Selberg's Lecture Series on the Analytic Theory of Prime Numbers

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Lecture Series on the Analytic Theory of the Prime Numbers

Lecture I

In this lecture we shall introduce Riemann zeta function, Dirichlet L-functions and prove their functional equations.

1.1. Dirichlet character mod q

Dirichlet character mod q is defined as follows:

$$\chi(n) = 0, \qquad \text{if } (n, q) > 1,$$

$$\chi(n) = \chi(m), \qquad \text{if } n \equiv m \pmod q,$$

$\chi(n)$ is totally multiplicative, that is

$$\chi(m)\chi(n) = \chi(mn),$$

and $\chi(1) = 1$. If $q = \prod_{i=1}^{s} p_i^{\alpha i}$ is the standard factorization into prime factors of q, and $\chi_i(n)$ is a character mod $p_i^{\alpha_i}$ for $1 \le i \le s$, then

$$\chi(n) = \prod_{i=1}^{s} \chi_i(n) \tag{1.1}$$

evidently satisfies all conditions for character mod q. Conversely if χ is a character mod q, then there exist unique characters χ_i mod $p_i^{\alpha_i}$, $1 \le i \le s$, such that (1.1) holds. In fact, for any integer n with $(n, q) = 1$, we may define n_i by the Chinese Remainder Theorem such that

$$\begin{cases} n_i \equiv n \pmod{p_i^{\alpha_i}} \\ n_i \equiv 1 \pmod{p_j^{\alpha_j}}, \quad j \ne i. \end{cases}$$

Therefore we have

$$\prod_{i=1}^{s} n_i \equiv n \pmod{p_i^{\alpha_i}}, \quad 1 \le i \le s$$

and consequently

$$\prod_{i=1}^{s} n_i \equiv n \pmod q.$$

Define

$$\chi_i(n) = \chi(n_i).$$

Then $\chi_i(n)$ is a character mod $p_i^{\alpha_i}$, and satisfies

$$\chi(n) = \chi\Big(\prod_{i=1}^{s} n_i\Big) = \prod_{i=1}^{s} \chi(n_i) = \prod_{i=1}^{s} \chi_i(n).$$

that is; (1.1) holds. Now we proceed to show that the expression (1.1) is unique. If there are two such expressions

$$\chi(n) = \prod_{i=1}^{s} \chi_i(n) = \prod_{i=1}^{s} \chi'_1(n),$$

then we take n such that

$$n \equiv n_i (\operatorname{mod} p_i^{\alpha_i})$$
$$n \equiv 1 (\operatorname{mod} p_j^{\alpha_j}), \quad j \neq i,$$

for $(n_i, p_i) = 1$, and so

$$\chi(n) = \chi_i(n_i) = \chi'_i(n_i),$$

and the uniqueness follows. If $\chi_i(n)$ for $(n, p_i) = 1$ does not coincide with a character mod $p_i^{\alpha_i - 1}$, we say that χ_i is a primitive character mod $p_i^{\alpha_i}$. Finally if in (1.1) every χ_i is primitive mod $p_i^{\alpha_i}$, we say χ is a primitive character mod q.

If $p > 2$ and $\alpha \geq 1$, we have for $(n, p) = 1$,

$$n^{\varphi(p^\alpha)} \equiv 1 \pmod{p^\alpha},$$

where $\varphi(n)$ is the Euler function. Therefore

$$\chi(n)^{\varphi(p^\alpha)} = 1,$$

that is; $\chi(n)$ is a root of unity. Since the multiplicative group of the reduced residue classes mod p^α is a cyclic group and generated by g, we have $\varphi(p^\alpha)$ distinct characters mod p^α, if $\chi(g)$ takes $\varphi(p^\alpha)$ distinct roots of unity $\chi_c(g) = e^{2\pi i c}/\varphi(p^\alpha)$, $1 \leq c \leq \varphi(p^\alpha)$. We may treat also the case mod 2^α similarly. If $\alpha = 2$, then we have 2 characters. If $\alpha \geq 3$, every odd number can be expressed as $n \equiv (-1)^{1/2(n-1)} 5^b \pmod{2^\alpha}$, where $0 \leq b < 2^{\alpha-2}$. Set $\chi_{a,c}(n) = (-1)^{\frac{1}{2}(n-1)a} e^{2\pi i c b/2^{\alpha-2}}$, where $a = 0, 1$ and $0 \leq c < 2^{\alpha-2}$. These are all $2^{\alpha-1}$ characters mod 2^α.

It follows from (1.1) that the number of distinct characters mod q is

$$\varphi(q) = q\prod_{p|q}(1-\frac{1}{p}),$$

where p runs over prime divisors of q. The number $\varphi^*(q)$ of primitive characters mod q is equal to $\varphi(q)$ minus the number of non-primitive characters mod q, that is;

$$\begin{aligned}
\varphi^*(q) &= \prod_{p|q}(\varphi(p^\alpha)-\varphi(p^{\alpha-1}))\\
&= \prod_{p^2|q}(\varphi(p^\alpha)-\varphi(p^{\alpha-1}))\prod_{\substack{p^2\nmid q\\ p|q}}(\varphi(p)-1)\\
&= \prod_{p^2|q}(p^\alpha-p^{\varphi-1}-p^{\alpha-1}+p^{\alpha-2})\prod_{\substack{p^2\nmid q\\ p|q}}(p-2)\\
&= q\prod_{\substack{p^2\nmid q\\ p|q}}\big(1-\tfrac{2}{p}\big)\prod_{p^2|q}(1-\tfrac{1}{p})^2.
\end{aligned}$$

The character $\chi(n) = 1$ if $(n, q) = 1$ is called the principal character mod q and denoted by $\chi_0(n)$. We have

$$\sum_{n \bmod \mathrm{q}} \chi(n) = \begin{cases} 0, & \text{for } \chi \neq \chi_0,\\ \varphi(q), & \text{for } \chi = \chi_0. \end{cases} \tag{1.2}$$

The second relation of (1.2) is obvious. Now we proceed to prove the first relation. Assume that $\chi \neq \chi_0$. Then there is an integer m such $(m, q) = 1$ and $\chi(m) \neq 1$. Therefore,

$$\chi(m)\sum_{n \bmod \mathrm{q}}\chi(n) = \sum_{n \bmod \mathrm{q}}\chi(nm) = \sum_{\ell \bmod \mathrm{q}}\chi(\ell),$$

$$(\chi(m)-1)\sum_{n \bmod \mathrm{q}}\chi(n) = 0.$$

The first relation follows. We also have

$$\sum_{\chi}\chi(n) = \begin{cases} 0, & \text{for } n \not\equiv 1 \pmod{q},\\ \varphi(q), & \text{for } n \equiv 1 \pmod{q}. \end{cases} \tag{1.3}$$

The second equation of (1.3) is clear. Now we show the first equation as follows: If $n \not\equiv 1 \pmod{p^\alpha}$, where $p > 2$ and $\alpha \geq 1$. Then $\chi_1(n) \neq 1$. Now we treat the case mod 2^α. If $\alpha = 2$, except the principal character. The other character χ satisfies

$\chi(1) = 1$, $\chi(3) = -1$. If $\alpha \geq 3$, then we have $\chi_{1,1}(n) \neq 1$ for $n \not\equiv 1 \pmod{2^\alpha}$ or $n \not\equiv -5^{2^{\alpha-3}}$, and $\chi_{0,1}(n) = -1$ for $n \equiv -5^{2^{\alpha-3}}$. For the general case $q > 2$ and $n \not\equiv 1 \pmod q$, we may take a character $\chi_i \bmod p_i^{\alpha_i}$ for $p_i > 2$ or $p_i = 2$ as above such that $\chi_i(n) \neq 1$, and other characters $\chi_j(n) \bmod p_j^{\alpha_i}$, the principal characters for $j \neq i$. Set $\widetilde{\chi}_(n) = \prod_i \chi_i(n)$. Then $\widetilde{\chi}(n)$ is a character mod q with $\widetilde{\chi}(n) \neq 1$, and

$$\widetilde{\chi}(n) \sum_\chi \chi(n) = \sum_\chi \widetilde{\chi}\chi(n) = \sum_\chi \chi(n).$$

The second equality of (1.3) is proved.

1.2. Gauss or Jacobi sums

Gauss or Jacobi sum is defined as follows:

$$\tau_\chi(n) = \sum_{\ell \bmod \mathrm{q}} \chi(\ell) e^{2\pi i n \ell/q}. \tag{1.4}$$

Suppose that $q = q_1 q_2$ and $\chi(n) = \chi_1(n)\chi_2(n)$, where $(q_1, q_2) = 1$, and χ_1, χ_2 and χ denote characters mod q_1, mod q_2 and mod q respectively. When ℓ_1 and ℓ_2 run over a complete system of residues mod q_1 and mod q_2 respectively, $\ell = q_1\ell_2 + q_2\ell_1$ runs over a complete residue system mod q. Hence we have

$$\begin{aligned}\tau_\chi(n) &= \chi_1(q_2)\chi_2(q_1) \sum_{\ell_1=1}^{q_1} \sum_{\ell_2=1}^{q_2} \chi_1(\ell_1)\chi_2(\ell_2) e^{2\pi i(q_1\ell_2+q_2\ell_1)/q_1q_2} \\ &= \chi_1(q_2)\chi_2(q_1)\tau_{\chi_1}(n)\tau_{\chi_2}(n).\end{aligned}$$

This means that the study of Gauss sum can be reduced to the case of prime power modulus.

First of all, we proceed to show that if χ is a primitive character mod q and $(n, q) > 1$, then $\tau_\chi(n) = 0$. In fact, it suffices to prove the assertion for the case $q = p^\alpha$. If $\alpha = 1$, then the assertion follows immediately from (1.2). Now assume that $\alpha \geq 2$. Set $\ell = x(1+p^{\alpha-1}y)$. Then ℓ runs over a reduced system of residues mod p^α if $1 \leq x \leq p^{\alpha-1}$, $p \nmid x$ and $1 \leq y \leq p$. Since $\chi(n)$ is a primitive character mod p^α, there exists an integer u such that $\chi(1 + p^{\alpha-1}u) \neq 1$. From $p|m$, we have

$$\begin{aligned}\tau_\chi(n) &= \sum_{\substack{x=1 \\ p\nmid x}}^{p^{\alpha-1}} \chi(x) e^{2\pi i n x/p\alpha} \sum_{y=1}^{p} \chi(1 + p^{\alpha-1}y) e^{2\pi i n x y/p} \\ &= \sum_{x=1}^{p^\alpha-1} \chi(x) e^{2\pi i n x/p\alpha} = \sum_{y=1}^{p} \chi(1 +^{\alpha-1} y).\end{aligned}$$

Since

$$\begin{aligned}\chi(1+p^{\alpha-1}u)\sum_{y=1}^{p}\chi(1+p^{\alpha-1}y) &= \sum_{y=1}^{p}\chi(1+p^{\alpha-1}(y+u)) \\ &= \sum_{y=1}^{p}\chi(1+p^{\alpha-1}y),\end{aligned}$$

we have

$$\sum_{y=1}^{p}\chi(1+p^{\alpha-1}y)=0,$$

and so

$$\tau_\chi(n)=0.$$

If $(n,q)=1$, then

$$\tau_\chi(n)=\overline{\chi}(n)\sum_{\ell \bmod q}\chi(n\ell)e^{2\pi i n\ell/q}=\overline{\chi}(n)\tau_\chi(1)$$

or writing τ_χ for $\tau_\chi(1)$.

(1.5) $$\tau_\chi(n)=\overline{\chi}(n)\tau_\chi.$$

Thus

$$\begin{aligned}\sum_{n \bmod q}\left|\tau_\chi(n)\right|^2 &= \sum_{n \bmod q}\tau_\chi(n)\overline{\tau_\chi(n)} \\ &= \sum_{n \bmod q}\chi(n)\overline{\chi}(n)\tau_\chi\,\overline{\tau}_\chi=\varphi(q)\left|\tau_\chi\right|^2.\end{aligned}$$

[Alternatively: $\sum_{n \bmod q}\left|\tau_\chi(n)\right|^2=\sum_{\substack{n=1\\(n,q)=1}}^{q}\left|\tau_\chi\right|^2=\varphi(q)\left|\tau_\chi\right|^2$.]

