Dirichlet L-function

$$L(s,\chi) = \sum_n \frac{\chi(n)}{n^s};$$

Always assume χ primitive to modul q.

Functional equation; $a = \frac{1-\chi(-1)}{2}$; $|\varepsilon| = 1$,

$$\phi(s,\chi) = \varepsilon \pi^{-\frac{s}{2}} q^{\frac{s}{2}} \Gamma\left(\frac{s+a}{2}\right) L(s,\chi)$$

Then

$$\phi(s;\chi) = \overline{\phi(1-\bar{s},\chi)}.$$

implies $\phi(s,\chi)$ real for $s=\frac{1}{2}+it$, t real.

For simplicity speak only of case $\chi(-1)=1$, so $a=0$.

If we have n such distinct $L(s,\chi_j)$ $j=1,2,\cdots,n$. and form

$$F(s) = \sum_{j=1}^{n} c_j \varepsilon_j q_j^{\frac{s}{2}-\frac{1}{4}} L(s,\chi_j); \quad c_j \text{ real and } \neq 0.$$

Then $\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)F(s)$ is real for $s=\frac{1}{2}+it$.

Apart from trivial zeros implied by the functional equation, the zeros of $F(s)$ are confined to some vertical strip if we let $N(T,F)$ denote the number of these with imaginary part in $(0,T)$, then for large T.

$$N(T,F) = \frac{T}{2\pi}(\log T + B) + O(\log T).$$

It is conjectured that almost all of these have real part $\frac{1}{2}$. A proof can be given assuming some other plausible but at present unverifiable conjectures. For the single L-function $L(s,\chi)$ it has been proved that a positive proportion of the zeros have real part $\frac{1}{2}$. More precisely, one can show if $N_0(T,L)$ denotes the zeros of $L(s,\chi)$ with real part $\frac{1}{2}$ and imaginary part in $(0,T)$, then

$$N_0(T,L) > c\,T\log T \quad \text{for } T > Aq^2$$

where c and A are $\underbrace{\text{absolute}}_{\text{positive}}$ constants. For a linear combination, the only results in the literature is for a linear combination of two

$$F(s) = \varepsilon L(s,\chi) + \bar{\varepsilon} L(s,\bar{\chi})$$

The latest result seems to be

$$N_0(T,F) > T(\log T)^{\frac{1}{2}} e^{-c\sqrt{\log\log T}}$$

for $T > T_0$. A.A. Karatsuba 1993.

For a more general linear combination he claims a much weaker and much more complicated result.

I shall sketch a proof that for the general linear combination

(1) $N_0(T,F) > c(n)\, T \log T$ for $T > T_0(F)$.

where the positive constant $c(n)$ depends on n only.

(2) If $\omega(t) \to \infty$ with t then $F(\frac{1}{2}+it)$ has a zero in the interval $\left(t, t+\frac{\omega(t)}{\log t}\right)$ for almost all t.

We go back to method used to show these results for a single L-function.

Write for $s = \frac{1}{2}+it$

$$\vartheta(t) = \arg \pi^{-\frac{s}{2}}\, \Gamma\left(\frac{s}{2}\right),$$

and put,

$$\varepsilon_\chi\, q^{\frac{it}{2}} e^{i\vartheta(t)} L(s,\chi) = X(t,\chi).$$

We need the "approximate functional-equation": assume $t>0$, then

$$L(s,\chi) = \sum_{n<\sqrt{\frac{tq}{2\pi}}} \chi(n)\, n^{-s} + \bar{\varepsilon}_\chi^{\,2}\, q^{-it} e^{-2i\vartheta(t)} \sum_{n<\sqrt{\frac{tq}{2\pi}}} \bar{\chi}(n)\, n^{s-1} + O\left(\left(\frac{q}{t}\right)^{\frac{1}{4}}\right).$$

We also write

$$\left(\zeta(s)\right)^{-\frac{1}{2}} = \sum_{n} \frac{\alpha_n}{n^s}\ ;\ \alpha_1 = 1\ ;\ \left(L(s,\chi)\right)^{-\frac{1}{2}} = \sum_{n} \frac{\chi(n)\,\alpha_n}{n^s}\ ;$$

and for $T \leq t \leq 2T$; $\xi^{2} = T^{\frac{1}{5}}$; $T > q^2$,

we write

$$\eta(t,\chi) = \sum_{n<\xi} \frac{\alpha_n}{n^s}\left(1 - \frac{\log n}{\log \xi}\right).$$

Now consider for $\frac{1}{\log T} \leq H < \frac{1}{\sqrt{\log T}}$,

the three expressions:

$$I_\chi(t,H) = \int_t^{t+H} X(u)\,|\eta(u,\chi)|^2\,du,$$

$$M_\chi(t,H) = \int_t^{t+H} L(\tfrac{1}{2}+iu,\chi)\,\eta^2(u,\chi)\,du - H,$$

and

$$J_\chi(t,H) = \int_t^{t+H} |X(u)\,\eta^2(u)|^2\,du.$$

We can show

$$\int_T^{2T} |I_\chi(t,H)|^2\,dt = O\left(T\,\frac{H^{\frac{3}{2}}}{\sqrt{\log T}}\right),$$

and
$$\int_T^{2T} |M_\chi(t,H)|^2\,dt = O\left(T\,\frac{H^{\frac{3}{2}}}{\sqrt{\log T}}\right).$$
Constants involved in O-symbols absolute.

