that $\beta > 1 - \frac{c}{\sqrt{q}\log^3 q}$, with a constant that could be effectively determined. It was also mentioned that the connection with the class number gives $\beta > 1 - \frac{c'}{\sqrt{q}\log^2 q}$, and that work of Goldfeld, Gross and Zagier gives $\beta > 1 - \frac{c'}{\sqrt{q}\log q}$, and I believe this has now been improved to $\beta > 1 - \frac{c'}{\sqrt{q}}$. In each case the result is obtained by finding a lower bound for $L(1,\chi)$ and then deriving a lower bound for $1-\beta$ by using that $L'(\sigma,\chi) = O(\log^2 q)$ for $\sigma > 1 - \frac{1}{\log q}$.

We shall now turn to Siegel's theorem, a result that is in some ways vastly superior in the order of the lower bound for $1-\beta$ in terms of q, but also very deficient for most purposes in that we have no way of estimating the constants that enter as coefficients.

We begin with the inequalities:

$$\zeta(s) = \frac{s}{s-1} + O\left((1+|t|)^{1-\sigma} \log(2+|t|)\right),$$

and

$$L(s,\chi) = 1 + O\left(q^{1-\sigma}(2+|t|)^{1-\sigma} \log q(2+|t|)\right).$$

We assume we have a primitive quadratic character mod q_1 such that $L(s,\chi_1)$ has a real zero $\beta_1 = 1-\delta$ with δ small, and shall see what we can conclude about $L(1,\chi)$ for a primitive quadratic character χ mod q, $q \neq q_1$.

We consider

$$f(s) = \zeta(s) L(s,\chi) L(s,\chi_1) L(s,\chi\chi_1),$$

We have
$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s}, \text{ with } a_n \geq 0, a_1 = 1.$$

We find for $\sigma \geq \frac{4}{5}$ that for $|s-1| \geq \frac{1}{10}$,
$$f(s) = O\left((1+|t|)^{\frac{5}{6}} (qq_1)^{\frac{1}{2}}\right).$$

For $x > 2$ we have
$$\frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{x^{s+1}}{s(s+1)} f(s+\beta_1)\, ds =$$
$$= \sum_{n=1}^{\infty} (x-n) \frac{a_n}{n^{\beta_1}} \geq x-1 > \frac{x}{2}.$$

Moving the path of integration to $\sigma = \frac{4}{5} - \beta_1 = -\frac{1}{5} + \delta$, we get that the integral above also equals (taking into account the residue at $s = 1-\beta_1 = \delta$)
$$\frac{x^{1+\delta} L(1,\chi) L(1,\chi_1) L(1,\chi\chi_1)}{\delta(1+\delta)} + O\left(x^{\frac{4}{5}+\delta} (qq_1)^{\frac{1}{2}}\right).$$

Comparison of the two results gives

$$\frac{x}{2} < x^{1+\delta} \frac{L(1,\chi)L(1,\chi_1)L(1,\chi\chi_1)}{\delta} + A x^{\frac{4}{5}+\delta} (q q_1)^{\frac{1}{2}}.$$

We now choose

$$x > 4A x^{\frac{4}{5}+\delta} (q q_1)^{\frac{1}{2}},$$

or

$$x^{\frac{1}{5}-\delta} > 4A (q q_1)^{\frac{1}{2}},$$

we may assume $\delta \leq \frac{1}{30}$, and so take

$$x = (4A)^6 (q q_1)^3.$$

Then we have

$$\frac{x}{4} < x^{1+\delta} \frac{L(1,\chi)L(1,\chi_1)L(1,\chi\chi_1)}{\delta}$$

or

$$L(1,\chi)L(1,\chi_1)L(1,\chi_1\chi) > \frac{\delta}{4} x^{-\delta} = \delta A' (q q_1)^{-3\delta}$$

since $L(1,\chi_1) = O(\log q_1)$, $L(1,\chi_1\chi) = O(\log q q_1)$

We get

$$L(1,\chi) > c(\delta) (q q_1)^{-4\delta}.$$

From
$$L'(\sigma, \chi) = O(\log^2 q),$$
for $\sigma > 1 - \frac{1}{\log q}$, we then get
$$1-\beta \geq C'(\delta)\, q^{-5\delta}.$$
Now, either the β have an absolute upper bound $\theta < 1$, or we can find q_1 with β_1 arbitrarily close to one. In the second case we may chose χ_1 such that $1-\beta_1 < \frac{\varepsilon}{5}$ for any given $\varepsilon > 0$. We then get
$$1-\beta > c(\varepsilon)\, q^{-\varepsilon},$$
while in the first case we of course have $1-\beta \geq 1-\theta > 0$. We could of course also phrase the result as
$$1-\beta > q^{-\varepsilon}, \quad \text{for } q > q_0(\varepsilon).$$
where ε can be chosen positive and arbitrarily small.

In either formulation, the constants $C(\varepsilon)$ and $q_0(\varepsilon)$ exist, but we can not give any bound for their size.

If on the other hand we could actually find a $L(s, \chi_1)$ with a β_1 very close to 1, we could obtain an effective estimate (and quite a bit better than indicated by our proof of Siegel's result, since we did not try to obtain the best factor in front of δ in the exponent of q).

This essentially concludes the material in analytic prime number theory that I had planned to talk about. I have not touched on developments like:

Better (that is: wider) zero free regions along $\sigma = 1$. These can be obtained using the theory of exponential sums to estimate sums of the type

$$\sum_{N}^{N'} n^{it}, \text{ or } \sum_{N}^{N'} \chi(n) n^{it},$$

Such estimate lead for instance for $\zeta(s)$ to a zero free region of the type

$$\sigma > 1 - \frac{\alpha}{(\log(1+|t|))^{\theta}}, \quad \alpha > 0,$$

with a $\theta < 1$. This gives us the remainder terms

$$\psi(x) = x + O\left(x\, e^{-\alpha' (\log x)^{\frac{1}{1+\theta}}}\right),$$

and

$$\pi(x) = \int_2^x \frac{dt}{\log t} + O\left(x\, e^{-\alpha' (\log x)^{\frac{1}{1+\theta}}}\right),$$

Another development of great importance for many applications are the socalled "density theorems".