But also

$$\begin{aligned}\sum_{n \bmod q}\left|\tau_\chi(n)\right|^2 &= \sum_{n \bmod q}\sum_{\ell,k \bmod q}\chi(\ell)\overline{\chi}(k)e^{2\pi i n(\ell-k)/q} \\ &= \sum_{\ell,k \bmod q}\chi(\ell)\overline{\chi}(k)\sum_{n \bmod q}e^{2\pi i n(\ell-k)/q} \\ &= q\sum_{\ell \bmod q}\left|\chi(\ell)\right|^2=q\varphi(q).\end{aligned}$$

So

$$\left|\tau_\chi\right|^2=q.$$

Since

$$\begin{aligned}\overline{\tau}_\chi &= \sum_{\ell \bmod \mathrm{q}} \overline{\chi}(\ell) e^{-2\pi i \ell/q} = \chi(-1) \sum_{-\ell \bmod \mathrm{q}} \overline{\chi}(-\ell) e^{-2\pi i \ell/q} \\ &= \chi(-1)\tau_{\overline{\chi}},\end{aligned}$$

we get

$$\tau_\chi \tau_{\overline{\chi}} = \chi(-1)q.$$

1.3. Poisson summation formula

Let

$$\hat{f}(v) = \int_{-\infty}^{\infty} f(u) e^{2\pi i u v} du.$$

Hereafter we always assume that $f(u)$ satisfies some smooth conditions when we use the formula later. $\hat{f}(v)$ is called to be the Fourier transform of $f(u)$.

Lemma 1.1 (Poisson summation formula). *The following relation holds*

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n).$$

Proof. The series

$$F(x) = \sum_{n \in \mathbb{Z}} f(x+n)$$

is absolutely convergent and is a periodic function of x with period 1. We expand $F(x)$ as the Fourier series

$$F(x) = \sum_{m \in \mathbb{Z}} a_m e(-mx), \quad e(x) = e^{2\pi i x},$$

where the Fourier coefficients equal to

$$\begin{aligned}a_m &= \int_0^1 F(x) e(mx) dx \\ &= \sum_{n \in \mathbb{Z}} \int_0^1 f(x+n) e(mx) dx \\ &= \sum_{n \in \mathbb{Z}} \int_n^{n+1} f(y) e(m(y-n)) dy \\ &= \sum_{n \in \mathbb{Z}} \int_n^{n+1} f(y) e(my) dy \\ &= \int_{-\infty}^{\infty} f(y) e(my) dy = \hat{f}(m).\end{aligned}$$

Hence we get

$$\sum_{n\in\mathbb{Z}} f(x+n) = \sum_{m\in\mathbb{Z}} \hat{f}(m)e(-mx).$$

Set $x=0$. The lemma follows.

There is a generalization: If χ is a primitive character mod q, then

$$\varepsilon_\chi \sum_n \chi(n) f\big(\frac{n}{\sqrt{q}}\big) = \overline{\varepsilon}_\chi \sum_n \overline{\chi}(n)\hat{f}\big(\frac{n}{\sqrt{q}}\big), \tag{1.6}$$

where $\overline{\varepsilon}_\chi = \varepsilon_{\overline{\chi}}$, $|\varepsilon_\chi| = 1$ and $\sum_n$ denotes a sum such that n runs over all integers.

In fact, we have

$$\begin{aligned}\sum_n \chi(n) f\big(\frac{n}{\sqrt{q}}\big) &= \sum_{\ell \bmod q} \chi(\ell) \sum_{n\equiv \ell \bmod q} f\big(\frac{n}{\sqrt{q}}\big)\\ &= \sum_{\ell \bmod q} \chi(\ell) \sum_n f\big(\frac{qn+\ell}{\sqrt{q}}\big).\end{aligned}$$

Since

$$\begin{aligned}\int_{-\infty}^{\infty} f\big(\frac{qu+\ell}{\sqrt{q}}\big)e^{2\pi i uv}du &= \frac{1}{\sqrt{q}}\int_{-\infty}^{\infty} f(w)e^{2\pi i}\big(\frac{\sqrt{q}w-\ell}{q}\big)v\,dw\\ &= \frac{1}{\sqrt{q}}e^{-2\pi i\ell v/q}\int_{-\infty}^{\infty} f(w)e^{2\pi i wv/\sqrt{q}}dw\\ &= \frac{1}{\sqrt{q}}e^{-2\pi i \ell v/q}\hat{f}\big(\frac{v}{\sqrt{q}}\big),\end{aligned}$$

we have from Lemma 1.1

$$\sum_n f\big(\frac{qn+\ell}{\sqrt{q}}\big) = \frac{1}{\sqrt{q}}\sum_n \hat{f}\big(\frac{n}{\sqrt{q}}\big)e^{-2\pi i\ell n/q}$$

so by (1.5)

$$\begin{aligned}\sum_n \chi(n) f\big(\frac{n}{\sqrt{q}}\big) &= \frac{1}{\sqrt{q}}\sum_n \hat{f}\big(\frac{n}{\sqrt{q}}\big)\sum_{\ell \bmod q}\chi(\ell)e^{-2\pi i\ell n/q}\\ &= \frac{1}{\sqrt{q}}\sum_n \hat{f}\big(\frac{n}{\sqrt{q}}\big)\tau_\chi(-n)\\ &= \frac{\chi(-1)\tau_\chi}{\sqrt{q}}\sum_n \overline{\chi}(n)\hat{f}\big(\frac{n}{\sqrt{q}}\big),\end{aligned}$$

which proves (1.6) with $\varepsilon_\chi^{-2} = \dfrac{\chi(-1)\tau_\chi}{\sqrt{q}}$.

1.4. Gamma function

If $Re\, s > 0$, we define Gamma function as follows:

$$\Gamma(s) = \int_0^\infty e^{-x} x^s \frac{dx}{x}.$$

The integral is uniformly convergent in the region $Re\, s \geq \alpha > 0$, and therefore $\Gamma(s)$ is analytic in the half plane $Re\, s > 0$.

Consider the function

$$f(s) = \int_C e^{-w}(-w)^{s-1} dw,$$

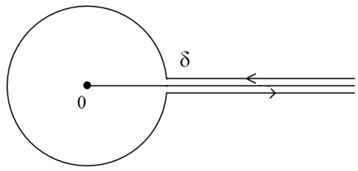

Figure 1.1

where C is the contour which goes from ∞ to δ along the real axis, then along the circle $|w| = \delta$ by the positive direction back to δ, and finally from δ to ∞ along the real axis as shown in Figure 1.1. We define the multi-valued function $(-w)^{s-1} = e^{(s-1)\log(-w)}$ by letting $\log(-w)$ to be real for $w = \delta$, and so $f(s)$ is convergent for all finite s. Hence $f(s)$ is holomorphic for all finite s.

Let $w = pe^{i\varphi}$. Then along the small circle, we have $\log(-w) = \log\delta + i(\varphi - \pi)$. Hence the integral along (δ, ∞) equals to

$$\int_\delta^\infty \{ - e^{-p+(s-1)\{\log p - i\pi\}} + e^{-p+(s-1)(\log p + i\pi)} \} dp$$

$$= -2i \sin \pi s \int_\delta^\infty e^{-p} p^{s-1} dp.$$

If $Re\, s = \sigma > 0$, then the integral along the circle $|w| = \delta$ has the estimation $\mathcal{O}(e^\delta \delta^\sigma) = \sigma(1)$ as $\delta \to 0+$. Hence we get

$$f(s) = -2i \sin \pi s \Gamma(s) \quad (Re\, s > 0).$$

as $\delta \to 0+$. The function $f(s)/-2i \sin \pi s$ may be regarded as the analytic continuation of $\Gamma(s)$ which is analytic except at the possible poles of $\dfrac{1}{\sin \pi s}$, that is; $s = 0,\ \pm 1,\ \pm 2, \cdots$. However $\Gamma(s)$ is analytic in the half-plane $Re\, s > 0$, and so its possible poles are only the points $s = 0,\ -1,\ -2,\ \cdots$. If s is a non-positive integer $-n$, then by Cauchy Theorem, we get

$$\begin{aligned} f(-n) &= \int_C e^{-w}(-w)^{-n-1}dw \\ &= \int_C \sum_{k=0}^{\infty} \frac{(-w)^k}{k!}(-w)^{-n-1}dw \\ &= -\frac{2\pi i}{n!}, \end{aligned}$$

and so

$$\lim_{s\to -n}(s+n)\Gamma(s) = \lim_{s\to -n}\frac{s+n}{-2i\sin\pi s}f(s) = \frac{(-1)^n}{n!}.$$

Thus we have proved that $\Gamma(s)$ is a meromorphic function in the s-plane and it has only simple poles at non-positive integers $-n$ with residues $(-1)^n/n!$.

If $\sigma > 0$ and $a > 0$, we have

$$\int_0^\infty x^s e^{-ax}\frac{dx}{x} = a^{-s}\int_0^\infty x^s e^{-x}\frac{dx}{x} = a^{-s}\Gamma(s). \tag{1.7}$$

1.5. Riemann zeta-function, Dirichlet L-function, and functional equations

For $s = \sigma + it;\ \sigma > 1$, we define Dirichlet L-function by

$$L(s,\chi) = \sum_{n=1}^{\infty}\chi(n)n^{-s} = \prod_p (1-\chi(p)p^{-s})^{-1}.$$

We assume χ is a primitive character mod q; and include the case $q = 1$, with $\chi(n) = 1$ for all integers n. In that case we may write $\zeta(s)$ instead of $L(s,\chi)$. $\delta(s)$ is the so-called Riemann zeta function.

If we want to extend the domain of definition beyond $\sigma > 1$, we can write $L(s,\chi)$ as a Stieltjes integral

$$L(s,\chi) = \int_{1/2}^{\infty}\frac{d\delta_\chi(x)}{x^s},$$

where we have written

$$s_\chi(x) = \sum_{n\le x}\chi(n).$$

Then by partial integration

$$\begin{aligned} L(s,\chi) &= \left[\frac{s_\chi(x)}{x^s}\right]_{1/2}^{\infty} + s\int_1^\infty \frac{s_\chi(x)}{x^{s+1}}dx \\ &= s\int_1^\infty \frac{s_\chi(x)}{x^{s+1}}dx. \end{aligned}$$

For $q > 1$, $|s_\chi(x)|$ is bounded by q by (1.2). So $L(s,\chi)$ is regular for $\sigma > 0$ and

$$|L(s,\chi)| \leq q|s|\int_1^\infty x^{-\sigma-1}dx$$

$$< A_\delta q(1+|t|) \tag{1.8}$$

for $\sigma > \delta > 0$. Hereafter we use A_δ to denote a constant depending only on δ.

If $q = 1$, we have $s_\chi(x) = [x]$, where $[x]$ denotes the integral part of x, so

$$\begin{aligned} \zeta(s) &= \sum_{n=1}^\infty n^{-s} = s\int_1^\infty \frac{[x]}{x^{s+1}}dx \\ &= s\sum_1^\infty \frac{dx}{x^s} - s\int_1^\infty \frac{x-[x]}{x^{s+1}}dx \\ &= \frac{s}{s-1} - s\int_1^\infty \frac{x-[x]}{x^{s+1}}dx. \end{aligned}$$

Thus $\zeta(s) - \dfrac{1}{s-1}$ is regular for $\sigma > \delta > 0$ and bounded by

$$A_\delta(1+|t|). \tag{1.9}$$

To effect analytic continuation of these functions in the whole complex plane, we go back to the generalized Poisson formula (1.6).