It is clear that
$$J_\chi(t,H) \geq H - |M_\chi(t,H)|,$$
and that if
$$J_\chi(t,H) > |I_\chi(t,H)|,$$
then $X(u,\chi)$ changes sign in $(t, t+H)$ and so has at least one zero there. Thus if
$$|M_\chi(t,H)| + |I_\chi(t,H)| < H,$$
there is a zero in $(t, t+H)$. This happens for instance if both $|M_\chi|$ and $|I_\chi|$ $\leq \frac{H}{3}$. But the measure of the set in $(T, 2T)$ for which one of these inequalities is false is seen to be
$$O\left(\frac{T}{\sqrt{H \log T}}\right).$$

Thus for all t in a subset of $(T, 2T-H)$ of measure $\geqslant T - O\left(\frac{T}{\sqrt{H \log T}}\right)$, the interval $(t, t+H)$ contains a zero. From this, choosing $H = \frac{\lambda}{\log T}$ with λ large enough, both statements made earlier follow easily.

We now have to try to adapt this idea to the general linear combination. For this we turn to some general results that can be proved about the value-distribution of $\log|L(s,\chi)|$ for $s = \frac{1}{2}+it$.

It can be shown that if a is a constant $0<a<1$, $T>q^2$, k a positive integer and $T^{\frac{a}{k}} \leq x \leq T^{\frac{1}{k}}$,

then

$$\int_T^{2T} \left| \log|L(s,\chi)| - R\sum_{p<x} \frac{\chi(p)}{p^s} \right|^{2k} dt =$$

$$= O\left(k^{4k} e^{Ak} T\right).$$

Again, constant implied by O is independent of χ.

From this it is possible to prove that

$$\frac{\log |L(s,\chi)|}{\sqrt{\pi \log\log t}}$$

has a normal Gaussian distribution. More precisely, if $\varkappa_{a,b}(u)$ denotes the characteristic function of the interval (a,b), then

$$\int_T^{2T} \varkappa_{a,b}\left(\frac{\log |L(s,\chi)|}{\sqrt{\pi \log\log T}}\right) dt = T\int_a^b e^{-\pi u^2}\,du + O\left(T\,\frac{(\log\log\log T)^2}{\sqrt{\log\log T}}\right).$$

If we have two L-functions $L(s,\chi)$ and $L(s,\chi')$ with $\chi \neq \chi'$, similar results hold for the difference

$$\log|L(s,\chi)| - \log|L(s,\chi')|,$$

only here we have to divide by

$$\sqrt{2\pi \log\log t}$$

to get the normal distribution.

From this we can derive that the subset of $(T, 2T)$ where

$$\left| \log|L(s,\chi)| - \log|L(s,\chi')| \right| \le$$
$$\le (\log\log T)^{\frac{1}{4}},$$

has measure

$$O\left(T (\log\log T)^{-\frac{1}{4}} \right).$$

Thus in $(T, 2T)$ we have for our n L-function that when $j \neq k$

$$\left| \log|L(s,\chi_j)| - \log|L(s,\chi_k)| \right| > (\log\log T)^{\frac{1}{4}},$$

except in a subset of measure $O(T(\log\log T)^{-\frac{1}{4}})$.

Outside of this exceptional subset one of the L-functions is decisively dominant in the linear combination. We shall see that this dominance is fairly stable over longer (compared with $\frac{1}{\log T}$) stretches

We have for $\frac{1}{\log T} \le h < \frac{1}{\sqrt{\log T}}$

that

$$\int_T^{2T} \left| \log|L(s+h,\chi)| - \log|L(s,\chi)| \right|^{2k}$$

$$= O\left(T k^k (\log(e+h\log T))^k\right) + O\left(T k^{\frac{4k}{e}} A^k\right).$$

Thus the amplitude of the oscillations of $X(t,\chi)$ is fairly stable over longer stretches.

Put $\Delta = \frac{1}{\log T} e^{(\log\log T)^{\frac{1}{5}}}$

and compare $\log|L(\frac{1}{2}+it,\chi)|$

with the average

$$\Delta(t,\chi) = \frac{1}{2\Delta} \int_{t-\Delta}^{t+\Delta} \log|L(\tfrac{1}{2}+iu,\chi)|\,du,$$

and we see that it deviates from this by less than $(\log\log T)^{\frac{1}{5}}$ except in a subset of $(T, 2T)$ of measure $O\left(\frac{T}{(\log\log T)^N}\right)$ for any fixed positive N.