If

$$f(u) = x^{\frac{1}{4}}e^{-\pi u^2 x},$$

then by

$$\int_{-\infty}^\infty e^{-x^2}dx = \sqrt{\pi}$$

we have

$$\begin{aligned}
\hat{f}(v) &= \int_{-\infty}^{\infty} x^{\frac{1}{4}} e^{-\pi u^2 x + 2\pi i u v} du \\
&= x^{\frac{1}{4}} e^{-\pi v^2/x} \int_{-\infty}^{\infty} e^{-(\sqrt{\pi x} u - \frac{\pi i v}{\sqrt{\pi x}})^2} du \\
&= \frac{x^{\frac{1}{4}}}{\sqrt{\pi x}} e^{-\pi v^2/x} \int_{-\infty}^{\infty} e^{-w^2} dw \\
&= x^{-\frac{1}{4}} e^{-\pi v^2/x},
\end{aligned}$$

and if

$$g(u) = x^{3/4} u e^{-\pi u^2 x},$$

then

$$\begin{aligned}
\hat{g}(v) &= \int_{-\infty}^{\infty} x^{\frac{3}{4}} u e^{-\pi u^2 x + 2\pi i u v} du \\
&= x^{\frac{3}{4}} e^{-\pi v^2/x} \int_{-\infty}^{\infty} u e^{-(\sqrt{\pi x} u - \frac{\pi i v}{\sqrt{\pi x}})^2} du \\
&= \frac{x^{\frac{3}{4}} e^{-\pi v^2/x}}{\sqrt{\pi x}} \int_{-\infty}^{\infty} (\frac{w}{\sqrt{\pi x}} + \frac{iv}{x}) e^{-w^2} dw \\
&= i x^{-\frac{3}{4}} v e^{-\pi v^2/x}.
\end{aligned}$$

For an even character χ, that is; $\chi(-1) = 1$, we put the above $f(u)$, $\hat{f}(u)$ in (1.6) and get for $q > 1$, since f and $\hat{f}$ are even,

$$(1.10) \qquad \varepsilon_\chi \sum_{n=1}^{\infty} \chi(n) x^{\frac{1}{4}} e^{-\pi \frac{n^2}{q} x} = \overline{\varepsilon}_\chi \sum_{n=1}^{\infty} \overline{\chi}(n) x^{-\frac{1}{4}} e^{-\pi \frac{n^2}{q} \cdot \frac{1}{x}}$$

which we may rewrite as

$$\varepsilon_\chi x^{\frac{1}{4}} \Theta_\chi(x) = \overline{\varepsilon}_\chi x^{-\frac{1}{4}} \Theta_{\overline{\chi}}(\frac{1}{x}),$$

where

$$\Theta_\chi(x) = \sum_{n=1}^{\infty} \chi(n) e^{-\pi \frac{n^2}{q} x}.$$

For an odd character when $\chi(-1) = -1$, we put the functions g and $\hat{g}$ in (1.6) and get when we multiply by $\sqrt{q}$,

$$(1.11) \qquad \varepsilon_\chi \sum_{n=1}^{\infty} \chi(n) x^{3/4} n e^{-\pi \frac{n^2}{q} x} = i\, \overline{\varepsilon}_\chi \sum_{n=1}^{\infty} \overline{\chi}(n) n x^{-3/4} e^{-\pi n^2/qx},$$

or

$$\varepsilon_\chi x^{3/4}\Theta^*_\chi(x) = i\,\overline{\varepsilon}_\chi x^{-3/4}\Theta^*_{\overline{\chi}}\big(\frac{1}{x}\big),$$

where we have put

$$\Theta^*_\chi(x) = \sum_{n=1}^{\infty}\chi(n)ne^{-\pi n^2 x/q}.$$

For $q = 1$, we get putting f and $\hat{f}$ in the classical Poisson formula (Lemma 1.1), that

$$x^{\frac{1}{4}}\big(1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2 x}\big) = x^{-\frac{1}{4}}\big(1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2/x}\big). \tag{1.12}$$

Here the presence of the constant term in the brackets will cause some difficulty.

Consider first case $q > 1$; χ even, we have by (1.7) for $\sigma > 1$

$$\begin{aligned}\int_0^\infty x^{\frac{s}{2}}\Theta_\chi(x)\frac{dx}{x} &= \int_0^\infty x^{\frac{s}{2}}\sum_{n=1}^{\infty}\chi(n)e^{-\pi n^2 x/q}\frac{dx}{x}\\ &= \sum_{n=1}^{\infty}\chi(n)\Gamma\big(\frac{s}{2}\big)\big(\frac{\pi n^2}{q}\big)^{-s/2}\\ &= \pi^{-s/2}q^{s/2}\Gamma\big(\frac{s}{2}\big)\sum_{n=1}^{\infty}\chi(n)n^{-s}\\ &= \pi^{-s/2}q^{s/2}\Gamma\big(\frac{s}{2}\big)L(s,\chi).\end{aligned}$$

Here the integral on the left-hand side exists for all complex s and shows that $\pi^{-s/2}q^{s/2}\Gamma\big(\frac{s}{2}\big)L(s,\chi)$ is an integral function, and consequently $L(s,\chi)$ is an integral function.

Furthermore we have by (1.10)

$$\begin{aligned}&\varepsilon_\chi\pi^{-s/2}q^{s/2}\Gamma\big(\frac{s}{2}\big)L(s,\chi)\\ =\ &\varepsilon_\chi\int_0^\infty x^{s/2-1/4}x^{1/4}\Theta_\chi(x)\frac{dx}{x}\\ =\ &\overline{\varepsilon}_\chi\int_0^\infty x^{s/2-1/4}x^{-\frac{1}{4}}\Theta_{\overline{\chi}}\big(\frac{1}{x}\big)\frac{dx}{x}\\ =\ &\overline{\varepsilon}_\chi\int_0^\infty y^{\frac{1-s}{2}}\Theta_{\overline{\chi}}(y)\frac{dy}{y}\\ =\ &\overline{\varepsilon}_\chi\pi^{\frac{s-1}{2}}q^{\frac{1-s}{2}}\Gamma\big(\frac{1-s}{2}\big)L(1-s,\overline{\chi})\end{aligned}$$

or

$$(1.13)\qquad \varepsilon_\chi \pi^{-s/2} q^{s/2}\Gamma\big(\frac{s}{2}\big)L(s,\chi) = \overline{\varepsilon}_\chi \pi^{\frac{s-1}{2}} q^{\frac{1-s}{2}}\Gamma\big(\frac{1-s}{2}\big)L(1-s,\overline{\chi}).$$

We now consider the case when χ is odd, and consider

$$\begin{aligned} & \sqrt{\frac{\pi}{q}}\int_0^\infty x^{\frac{s+1}{2}}\Theta_\chi^*(x)\frac{dx}{x} = \sqrt{\frac{\pi}{q}}\int_0^\infty x^{\frac{s+1}{2}}\sum_{n=1}^\infty \chi(n)ne^{-\pi n^2 x/q}\frac{dx}{x} \\ = & \sqrt{\frac{\pi}{q}}\sum_{n=1}^\infty n\chi(n)\int_0^\infty x^{\frac{s+1}{2}}e^{-\pi n^2 x/q}\frac{dx}{x} \\ = & \sqrt{\frac{\pi}{q}}\sum_{n=1}^\infty \chi(n)n\Gamma\big(\frac{s+1}{2}\big)\big(\frac{\pi n^2}{q}\big)^{-\frac{s+1}{2}} \\ = & \pi^{-s/2}q^{s/2}\Gamma\big(\frac{s+1}{2}\big)L(s,\chi). \end{aligned}$$

Using now (1.11) in the same way as we define used (1.10), we get further for $\chi(-1) = -1$ that

$$\begin{aligned} & \varepsilon_\chi \pi^{-s/2}q^{s/2}\Gamma\big(\frac{s+1}{2}\big)L(s,\chi) \\ = & \varepsilon_\chi \sqrt{\frac{\pi}{q}}\int_0^\infty x^{\frac{s+1}{2}-\frac{3}{4}}x^{3/4}\Theta_\chi^*(x)\frac{dx}{x} \\ = & i\,\overline{\varepsilon}_\chi \sqrt{\frac{\pi}{q}}\int_0^\infty x^{\frac{s+1}{2}-\frac{3}{2}}\Theta_\chi^*\big(\frac{1}{x}\big)\frac{dx}{x} \\ = & i\,\overline{\varepsilon}_\chi \sqrt{\frac{\pi}{q}}\int_0^\infty x^{\frac{2-s}{2}}\Theta_{\overline{\chi}}^*(x)\frac{dx}{x} \\ = & i\,\overline{\varepsilon}_\chi \pi^{\frac{s-1}{2}}q^{\frac{1-s}{2}}\Gamma\big(\frac{2-s}{2}\big)L(1-s,\overline{\chi}). \end{aligned}$$

Writing $\varepsilon'_\chi = e^{-i\pi/4}\varepsilon_\chi$, this takes the form

$$(1.14)\qquad \varepsilon'_\chi \pi^{-s/2}q^{s/2}\Gamma\big(\frac{s+1}{2}\big)L(s,\chi) = \overline{\varepsilon'_\chi}\pi^{\frac{s-1}{2}}q^{\frac{1-s}{2}}\Gamma\big(\frac{2-s}{2}\big)L(1-s,\overline{\chi}).$$

Consequences: Writing $\xi(s,\chi)$ for the left-hand side of (1.13) and (1.14), the functional equation can be written in the form

$$(1.15)\qquad \xi(s,\chi) = \overline{\xi(1-\overline{s},\chi)}.$$

We have proved that $\xi(s,\chi)$ is an integral function of s, and now we see by (1.15) that $\xi(s,\chi)$ is real on the line $\sigma = 1/2$. Since the zeros of $\dfrac{1}{\Gamma(s)}$ are at $s = 0,-1,-2,\cdots$ only and

$$|L(s,\chi)| \geq \prod_p (1+|\chi(p)p^{-s}1)^{-1} \geq \prod_p (1+p^{-\sigma})^{-1} > 0,$$

for $\sigma > 1$, all zeros of $\xi(s,\chi)$ lie in the strip $0 \le \sigma \le 1$. By the poles of $\Gamma(s)$, it follows that $s = 0, -2, -4, \cdots$ are simple zeros of $L(s,\chi)$ for $\chi(-1) = 1$, and that $s = -1, -3, -5, \cdots$ are simple zeros of $L(s,\chi)$ for $\chi(-1) = -1$. These zeros are called the trivial zeros of $L(s,\chi)$.

We still need to deal with the case $q = 1$, that is $\zeta(s)$. We write formula (1.12) by

$$x^{\frac{1}{4}}\big(1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2 x}\big) = x^{-\frac{1}{4}}\big(1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2/x}\big) \tag{1.16}$$

which cannot be dealt with in the same way because of the presence of the constant terms in the brackets.

Now let us show the following operator relation

$$x^{1-\alpha}\frac{d}{dx}x^{\alpha} = x\frac{d}{dx} + \alpha.$$

In fact,

$$x^{1-\alpha}\frac{d}{dx}(f(x)x^{\alpha}) = x\frac{df}{dx} + \alpha x^{1-\alpha}x^{\alpha-1}f = \big(x\frac{d}{dx} + \alpha\big)f.$$

Therefore

$$\begin{aligned} & \big(x^{\frac{5}{4}}\frac{d}{dx}x^{-\frac{1}{4}}\big)\big(x^{\frac{3}{4}}\frac{d}{dx}x^{\frac{1}{4}}\big) \\ = \ & \big(x\frac{d}{dx} - \tfrac{1}{4}\big)\big(x\frac{d}{dx} + \tfrac{1}{4}\big) \\ = \ & \big(x^{\frac{3}{4}}\frac{d}{dx}x^{\frac{1}{4}}\big)\big(x^{\frac{5}{4}}\frac{d}{dx}x^{-\frac{1}{4}}\big) = \big(x\frac{d}{dx}\big)^2 - \frac{1}{16}. \end{aligned}$$

The operator

$$D = \delta^2 - \frac{1}{16}, \quad \delta = x\frac{d}{dx},$$

so, $\big(x^{1-\alpha}\frac{d}{dx}x^{\alpha}\big)\big(x^{1+\alpha}\frac{d}{dx}x^{-\alpha}\big) = \big(x\frac{d}{dx} + x\big)\big(x\frac{d}{dx} - \alpha\big) = \big(x\frac{d}{dx}\big)^2 - \alpha^2$ which annihilates $x^{\pm\alpha}$ will annihilate the constant terms on both sides of (1.16).

We assume f and δf are vanishing at 0 and ∞. Since

$$\int_0^{\infty} fDg\frac{dx}{x} = \int_0^{\infty} f\delta^2 g\frac{dx}{x} - \frac{1}{16}\int_0^{\infty} fg\frac{dx}{x}$$

and

$$\int_0^\infty f\delta^2 g\frac{dx}{x} = \int_0^\infty fx\frac{dx}{x}\delta g\frac{dx}{x} = \int_0^\infty fd(\delta g)$$

$$\begin{aligned} &= f\delta g\int_0^\infty - \int_0^\infty \delta g df \\ &= -\int_0^\infty \delta g\delta f\frac{dx}{x} = -\int_0^\infty \delta f dg \\ &= -g\delta f\int_0^\infty + \int_0^\infty g d\delta f = \int_0^\infty g\delta^2 f\frac{dx}{x}, \end{aligned}$$

we have

$$\int_0^\infty fDg\frac{dx}{x} = \int_0^\infty gDf\frac{dx}{x},$$

that is; D is self-adjoint with respect to the measure $\frac{d}{dx}$. Writing

$$\Theta(x) = 1 + 2\sum_{n=1}^\infty e^{-\pi n^2 x},$$

we now consider, first for $\sigma > 1$,

$$\int_0^\infty x^{\frac{s}{2}-\frac{1}{4}} D\big(x^{\frac{1}{4}}\Theta(x)\big)\frac{dx}{x} = 2\sum_{n=1}^\infty \int_0^\infty x^{s/2-1/4} D\big(x^{\frac{1}{4}} e^{-\pi n^2 x}\big)\frac{dx}{x}.$$

Here for $a > 0$, we have by (1.7)

$$\int_0^\infty x^{\frac{s}{2}-\frac{1}{4}} D\big(x^{\frac{1}{4}} e^{-ax}\big)\frac{dx}{x} = \int_0^\infty x^{\frac{1}{4}} e^{-ax} D\big(x^{\frac{s}{2}-\frac{1}{4}}\big)\frac{dx}{x}$$

$$\begin{aligned} &= \int_0^\infty x^{\frac{1}{4}} e^{-ax}\big(\delta - \frac{1}{4}\big)\,\big(\delta + \frac{1}{4}\big)\,\big(x^{\frac{s}{2}-\frac{1}{4}}\big)\,\frac{dx}{x} \\ &= \frac{s}{2}\,\big(\frac{s}{2} - \frac{1}{2}\big)\int_0^\infty x^{\frac{1}{4}} e^{-ax} x^{-\frac{1}{4}+\frac{s}{2}}\,\frac{dx}{x} \\ &= \frac{s(s-1)}{4}\int_0^\infty x^{\frac{s}{2}} e^{-ax}\,\frac{dx}{x} \\ &= \frac{s(s-1)}{4}\Gamma\big(\frac{s}{2}\big)a^{-s/2}. \end{aligned}$$

Thus

$$\int_0^\infty x^{\frac{s}{2}-\frac{1}{4}} D\big(x^{\frac{1}{4}}\Theta(x)\big)\frac{dx}{x} = \frac{s(s-1)}{2}\pi^{-s/2}\Gamma\big(\frac{s}{2}\big)\sum_{n=1}^\infty n^{-s}$$

$$= \frac{s(s-1)}{2}\pi^{-s/2}\Gamma\big(\frac{s}{2}\big)\zeta(s). \tag{1.17}$$

Since the left-hand side of (1.17) is well defined for all s, we see that

$$\xi(s) = s(s-1)\pi^{-s/2}\Gamma\big(\frac{s}{2}\big)\zeta(s)$$

is an integral function. Also changing x into $\frac{1}{x}$ in the integral on the left-hand side of (1.17) and using the relation $x^{\frac{1}{4}}\Theta(x) = x^{-\frac{1}{4}}\Theta\big(\frac{1}{x}\big)$ (see (1.16)), and the fact that D remains invariant (under the transformation $x \to \frac{1}{x}$) we get

$$\int_0^\infty x^{\frac{1}{4}-\frac{s}{2}} D\big(x^{-\frac{1}{4}}\Theta\big(\frac{1}{x}\big)\big)\frac{dx}{x}$$

$$= \frac{s(s-1)}{2}\pi^{-(1-s)/2}\Gamma\big(\frac{1-s}{2}\big)\zeta(1-s) = \xi(1-s),$$

and consequently the functional equation

$$\xi(s) = \xi(1-s).$$

Again we see that $\xi(s)$ is real on the line $\sigma = 1/2$ since $\overline{\xi(s)} = \xi(\overline{s})$, and that all zeros of $\xi(s)$ must lie in the strip $0 \le \sigma \le 1$. Besides, $\zeta(s)$ has simple zeros at $s = -2, -4, \cdots$.

Now we write the main result of this lecture as follows:

Theorem 1.1. *we have*

$$\xi(s,\chi) = \overline{\xi(1-\overline{s},\chi)}$$

and

$$\xi(s) = \xi(1-s),$$

where $q > 1$ and χ is a primitive character mod q.

The growth of $\xi(s)$ as well as $\xi(s,\chi)$ is well-established by the fact that $L(s,\chi)$ and $\zeta(s)$ are bounded for $\sigma > 1+\delta; \delta > 0$ and the earlier estimations given for $\sigma > \delta$ (see (1.8), (1,9)). Combined with the functional equation (or the symmetry around $\sigma = 1/2$), this will enable us to get more precise information of the distribution of zeros.

Lecture II

In this lecture we shall introduce Chebyshev's function $\psi(x)$ and its generalization $\psi_{q,\ell}(x)$, and establish their asympotic formula.

2.1. Estimation of the number of zeros of an analytic function

Lemma 2.1. *If $f(z)$ is analytic in $|z| \le R$, $f(0) \ne 0$ and $|f(z)| < M$ for $|z| \le R$, then the number of zeros of $f(z)$ in the circle $|z| < r$ where $r < R$ is bounded by*

$$\frac{\log \frac{M}{|f(0)|}}{\log \frac{R}{r}}.$$

Proof. Denote the zeros of $f(z)$ in the circle $|z| \le r$ by α_i and write $r_i = |\alpha_i|$, write

$$g(z) = \prod_i \frac{z - \alpha_i}{R - \frac{\overline{\alpha_i} z}{R}}.$$

We have

$$|g(0)| = \prod_i \frac{r_i}{R} \le \left(\frac{r}{R}\right)^n$$

if n is the number of zeros. Since $\dfrac{f(z)}{g(z)}$ is regular in $|z| \le R$ and $|g(z)| = 1$ on the boundary, we have

$$\left(\frac{R}{r}\right)^n |f(0)| \le \frac{|f(0)|}{|g(0)|} \le M$$

by maximum modules theorem. The lemma follows.

2.2. Stirling's formula

1) We assume first $\operatorname{Re} z > 1$. Then by integration by parts we have

$$\begin{aligned} \Gamma(z) &= \int_0^\infty t^z e^{-t} \frac{dt}{t} = (z-1) \int_0^\infty t^{z-1} e^{-t} \frac{dt}{t} \\ &= (z-1)\Gamma(z-1). \end{aligned}$$

Since $\Gamma(1) = 1$, we get

$$\Gamma(n) = (n-1)!$$

if n is a positive integer. Now we proceed to find an asymptotic formula for $\Gamma(n)$. Since

$$\begin{aligned}\int_{\nu-\frac{1}{2}}^{\nu+\frac{1}{2}} \log t\,dt &= \int_0^{\frac{1}{2}} \big(\log(\nu+t)+\log(\nu-t)\big)\,dt \\ &= \int_0^{\frac{1}{2}} \big(\log \nu^2 + \log\big(1-\frac{t^2}{\nu^2}\big)\,dt \\ &= \log \nu + C_\nu,\end{aligned}$$

where $C_\nu = \mathcal{O}(\nu^{-2})$, we have

$$\begin{aligned}\log\Gamma(n) &= \sum_{\nu=1}^{n-1}\log\nu = \int_{\frac{1}{2}}^{n-\frac{1}{2}}\log t\,dt - \sum_{\nu=1}^{n-1} C_\nu \\ &= (n-\frac{1}{2})\log(n-\frac{1}{2}) - \frac{1}{2}\log\frac{1}{2} - \big(n-\frac{1}{2}\big) + \frac{1}{2} + C' + o(1) \\ &= (n-\frac{1}{2})\log n - n + C + o(1) \quad (\text{as } n\to\infty), \qquad (2.1)\end{aligned}$$

where $C = \frac{1}{2} + \frac{1}{2}\log 2 - \sum_{\nu=1}^{\infty} C_\nu$ is a constant.

2) Suppose that $\operatorname{Re} z > 0$ and $\operatorname{Re} w > 0$. We have

$$\begin{aligned}\Gamma(z)\Gamma(w) &= \int_0^\infty t^{z-1}e^{-t}\,dt \int_0^\infty q^{w-1}e^{-q}\,dq \ (q=t\mu) \\ &= \int_0^\infty \mu^{w-1}\,d\mu \int_0^\infty t^{z+w-1}e^{-t(1+\mu)}\,dt \\ &= \Gamma(z+w)\int_0^\infty \frac{\mu^{w-1}}{(1+\mu)^{z+w}}\,du \quad (\mu = \frac{1}{\lambda}-1) \\ &= \Gamma(z+w)\int_0^1 (1-\lambda)^{w-1}\lambda^{z-1}\,d\lambda.\end{aligned}$$

If $\operatorname{Re} z > h > 0$, we obtain

$$\begin{aligned}\frac{\Gamma(z-h)\Gamma(h)}{\Gamma(z)} &= \int_0^1 (1-\lambda)^{z-h-1}\lambda^{h-1}d\lambda \\ &= \frac{1}{h} + \int_0^1 \big((1-\lambda)^{z-h-1}-1\big)\lambda^{h-1}d\lambda \\ &= \frac{1}{h} + \int_0^1 \big((1-\lambda)^{z-1}-1\big)\lambda^{-1}d\lambda + o(1) \quad (\text{as } h\to\infty).\end{aligned}$$

Expanding the left hand side by Taylor series, we have

$$\frac{1}{\Gamma(z)}\big(\Gamma(z) - h\Gamma'(z) + \cdots\big)\,\big(\frac{1}{h} + T'(1) + \cdots\big),$$

so that

$$\frac{\Gamma'}{\Gamma}(z) = -\int_0^1 \big((1-\lambda)^{z-1} - 1\big)\frac{d\lambda}{\lambda} + \Gamma'(1)$$

Substituting $\frac{1}{\lambda}$ by the series

$$\frac{1}{\lambda} = \sum_{n=0}^{\infty}(1-\lambda)^n$$

and integration term by term we get

$$\frac{\Gamma'}{\Gamma}(z) = \sum_{n=0}^{\infty}\big(\frac{1}{n+1} - \frac{1}{n+z}\big) + \Gamma'(1)$$

or

$$\begin{aligned}
\frac{\Gamma'}{\Gamma}(z) + \frac{1}{z} &= \sum_{n=1}^{\infty}\big(\frac{1}{n} + \frac{1}{n+1} - \frac{1}{n} - \frac{1}{n+z}\big) + 1 + \Gamma'(1) \\
&= \sum_{n=1}^{\infty}\big(\frac{1}{n} - \frac{1}{n+z}\big) + 1 + \Gamma'(1) - \sum_{n=1}^{\infty}\big(\frac{1}{n} - \frac{1}{n+1}\big) \\
&= \sum_{n=1}^{\infty}\big(\frac{1}{n} - \frac{1}{n+z}\big) + \Gamma'(1)\,.
\end{aligned}$$

Therefore

$$\frac{1}{\Gamma(z)} = e^{-\Gamma'(1)z} z \prod_{n=1}^{\infty}\big(1 + \frac{z}{n}\big)e^{-\frac{z}{n}}. \tag{2.2}$$

Take $z = 1$. We have

$$\begin{aligned}
-\Gamma'(1) &= -\log \prod_{n=1}^{\infty}\big(1 + \frac{1}{n}\big)e^{-\frac{1}{n}} \\
&= \lim_{n\to\infty}\big(\sum_{n=1}^{N}\frac{1}{n} - \log(N+1)\big) = \gamma.
\end{aligned}$$

γ is called to be the Euler's constant.