We can conclude from this that $(T, 2T)$, apart from a subset of measure

$$O\left(T(\log\log T)^{-\frac{1}{4}}\right),$$

which we call S_E,

can be divided into n sets S_j in each of which $\log|L(\frac{1}{2}+it, \chi_j)|$ exceeds all the other $\log|L(\frac{1}{2}+it, \chi_k)|$ by at least $(\log\log T)^{\frac{1}{4}}$ and furthermore such that in the interval $(t, t+H)$ where $\frac{1}{\log T} < h < \frac{\log\log T}{\log T}$ and t in S_j,

$$\log|L(\tfrac{1}{2}+it', \chi_j)| > \log|L(\tfrac{1}{2}+it', \chi_k)| + \tfrac{1}{2}(\log\log T)^{\frac{1}{4}}$$

for $t < t' < t+H$, except in a subset of $(t, t+H)$ of measure $O\left(\frac{H}{(\log\log T)^{\frac{1}{8}}}\right)$.

The measure of S_j can be shown to be $\geq \frac{T}{n} - O\left(T(\log\log T)^{-\frac{1}{4}}\right)$ (each $|L(s,\chi_j)|$ dominates about equally often).

We also have for all j

$$\int_T^{2T} |L(\tfrac{1}{2}+it, \chi_j)\,\eta^2(t, \chi_j)|^2\,dt = O(T).$$

From this we conclude that

except for a subset S_E' of measure $O(T(\log\log T)^{-\frac{1}{6}})$ we have

$$\int_t^{t+H} |L(\tfrac{1}{2}+iu, \chi_j)\,\eta^2(u, \chi_j)|^2\,du = O(H(\log\log T)^{\frac{1}{6}}).$$

If we exclude the points S_E' also from S_j we get S_j' of measure

$$\geq \frac{T}{m} - O(T(\log\log T)^{-\frac{1}{6}})$$

and such that for t in S_j'

$$\int_t^{t+H} |L(\tfrac{1}{2}+iu, \chi_j)\,\eta^2(u, \chi_j)|^2 = O(H(\log\log T)^{\frac{1}{6}})$$

and except for an exceptional set of measure $O(H(\log\log T)^{-\frac{1}{4}})$ we have

$$\log|L(\tfrac{1}{2}+iu, \chi_j)| > \log|L(\tfrac{1}{2}+iu, \chi_k)| + \tfrac{1}{2}(\log\log T)^{\frac{1}{4}}$$

in $(t, t+H.)$

This exeptional set in $(t, t+H)$ can contribute at most

$$O\left(\sqrt{\frac{H}{(\log\log T)^{\frac{1}{4}}}}\right) O\left(\sqrt{H(\log\log T)^{\frac{1}{6}}}\right)$$

$$= O\left(H(\log\log T)^{-\frac{1}{24}}\right).$$

to $I_{\chi_j}(t,H)$, $M_{\chi_j}(t,H)$ and $J_{\chi_j}(t,H)$

Calling these integrals with the exceptional subset excluded

$$I^*_{\chi_j}(t,H),\ M^*_{\chi_j}(t,H) \text{ and } J^*_{\chi_j}(t,H)$$

We see that we have sign change of $X_{\chi_j}(t)$ in $(t, t+H)$ outside the excluded subset if

$$J^*_{\chi_j}(t,H) > |I^*_{\chi_j}(t,H)|,$$

which turns out to be equivalent to

$$H > |I_{\chi_j}(t,H)| + |M_{\chi_j}(t,H)| + + O\left(H \cdot (\log\log T)^{-\frac{1}{24}}\right)$$

For large enough T this inequality holds in $(T, 2T)$ outside of a set of measure $O\left(\frac{T}{\sqrt{H \log T}}\right)$, and so in most of S_j' if $H = \frac{\lambda m^2}{\log T}$ with λ a large enough constant. This produces $> \frac{c}{m^3} T \log T$ sign changes in $\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})F(s)$ for $s=\frac{1}{2}+it$ and t in S_j', adding up over the j we get all in all

$$> \frac{c}{n^2} T \log T$$

signchanges or zeros for $T > T(F)$.

Dependence on n can be improved by sharpening estimations, showing

$$\int_T^{2T} |I_\chi(t,H)|^2 dt = O(T \frac{H}{\log T})$$

and

$$\int_T^{2T} |M_\chi(t,H)|^2 dt = O(T \frac{H}{\log T})$$

which gives $> \frac{c}{n} T \log T$ zeros.

Problem of generalizations.