3) We take the principal value of logarithm, and we obtain

$$
\begin{aligned}
\int_0^N \frac{[u]-u+\frac{1}{2}}{u+z}\,du &= \sum_{n=0}^{N-1}\int_n^{n+1}\big(\frac{n+\frac{1}{2}+z}{u+z}-1\big)\,du \\
&= \sum_{n=0}^{N-1}\big(\log(n+1+z)-\log(n+z)\big)\,\big(n+\frac{1}{2}+z\big)-N \\
&= -\sum_{n=1}^{N-1}\log(n+z)+\big(N-\frac{1}{2}+z\big)\log(N+z)-\big(\frac{1}{2}+z\big)\log z-N \\
&= \sum_{n=1}^{N-1}\big(\frac{z}{n}-\log(1+\frac{z}{n})\big)-\log(N-1)!-z\sum_{n=1}^{N-1}\frac{1}{n} \\
&\quad +\big(N-\frac{1}{2}+z\big)\log(N+z)-\big(\frac{1}{2}+z\big)\log z-N.
\end{aligned}
$$

Since

$$\log(n+z)=\log n+\frac{z}{n}+\mathcal{O}\big(\frac{1}{n^2}\big)\quad(\text{as } n\to\infty)$$

we have by (2.2)

$$
\begin{aligned}
\log\Gamma(z) &= \sum_{n=1}^{\infty}\big(\frac{z}{n}-\log(1+\frac{z}{n})\big)-\gamma z-\log z \\
&= \log(N-1)!+z\sum_{n=1}^{N-1}\frac{1}{n}-\big(N-\frac{1}{2}+z\big)\log(N+z) \\
&\quad +\big(\frac{1}{2}+z\big)\log z+N-\gamma z-\log z+\int_0^N\frac{[u]-u+\frac{1}{2}}{u+z}\,du+\mathcal{O}\big(\frac{1}{N}\big) \\
&= \big(z+\frac{1}{z}\big)\log z-\big(N-\frac{1}{2}+z\big)\,\big(\log N+\frac{z}{N}+\mathcal{O}(N^{-2})\big)+N-\gamma z \\
&\quad -\log z+\log(N-1)!+z\big(\log N+\gamma+\mathcal{O}(N^{-1})\big) \\
&\quad +\int_0^N\frac{[u]-u+\frac{1}{2}}{u+z}\,du+\mathcal{O}(N^{-1}),
\end{aligned}
$$

and by (2.1)

$$
\begin{aligned}
\log\Gamma(z) = \big(z-\frac{1}{2}\big)\log z-\big(N-\frac{1}{2}+z\big)\,\big(\log N+\frac{z}{N}+\mathcal{O}(N^{-2})\big) \\
+N-\gamma z+\big(N-\frac{1}{2}\big)\log N-N+C
\end{aligned}
$$

$$+z\big(\log N+\gamma+o(1)\big)+\int_0^N \frac{[u]-u+\frac12}{u+z}\,du+o(1)\quad (\text{as } N\to\infty).$$

Let $N\to\infty$. Then

$$\log\Gamma(z)=\big(z-\frac12\big)\log z-z+C+\int_0^\infty \frac{[u]-u+\frac12}{u+z}\,du.$$

Now we treat the integral in the right-hand side. Let

$$\varphi(u)=\int_0^u \big([v]-v+\frac12\big)\,dv.$$

We have $\varphi(n+1)=\varphi(n)$ if n is an integer and so $\varphi(u)$ is bounded. Hence

$$\int_0^\infty \frac{[u]-u+\frac12}{u+z}\,du=\int_0^\infty \frac{\varphi'(u)}{u+z}\,du$$

$$=\int_0^\infty \frac{\varphi(u)}{(u+z)^2}\,du=\mathcal{O}\Big(\int_0^\infty \frac{du}{(u+z)(\overline{u+z})}\Big).$$

Write $z=re^{i\theta}$, where $-\pi+\delta\le\theta\le\pi-\delta;\ \delta>0$. Then

$$\int_0^\infty \frac{[u]-u+\frac12}{u+z}\,du=\mathcal{O}\Big(\int_0^\infty \frac{du}{u^2+r^2-2ur\cos\delta}\Big)$$

$$=\mathcal{O}\Big(\frac1r\int_0^\infty \frac{d\nu}{\nu^2+1-2\nu\cos\delta}\Big)=\mathcal{O}\Big(\frac1r\Big).$$

Hence we have proved the following

Lemma 2.2. (Stirling's Formula.) *Let $a>0$. Then*

$$\log\Gamma(z+a)=\big(z+a-\frac12\big)\log z-z+C+\mathcal{O}\Big(\frac{1}{|z|}\Big)\quad (\text{as } z\to\infty) \tag{2.3}$$

holds uniformly for $-\pi+\delta\le\arg z\le\pi-\delta;\ \delta>0$.

Remark. The constant C may be calculated to be equal to $\log\sqrt{2}\pi$.

2.3. Zero estimations for $\xi(s,\chi)$ and $\xi(s)$

By (1.8) and (1.9), we see that

$$|L(s,\chi)|<A_\delta q(1+|t|)\text{ and }|\zeta(s)|<A_\delta(1+|t|)$$

for $\sigma\ge\delta>0$, and then by (2.3) and the functional equations of $\xi(s,\chi)$ and $\xi(s)$, we see that $\xi(s,\chi)$ and $\xi(s)$ are bounded by

$$\mathcal{O}(q^{R+1}R^R)\text{ and }\mathcal{O}(R^R)$$

respectively for $|s-2|<2R$. Hence by Lemma 2.1, it follows that the the number of zeros of $\xi(s,\chi)$ and $\xi(\chi)$ is $|s-2|<R$ are not exceeding

$$\mathcal{O}(R\log qR) \text{ and } \mathcal{O}(R\log R)$$

respectively.

To obtain more precise results, we need to use the so-called "argument principle": Let $f(z)$ be an analytic function on a simply connected domain D with boundary C. Assume $f(z)$ has no singular points on C. If $f(z)$ has a zero α of degree m on D, then $\frac{f'}{f}(z)$ has a simple pole with residue m on D, and the integral along C is equal to $2\pi im$ with this α. Therefore the total number n of zeros of $f(z)$ on D is equal to the increase of argument of $f(z)$ along C times $(2\pi)^{-1}$, that is,

$$n=\frac{1}{2\pi}\big[\arg f(z)\big]_C.$$

Take D to be a rectangle with boundary C and four vertices $2,\ 2+iT,\ -1+iT,\ -1$ as shown in Figure 2.1.

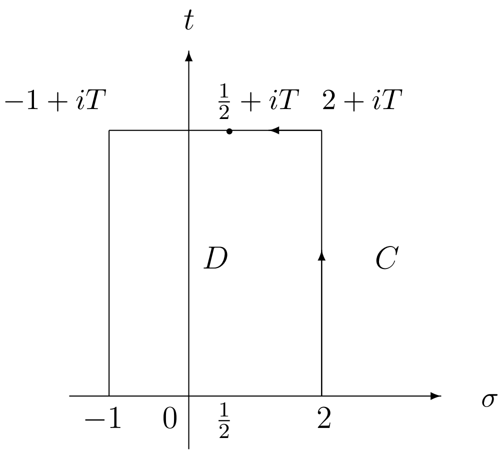

Figure 2.1

We shall denote the number of zeros in this rectangle by $N(T)$ for $\xi(s)$ (or $(\xi(s))$ and $N(T,\chi)$ for $\xi(s,\chi)$ (or $L(s,\chi)$). Then

$$N(T,\chi)=\frac{1}{2\pi}\big[\arg\xi(s,\chi)\big]_C.$$

We omit here the similar expression for $N(\Gamma)$. We may consider the variation of argument of $\xi(s,\chi)$ from $\frac{1}{2},\ 2,\ 2+iT,\ \frac{1}{2}+iT$, and the left half of the rectangle will add the same amount by virtue of the functional equation.

Now suppose that T is not the vertical coordinate of a zero of $\xi(z,\chi)$. It follows from Lemma 2.2 that a factor

$$\pi^{-\frac{s}{2}}\, q^{\frac{s}{2}}\Gamma\big(\frac{s}{2}\big) \text{ or } \pi^{-\frac{s}{2}}\, q^{\frac{s}{2}}\Gamma\big(\frac{s+1}{2}\big)$$

gives in both cases that the argument increases from 2 to $2+iT$ by

$$\frac{T}{2}\Big(\log\frac{qT}{2\pi}-1\Big)+\mathcal{O}(\log T).$$

Also the variation of the argument $L(s,\chi)$ or $\xi(s)$ on $\sigma=2$ is seen to be bounded since

$$\begin{aligned}\operatorname{Re} L(s,\chi) &\geq 1-\sum_{n=2}^{\infty}\frac{1}{n^2}\\ &\geq 1-\frac{1}{4}-\sum_{n=3}^{\infty}\big(\frac{1}{n-1}-\frac{1}{n}\big)=\frac{1}{4}.\end{aligned}$$

It remains to estimate the variation of $\zeta(z)$ or $L(s,\chi)$ on the stretch $2+iT$, $\frac{1}{2}+iT$ and $\frac{1}{2}$, 2. Let

$$f(z)=L(2+iT-z,\,\chi)+\overline{L(2+iT-\overline{z},\,\chi)}.$$

We have $f(0)>1/2$ and in the circle $|z|\leq\frac{7}{4}$ we have by (1.8)

$$|f(z)|<cq(1+T).$$

Thus by Lemma 2.1, the number n of zeros of $f(z)$ in $|z|\leq 3/2$ is bounded by

$$n<c'\log q(2+T).$$

But the zeros on the positive real axis are simply the points where $L(2+iT-z,\,\chi)$ is purely imaginary. Between these the argument of L can vary at most π so the total variation of $L(\sigma+iT,\,\chi)$ where σ goes from $2\to\frac{1}{2}$ is

$$\leq(n+2)\pi=\mathcal{O}\big(\log q(T+2)\big).$$

Consider

$$g(z)=L(2-z,\,\chi)+\overline{L(2-\overline{z},\,\chi)}\,.$$

Then by similar argument, we know that total variation of $L(\sigma, \chi)$ when σ goes from $\frac{1}{2}$ to 2 is $\mathcal{O}(\log q)$. Hence the total variation around the rectangle is thus

$$T\big(\log \frac{qT}{2\pi} - 1\big) + \mathcal{O}\big(\log q(2+T)\big),$$

and we have

Theorem 2.3. *We have*

$$N(T) = \frac{T}{2\pi}\big(\log \frac{T}{2\pi} - 1\big) + \mathcal{O}\big(\log(2+T)\big)$$

and

$$N(T,\, \chi) = \frac{T}{2\pi}\big(\log \frac{qT}{2\pi} - 1\big) + \mathcal{O}\big(\log q(2+T)\big)\,,$$

where χ is even or odd.

2.4. Product formulas of $\xi(s, \chi)$ and $\xi(s)$

Lemma 2.4. *Suppose $f(z)$ is an integral fuction without zeros, and $\{\gamma_m\}$ is a sequence of positive numbers such that $\gamma_m \to \infty$ (as $m \to \infty$) and*

$$\log |f(z)| < C_\varepsilon \gamma_m^{\alpha+\varepsilon}, \quad |z| = \gamma_m,$$

where α is a positive constant and ε is any preassigned positive constant. Then $f(z) = e^{P(z)}$, where $P(z)$ is a polynomial of degree $\le \alpha$.

Proof. We always take the principal value of logarithm $P(z) = \log f(z)$. Now we proceed to show that $P(z)$ is a polynomial of degree $\le \alpha$. Since $P(z)$ is also an integral function. We expand it by power series

$$P(z) = \sum_{n=0}^{\infty} c_n\, z^n, \quad |z| < \infty.$$

Write $c_n = |c_n| e^{i\theta n}$ and $z = \gamma e^{i\theta}$, where $\gamma \in \{r_m\}$. Then

$$\text{Re}\,\big(P(z) - c_0\big) = \sum_{n=1}^{\infty} |c_n| r^n \cos(\theta_n + n\theta).$$

We have

$$\int_0^{2\pi} \text{Re}\,\big(P(z) - c_0\big)\, d\theta = 0$$

and

$$\int_0^{2\pi} \cos(\theta_n + n\theta)\big(\mathrm{Re}\,(P(z) - c_0)\big)\, d\theta = \pi |c_n| r^n, \;\; n = 1, 2, \cdots .$$

Therefore

$$\begin{aligned} \pi |c_n| r^n &= \int_0^{2\pi} \big(1 + \cos(\theta_n + n\theta)\big)\, \mathrm{Re}\,\big(P(z) - c_0\big)\, d\theta \\ &\le C_\varepsilon r^{\alpha+\varepsilon} . \end{aligned}$$

Let $r = r_m \to \infty$. Then $|c_n| = 0$ if $n > \alpha$, and so $P(z)$ is a polynomial with degree $\le \alpha$. The lemma is proved.

Write

$$f(s) = \prod_\rho (1 - \frac{s}{\rho}) e^{\frac{s}{\rho}},$$

where $\rho = \sigma + i\gamma$ runs over all zeros of $\xi(s,\,\chi)$. We arrange a order for $\rho_1, \;\rho_2, \;\cdots$ according to the increasing order of $|\gamma|$. Now we proceed to show that

$$\sum_n \frac{1}{|\rho_n|^\alpha} \text{ is } \begin{cases} \text{convergent, if } \alpha > 1; \\ \text{divergent, if } \alpha \le 1. \end{cases} \tag{2.4}$$

In fact, it follows from Theorem 2.3 that the number of zeros of $\xi(s, \chi)$ satisfying $T \le |\gamma_m| < T + 1$ is at most $\mathcal{O}\big(\log q(T+2)\big)$, and so

$$\begin{aligned} \sum_n \frac{1}{|\rho_n|^\alpha} &= \mathcal{O}\big(\sum_{m=0}^{\infty} \sum_{m \le \gamma_n < m+i}^{\infty} \frac{1}{|\gamma_n|^\alpha}\big) \\ &= \mathcal{O}\big(\sum_{m=0}^{\infty} \frac{\log q(m+2)}{(m+1)^\alpha}\big) = \mathcal{O}(\log q) \end{aligned}$$

if $\alpha > 1$. Similarly we may prove that the series is divergent if $\alpha \le 1$. Now let us show that $f(s)$ is an analytic function with only zeros at the ρ's. Write

$$f(R, s) = \prod_{|p| > 2R} (1 - \frac{s}{\rho}) e^{\frac{s}{\rho}}.$$

We have, for $|\rho| > 2R$ and $|s| \le R$.

$$(1 - \frac{s}{\rho}) e^{\frac{s}{\rho}} = e^{-\sum_{\ell=2}^{\infty} \frac{1}{(\ell-1)} (\frac{s}{\rho})^\rho} = 1 + u_\rho(s) \ne 0$$

and

$$\Big| \sum_{\ell=2}^{\infty} \frac{1}{(\ell-1)} \big(\frac{s}{\rho}\big)^\ell \Big| \le 2 \Big| \frac{s}{\rho} \Big|^2 < 1.$$

Therefore

$$\big|u_\rho(s)\big| \le 4\big|\frac{s}{\rho}\big|^2,$$

and by (2.4), $f(R,s)$ is an analytic function without zeros in $|s| \le R$. It is obvious that $\prod\limits_{|\rho|\le 2R}\big(1-\dfrac{s}{\rho}\big)e^{\frac{s}{\rho}}$ is an analytic function in $|s| \le R$ which has only zeros at ρ's with $|\rho| \le R$. Since R is arbitrary, $f(s)$ is thus an analytic function with zeros only at the ρ's.

By (2.4), we can always find a sequence of increasing positive numbers $\{r_m\}$, where $r_m \to \infty$, such that

$$\big|r_m - |\rho|\ \big| > |\rho|^{-2} \tag{2.5}$$

for all m and ρ. In fact, it follows by (2.4) the total length of the intervals with center ρ and length $2|\rho|^{-2}$ is finite, so that the r not belonging to these intervals satisfies our requirement. Take $r \in \{r_m\}$ and write $f(s)$ as

$$f(s) = \prod_{|\rho|\le r/2} \cdot \prod_{r/2<|\rho|\le 2r} \cdot \prod_{|\rho|>2r} = f_1(s)f_2(s)f_3(s).$$

(a) Suppose $|\rho| \le r/2$. Then

$$\Big|\big(1-\frac{s}{\rho}\big)e^{\frac{s}{\rho}}\Big| \ge \big(\frac{r}{|\rho|}-1\big)e^{-\frac{r}{|\rho|}} \ge e^{-(\frac{r}{|\rho|})^{1+\varepsilon}}$$

for $|s| = r$, and by (2.4) we obtain

$$|f_1(s)| \ge e^{-c_\varepsilon r^{1+\varepsilon}}.$$

(b) Suppose $\dfrac{r}{2} < \rho| \le 2r$. Then by (2.5)

$$\Big|\big(1-\frac{s}{\rho}\big)e^{\frac{s}{\rho}}\Big| \ge \frac{\big||\rho|-r\big|}{|\rho|}e^{-2} \ge e^{-2}\frac{1}{|\rho|^3} \ge (8e^2r^3)^{-1}$$

for $|s| = r$. Since by Theorem 2.3, the number of ρ's satisfying $\dfrac{r}{2} < |\rho| \le 2r$ is $\mathcal{O}(r\log q\,r)$, we have

$$|f_2(s)| \ge r^{-c\,r\log qr}.$$

(c) Finally suppose $|\rho| > 2r$. Then

$$\log\Big|(1-\frac{s}{\rho})e^{\frac{s}{\rho}}\Big| = \mathrm{Re}\Big(-\sum_{j=2}^{\infty}\frac{1}{j}\big(\frac{s}{\rho}\big)^j\Big) \ge -2\big(\frac{r}{|\rho|}\big)^2$$

for $|s| = r$, and by the Theorem 2.3 we get

$$\begin{aligned} |f_3(s)| &\geq e^{-2r^2 \sum\limits_{|\rho|>2r} |\rho|^{-2}} \geq e^{-2r^2 \sum\limits_{n \geq r} n^{-2} \log qn} \\ &\geq e^{-c_\varepsilon r^{1+\varepsilon} \log q}. \end{aligned}$$

Combining (a), (b), (c), we have

$$|f(s)|^{-1} \leq e^{c_\varepsilon r^{1+\varepsilon} \log q}.$$

On the other hand, it yields by the functional equation of $\xi(s, \chi)$ that

$$\begin{aligned} \big|\xi(s, \chi)\big| &< e^{cr(\log q + \log r)} \log qr \\ &\leq e^{c_\varepsilon r^{1+\varepsilon} \log q}. \end{aligned}$$

Therefore

$$\left| \frac{\xi(s, \chi)}{f(s)} \right| \leq e^{c_\varepsilon r^{1+\varepsilon} \log q},$$

and by Lemma 2.4, we have

$$\xi(s, \chi) = e^{a+bs} f(s).$$

We may treat similarly the case $q = 1$. We have thus the following product formulas

Theorem 2.5. *We have*

$$\xi(s) = c' e^{cs} \prod_\rho (1 - \frac{s}{\rho}) e^{\frac{s}{\rho}}$$

and

$$\xi(s, \chi) = c'_\chi e^{c_\chi s} \prod_\rho (1 - \frac{s}{\rho}) e^{\frac{s}{\rho}},$$

where in each case ρ runs through the zeros of the function on the left-hand side and c' and c are constants and the c'_χ and c_χ are constants depending on χ only. By theorem 2.5 and (2.2), we obtain the product formula for $\xi(s)$

$$\zeta(s) = (s-1)^{-1} e^{A+Bs} \prod_{n=1}^{\infty} (1 + \frac{s}{2n}) e^{-\frac{s}{2n}} \prod_\rho (1 - \frac{s}{\rho}) e^{\frac{s}{p}}.$$

Similarly we have for primitive and even character $\chi \mod q$

$$L(s, \chi) = (\frac{s}{2}) e^{A_\chi + B_\chi s} \prod_{n=1}^{\infty} (1 + \frac{s}{2n}) e^{-\frac{s}{2n}} \prod_\rho (1 - \frac{s}{\rho}) e^{\frac{s}{\rho}},$$

and for primitive and odd character $\chi \mod q$

$$L(s,\chi) = (\frac{s+1}{2})e^{C_\chi + D_\chi s}\prod_{n=1}^{\infty}(1+\frac{s+1}{2n})e^{-\frac{s+1}{2n}}\prod_{\rho}(1-\frac{s}{\rho})e^{\frac{s}{\rho}},$$

where A_χ, B_χ, C_χ, D_χ *are constants depending on* χ *only.*

2.5. Average formulas of $\psi_\chi(x)$ and $\psi(x)$

Taking the logarithmic derivatives for $\zeta(s)$ and $L(s,\chi)$, we obtain

$$\frac{\zeta'}{\zeta}(s) = c'' - \frac{1}{s-1} + \sum_{\rho}(\frac{1}{s-\rho}+\frac{1}{\rho}) + \sum_{n=1}^{\infty}(\frac{1}{s+2n}-\frac{1}{2n}), \tag{2.6}$$

and for χ primitive and even

$$\frac{L'}{L}(s,\chi) = c''_\chi + \sum_{\rho}(\frac{1}{s-\rho}+\frac{1}{\rho}) + \frac{1}{s} + \sum_{n=1}^{\infty}(\frac{1}{s+2n}-\frac{1}{2n}), \tag{2.7}$$

and χ primitive and odd

$$\frac{L'}{L}(s,\chi) = c''_\chi + \sum_{\rho}(\frac{1}{s-\rho}+\frac{1}{\rho}) + \sum_{n=0}^{\infty}(\frac{1}{s+(2n+1)}-\frac{1}{2n+2}). \tag{2.8}$$

By Theorem 2.3 it is easy to show for $s=\sigma+it;\ A>\sigma\geq 1+\delta$.

$$\sum_{\rho}\left|\frac{1}{s-\rho}+\frac{1}{\rho}\right| = \mathcal{O}\big(\log^2 q(2+|t|)\big)$$

and

$$\sum_{n=1}^{\infty}\left|\frac{1}{s+n}-\frac{1}{n}\right| = \mathcal{O}\big(\log(2+|t|)\big)\,.$$

In fact,

$$\begin{aligned}\sum_{\rho}\left|\frac{1}{s-\rho}+\frac{1}{\rho}\right| &= \sum_{\rho}\left|\frac{|s|}{|s-\rho|\,|\rho|}\right| = \sum_{|\rho|\leq 2|t|} + \sum_{|\rho|>2|t|} \\ &= \mathcal{O}\Big(\sum_{1\leq n\leq 2|t|}\frac{|t|\log q(n+2)}{|t|n}\Big) + \mathcal{O}\Big(\sum_{n>|t|}\frac{|t|}{n^2}\log q(|t|+2)\Big) \\ &= \mathcal{O}\big(\log^2 q(|t|+2)\big).\end{aligned}$$

The latter formula can be proved similarly.

Lemma 2.6. *We have*

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\frac{x^{s+1}}{s(s+1)}\,ds=\begin{cases}0, & \text{for } 0\le x\le 1,\\ x-1, & \text{for } x\ge 1.\end{cases}$$

Proof. If $0\le x\le 1$, we take the contour C as shown by Figure 2.2, and so by Cauchy theorem

$$\int_C \frac{x^{s+1}}{s(s+1)}ds=0.$$

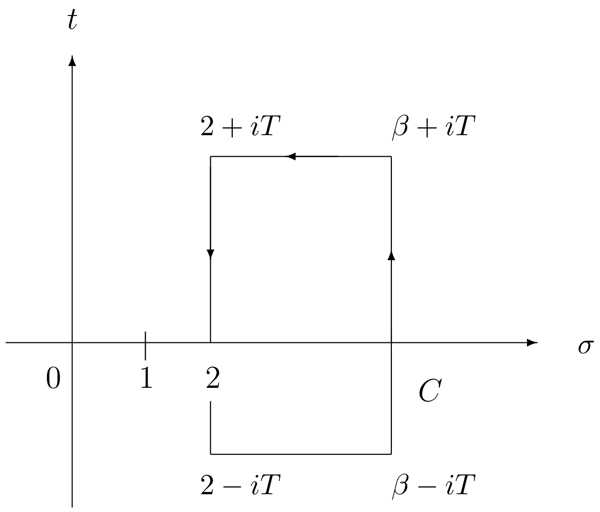

Figure 2.2

Since

$$\int_{\beta+iT}^{2+iT}\frac{x^{s+1}}{s(s+1)}ds=\mathcal{O}\big(\frac{|x|^{2+1}}{T^2}|\beta-2|\big)$$

(the integral from $2-iT$ to $\beta-iT$ has the same bound) and

$$\int_{\beta-iT}^{\beta+iT}\frac{x^{s+1}}{s(s+1)}ds=\mathcal{O}\big(|x|^{\beta+1}\big),$$

we obtain the first assertion immediately if we put $\beta=T\to\infty$.

If $x>1$, we take the contour C' as shown by Figure 2.3, and so

$$\int_{C'}\frac{x^{s+1}}{s(s+1)}ds=x-1.$$

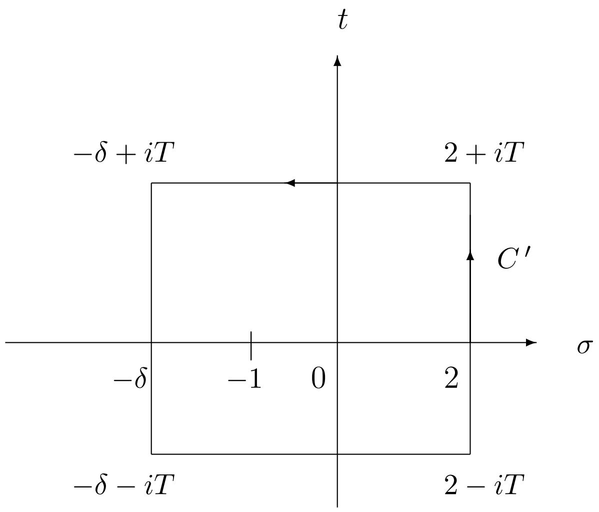

Figure 2.3

We thus have the second assertioin of the lemma by the similar argument. If

$$f(s) = \sum_{n=1}^{\infty} \frac{c_n}{n^s}$$

is absolutely convergent for $\sigma > 1$, then by Lemma 2.6 we get

$$\frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{x^{s+1}}{s(s+1)} f(s) ds = \frac{1}{2\pi i} \sum_{n=1}^{\infty} n\, c_n \int_{2-i\infty}^{2+i\infty} \frac{(\frac{x}{n})^{s+1}}{s(s+1)} ds$$

$$= \sum_{n \le x} (x - n) c_n = \int_0^x \big(\sum_{n \le t} c_n \big)\, dt.$$

We define the von Mangoldt function

$$\wedge(n) = \begin{cases} \log p, & \text{for } n = p^r,\ r \ge 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\sum_{n=1}^{\infty} \frac{\wedge(n)}{n^s} = -\frac{\zeta'}{\zeta}(s)$$

and

$$\sum_{n=1}^{\infty} \chi(n) \frac{\wedge(n)}{n^s} = -\frac{L'}{L}(s, \chi).$$

Denote by

$$\psi(x) = \sum_{n \le x} \wedge(n),$$

the Chebyshev function, and

$$\psi_\chi(x) = \sum_{n \le x} \chi(n) \wedge (n).$$

Then by taking $c_n = \wedge(n)$, we obtain

$$\int_0^x \psi(t)dt = \sum_{n \le x} (x \cdot n) \wedge (n) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{x^{s+1}}{s(s+1)} \big(- \frac{\zeta'}{\zeta}(s) \big)\, ds.$$

Here we may use the precious expansion for $-\dfrac{\zeta'}{\zeta}$ and can integrate by term since integral exists if we take absolute value everywhere. We take contour C' shown by Figure 2.3 and let $\delta = T \to \infty$. Then the integral on the right-hand side equals to the sum of residues at $s = 0, -1, \rho$'s, $1, -2n$, that is;

$$\int_0^x \psi(t)dt = \frac{x^2}{2} - \sum_\rho \frac{x^{\rho+1}}{\rho(\rho+1)} + \mathcal{O}(x). \tag{2.9}$$

Similarly, we have

$$\int_0^x \psi_\chi(t)dt = - \sum_\rho \frac{x^{\rho+1}}{\rho(\rho+1)} + \mathcal{O}(x). \tag{2.10}$$

where the ρ run through the zeros of $\xi(s)$ or $\xi(s, \chi)$ respectively.

2.6. Asymptotic estimations for $\psi_{q,\ell}(x)$ and $\psi(x)$

First, we need to show that neither $\zeta(s)$ or any of the $L(s, \chi)$ have a zero with real part 1. The proof is based on the simply inequality

$$3 + 4\cos\varphi + \cos 2\varphi = 2(1 + \cos\varphi)^2 \ge 0.$$

For $\sigma > 1$ we have

$$\begin{aligned} & \mathrm{Re} \big(- 3\frac{\zeta'}{\zeta}(\sigma) - 4\frac{L'}{L}(\sigma + it, \chi) - \frac{L'}{L}(\sigma + 2it, \chi^2) \big) \\ = & \sum_{n=1}^{\infty} \frac{\wedge(n)}{n^\sigma} (3 + 4\cos(t \log n - \theta_n) + \cos(2t \log n - 2\theta_n)\big) \\ \ge & \ 0, \end{aligned}$$

where $\chi(n) = e^{i\theta_n}$. If either $t \neq 0$ for $\chi^2 \neq \chi_0$, and if $1 + it$ is a zero of $L(s, \chi)$, then by letting $\sigma \to 1$, the left-hand side of the above inequality would tend to $-\infty$. This leads to a contradiction and so $L(1 + it, \chi) \neq 0$.

In the case $t = 0,\ \chi^2 = \chi_0$, we look at $\zeta(s)L(s, \chi)$ and see that

$$\zeta(s)L(s,\chi) = \big(\sum_{n=1}^{\infty} \frac{1}{n^s}\big)\big(\sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}\big) = \sum_{n=1}^{\infty} \frac{c_n}{n^s},$$

where

$$c_n = \sum_{m|n} \chi(m) = \sum_{m|p_1^{\alpha_1}\cdots p_r^{\alpha_r}} \chi(m) = \prod_{i=1}^{r} \big(1 + \chi(p_i) + \cdots + \chi(p_i^{\alpha_i})\big) \geq 0$$

and

$$c_{n^2} = \sum_{m|n^2} \chi(m) = \sum_{i=1}^{r} \big(1 + \chi(p_i) + \cdots + \chi(p_i^{2\alpha_i})\big) \geq 1.$$

If $L(s, \chi)$ has a zero at $s = 1$ then $\zeta(s)L(s, \chi)$ is an integral function, so its power series around $s = 2$ converges everywhere. The k-th coefficient is in absolute value

$$\begin{aligned} \frac{1}{k!}(-1)^k\big(\zeta(2)L(2,\chi)\big)^{(k)} &= \frac{1}{k!}\sum_{n=1}^{\infty} \frac{c_n \log^k n}{n^2} \\ &\geq \frac{1}{k!}\sum_{n=1}^{\infty} \frac{2^k \log^k n}{n^4}. \end{aligned}$$

Comparing this to the k-th coefficient in the power series of $\zeta(2s)$ around $s = 2$ we find that to be in absolute value

$$\frac{1}{k!}\sum_{k=1}^{\infty} \frac{2^k \log^k n}{n^4}.$$

But this power series cannot converge beyond $s = \frac{1}{2}$ since $\zeta(2s)$ has a pole there. This gives a contradiction, so $L(1, \chi) \neq 0$.

From this we now easily conclude

Lemma 2.7. *We have*

$$\int_0^x \psi(t)dt = \frac{x^2}{2} + o(x^2)$$

and

$$\int_0^x \psi_\chi(t)dt = o(x^2).$$

Proof. By (2.9) and (2.10), we know all that is to show that in each case

$$\sum_{\rho} \frac{x^{\beta+1}}{|\rho(\rho+1)|} = o(x^2),$$

where $\rho = \beta + i\gamma$.

By Theorem 2.3, we can always choose a T so large that

$$\sum_{|r|\geq T} \frac{1}{|\rho(\rho+1)|} < \frac{\varepsilon}{2}$$

if ε is a given positive quantity. Then

$$\sum_{\rho} \frac{x^{\beta+1}}{|\rho(\rho+1)|} < \sum_{|r|<T} \frac{x^{\beta+1}}{|\rho(\rho+1)|} + \frac{\varepsilon}{2}x^2.$$

Since in the finite sum $|\gamma| < T$ all exponents $\beta + 1 < 2$, this will be $< \frac{\varepsilon}{2}x^2$ for x sufficiently large. So

$$\sum_{\rho} \frac{x^{\beta+1}}{|\rho(\rho+1)|} < \varepsilon x^2$$

for $x > x_0$, which proves the lemma.

Write by

$$\psi_{q,\ell}(x) = \sum_{\substack{n\leq x \\ n\equiv \ell(q)}} \wedge(n), \quad (\ell, q) = 1.$$

Then by (1.3) we get

$$\begin{aligned} \psi_{q,\ell}(x) &= \frac{1}{\varphi(q)} \sum_{n\leq x} \sum_{\chi} \overline{x}(\ell)\chi(n) \wedge (n) \\ &= \frac{1}{\varphi(q)} \sum_{\chi} \overline{x}(\ell) \sum_{n\leq x} \chi(n) \wedge (n). \end{aligned}$$

For any $\chi \mod q$, it corresponds a unique primitive character $\chi_{q'} \mod q'$. Since

$$\begin{aligned} &\left| \sum_{n\leq x} \chi(n) \wedge (n) - \sum_{n\leq x} \chi_{q'}(n) \wedge (n) \right| \\ = &\left| - \sum_{\substack{n\leq x \\ (n,q)>1}} \chi_{q'}(n) \wedge (n) \right| \leq \sum_{p^{\ell}\leq x} \sum_{p|q} \log p \leq \nu(q) \log x, \end{aligned}$$

where $\nu(q)$ denotes the number of prime factors of q. Hence

$$\psi_{q,\ell}(x) = \frac{1}{\varphi(q)} \sum_{q'|q} \sum_{\chi_{q'}} \overline{\chi}(\ell)\psi_{\chi_{q'}}(x) + \mathcal{O}(\log q \log x)$$

and by Lemma 2.7

$$\begin{aligned}
\int_0^x \psi_{q,\ell}(t)dt &= \frac{1}{\varphi(q)} \int_0^x \psi_{\chi_0}(t)dt + \frac{1}{\varphi(q)} \sum_{q'|q} \sum_{\substack{\chi_{q'} \\ \chi'_q \neq \chi_0}} \overline{\chi}(\ell) \int_0^x \psi_{\chi'_q}(t)dt + o(x^2) \\
&= \frac{1}{\varphi(q)} \frac{x^2}{2} + o(x^2).
\end{aligned}$$

Since $\psi(x)$ is a non-decreasing function, we obtain by Lemma 2.7

$$\frac{1}{h} \int_{x-h}^{x} \psi(t)dt \le \psi(x) \le \frac{1}{h} \int_{x}^{x+h} \psi(t)dt,$$

and so

$$x - \frac{h}{2} - \frac{1}{h} o(x^2) \le \psi(x) \le x + \frac{h}{2} + \frac{1}{h} o(x^2).$$

Take $h = \varepsilon x$ and x so large that $\frac{o(x^2)}{\varepsilon x} < \frac{\varepsilon}{2} x$, and we get

$$x - \varepsilon x \le \psi(x) \le x + \varepsilon x$$

for $x \ge x_0$. Similarly for $\psi_{q,\ell}(x)$. We write the result as follows.

Theorem 2.8. *We have*

$$\psi(x) = x + o(x)$$

and

$$\psi_{q,\ell}(x) = \frac{1}{\varphi(q)} x + o(x).$$

Lecture III

In this lecture we shall prove the Prime Number Theorem (P.N.T) and more general the P.N.T. for the arithmetic progression. And we shall introduce the Riemann Hypothesis (R.H.) for $\zeta(s)$ and the similar hypothesis the so-called Generalized Riemann Hypothesis (G.R.H.) for $L(s,\chi)$, and then we shall give the error estimations for the P.N.T. and the P.N.T. for the arithmetic progression under the assumption of R.H. or G.R.H., and without any suppositions. Besides we shall derive a zero-free region for $\zeta(s)$ by the error estimation for P.N.T.

3.1. Prime Number Theorem

Define

$$\mathcal{J}(x)=\sum_{p\le x}\log p,\quad \mathcal{J}_{q,\ell}(x)=\sum_{\substack{p\equiv\ell(q)\\ p\le x}}\log p,$$

and

$$\pi(x)=\sum_{p\le x}1,\quad \pi_{q,\ell}(x)\sum_{\substack{p\equiv\ell(q)\\ p\le x}}1.$$

We have

$$\mathcal{J}(x)=\psi(x)+\mathcal{O}(\psi\big(x^{1/2})\big)+\mathcal{O}\big(\psi(x^{1/3})\big)+\cdots$$

$$\psi(x)+\mathcal{O}(\sqrt{x})=x+o(x),$$

and generally

$$\mathcal{J}_{q,\ell}(x)=\psi_{q,\ell}(x)+\mathcal{O}(\sqrt{x})=\frac{1}{\varphi(q)}x+0(x)$$

by Theorem 2.8. Now we write

$$\pi(x)=\int_{3/2}^{x}\frac{1}{\log t}d\,\mathcal{J}(t)=\frac{\mathcal{J}(x)}{\log x}+\int_2^x\frac{\mathcal{J}(t)}{t\log^2 t}dt.$$

Here

$$\int_2^x\frac{\mathcal{J}(t)}{t\log^2 t}dt=\mathcal{O}\Big(\int_2^x\frac{dt}{\log^2 t}\Big)=\mathcal{O}\Big(\frac{x}{\log^2 x}\Big).$$

So

$$\begin{aligned}\pi(x)&=\frac{\mathcal{J}(x)}{\log x}+\mathcal{O}\Big(\frac{x}{\log^2 x}\Big)\\&=\frac{x}{\log x}+o\Big(\frac{x}{\log x}\Big).\end{aligned}$$

In the same way, we may treat $\pi_{q,\ell}(x)$. So we obtain the following

Theorem 3.1. *We have the P.N.T.*

$$\pi(x) = \frac{x}{\log x} + o\Big(\frac{x}{\log x}\Big)$$

and the P.N.T. for the arithmetic progression

$$\pi_{q,\ell}(x) = \frac{1}{\varphi(q)} \cdot \frac{x}{\log x} + o\Big(\frac{x}{\log x}\Big).$$

3.2. Prime Number Theorem with remainder term

For later use, we also note that if we write

$$\mathcal{J}(x) = x + \rho(x),$$

then

$$\pi(x) = \int_2^x \frac{dt}{\log t} + \frac{\rho(x)}{\log x} + \int_2^x \frac{\rho(t)}{t \log^2 t} dt. \tag{3.1}$$

To obtain sharper estimations we need to have more information about how close the zeros ρ may come to the line $\sigma = 1$.

We had for $\zeta(s)$

$$-\frac{\zeta'}{\zeta}(s) = c + \frac{1}{s-1} - \sum_\rho \Big(\frac{1}{s-\rho} + \frac{1}{\rho}\Big) - \sum_{n=1}^\infty \Big(\frac{1}{2+2n} - \frac{1}{2n}\Big) \tag{3.2}$$

(see (2.6)). We may assume $s = \sigma + it$ with $1 < \sigma < 5$, and $|t| \geq 1$, since there exists a constant c such that $\zeta(s)$ has no zero in $|r| \leq t, \sigma \geq 1 - c$ if $\rho = \beta + ir$ denotes the zeros, and so we may suppose $|\gamma| > 1$. By Theorem 2.3 we obtain

$$\mathrm{Re}\Big(\sum_\rho \frac{1}{\rho}\Big) = \sum_\rho \frac{\beta}{|\rho|^2} = \mathcal{O}(1),$$

and

$$\mathrm{Re}\Big(-\frac{\zeta'}{\zeta}(s)\Big) = -\sum_\rho \frac{\sigma - \beta}{(\sigma-\beta)^2 + (t-\gamma)^2} + \mathcal{O}\big(\log(2+|t|)\big).$$

We consider a zero $\rho_0 = \beta_0 + i\gamma_0$, and consider the expression:

$$\mathrm{Re}\Big\{-3\frac{\zeta'}{\zeta}(\sigma) - 4\frac{\zeta'}{\zeta}(\sigma + i\gamma_0) - \frac{\zeta'}{\zeta}(\sigma + 2i\gamma_0)\Big\} \geq 0$$

since

$$-\frac{\zeta}{\zeta}(s) = \sum_{n=1}^{\infty} \frac{\wedge(n)}{n^s}, \qquad \text{and} \quad 3 + 4\cos\theta + \cos 2\theta \geq 0.$$

(see §2.6). Here

$$-\frac{\zeta'}{\zeta}(\sigma) = \frac{1}{\sigma - 1} + \mathcal{O}(1),$$

$$\mathrm{Re}\Big(-\frac{\zeta'}{\zeta}(\sigma + 2i\gamma_0)\Big) \leq \mathcal{O}\big(\log(2 + |\gamma_0|)\big),$$

and

$$\mathrm{Re}\Big(-\frac{\zeta'}{\zeta}(\sigma + i\gamma_0)\Big) \leq \frac{-1}{\sigma - \beta_0} + \mathcal{O}\big(\log(2 + |r_0|)\big),$$

thus

$$\frac{3}{\sigma - 1} - \frac{4}{\sigma - \beta_0} + \mathcal{O}\big(\log(2 + |\gamma_0|)\big) \geq 0.$$

or

$$\frac{4}{\sigma - \beta_0} - \frac{3}{\sigma - 1} \leq a\log(2 + |\gamma_0|),$$

with some absolute constant $a > 0$. If we put $\sigma = 1 + 6(1 - \beta_0)$, we get

$$\frac{1}{14} \cdot \frac{1}{1 - \beta_0} \leq a\log(2 + |\gamma_0|),$$

or

$$1 - \beta_0 \geq \frac{1}{14a\log(2 + |\gamma_0|)},$$

or

$$\beta_0 \leq 1 - \frac{\alpha}{\log(2 + |\gamma_0|)},$$

with some absolute positive constant α. Thus we have proved the following

Theorem 3.2. *$\zeta(s)$ has no zeros in the region*

$$\sigma > 1 - \frac{\alpha}{\log(2 + |t|)}.$$

We showed in §2.5 that

$$\int_1^x \psi(x)dt = \frac{x^2}{2} - \sum_{\rho} \frac{x^{\rho+1}}{\rho(\rho + 1)} + \mathcal{O}(x)$$

(see (2.9)). Now by Theorem 2.3 and 3.2, we have for $x > 1$,

$$\begin{aligned}
\Big|\sum_\rho \frac{x^{\rho+1}}{\rho(\rho+1)}\Big| &\leq gx^2 \sum_\gamma \frac{e^{-\frac{\alpha \log x}{\log(2+|\gamma|)}}}{(2+|\gamma|)^2} \\
&= gx^2 \sum_\gamma \frac{1}{(2+|\gamma|)^{3/2}} e^{-\frac{\alpha \log x}{\log(2+|\gamma|)} - \frac{1}{2}\log(2+|\gamma|)} \\
&< gx^2 e^{-\sqrt{2\alpha \log x}} \sum_\gamma \frac{1}{(2+|r|)^{3/2}} \\
&= \mathcal{O}(x^2 e^{-\sqrt{2\alpha \log x}}).
\end{aligned}$$

Thus

$$\int_1^x \psi(t)dt = \frac{x^2}{2} + \mathcal{O}(x^2 e^{-\sqrt{2\alpha \log x}}).$$

From this

$$\begin{aligned}
\psi(x) &\leq \frac{1}{h}\int_x^{x+h} \psi(t)dt = x + \frac{h}{2} + \mathcal{O}\Big(\frac{x^2}{h} e^{-\sqrt{2\alpha \log x}}\Big) \\
&= x + \mathcal{O}\Big(xe^{-\frac{1}{2}\sqrt{2\alpha \log x}}\Big),
\end{aligned}$$

where we have chosen $h = xe^{-\frac{1}{2}\sqrt{2\alpha \log x}}$. Considering in the same way

$$\psi(x) \geq \frac{1}{h}\int_{x-h}^x \psi(t)dt,$$

we get

$$\psi(x) = x + \mathcal{O}(xe^{-\alpha'\sqrt{\log x}}) \tag{3.3}$$

with $\alpha' = \frac{1}{2}\sqrt{2\alpha}$.

This gives of course

$$\mathcal{J}(x) = x + \mathcal{O}(xe^{-\alpha'\sqrt{\log x}})$$

and finally by (3.1)

$$\pi(x) = \int_2^x \frac{dt}{\log t} + \mathcal{O}(xe^{-\alpha'\sqrt{\log x}}).$$

This is the Prime Number Theorem with the remainder term first proved by de la Vallee Poussin in 1898. He was the first to obtain a zero-free region near $\sigma = 1$, and so an explicit remainder term.

3.3. Prime Number Theorem with remainder term for the arithmetic progression

When we use the same approach for the $L(s,\chi)$, there are some new difficulties in the case of quadratic characters χ.

Let us consider the case χ even for simplicity, χ odd can be handled in a similar way.

We have proved

$$-\frac{L'}{L}(s,\chi) = c_\chi - \sum_\rho \Big(\frac{1}{s-\rho} + \frac{1}{\rho}\Big) - \frac{1}{s} - \sum_{n=1}^{\infty} \Big(\frac{1}{s+2n} - \frac{1}{2n}\Big)$$

$$= c'_\chi - \sum_{|\gamma|\le 1} \frac{1}{s-\rho} - \sum_{|\gamma|>1} \Big(\frac{1}{s-\rho} + \frac{1}{\rho}\Big) - \frac{1}{s} - \sum_{n=1}^{\infty} \Big(\frac{1}{s+2n} - \frac{1}{2n}\Big),$$

where the ρ runs over the zeros of $L(s,\chi)$ in the strip $0 < \sigma < 1$, and the constants c_χ and c'_χ depending on χ only (see (2.7)). Putting $s = 2$, we observe that

$$\Big|\frac{L'}{L}(2,\chi)\Big| \le -\frac{\zeta'}{\zeta}(2).$$

Also from Theorem 2.3 we get

$$\sum_{|\gamma|\le 1} \frac{1}{2-\rho} = \mathcal{O}(\log q)$$

and

$$\sum_{|\gamma|>1} \frac{2}{(2-\rho)\rho} = \mathcal{O}(\log q).$$

From this we find $c'_\chi = \mathcal{O}(\log q)$. We may now consider

$$\text{(3.4)} \qquad \mathrm{Re}\Big(-3\frac{\zeta'}{\zeta}(\sigma) - 4\frac{L'}{L}(\sigma + i\gamma_0, \chi) - \frac{L'}{L}(\sigma + 2i\gamma_0, \chi^2)\Big) \ge 0$$

(see §2.6). We assume that $\gamma_0 \ne 0$ if χ is a quadratic character, at first assume that $|\gamma_0| \ge \dfrac{1}{\log q}$ if χ is quadratic, then for $1 < \sigma < 5$

$$-\frac{\zeta'}{\zeta}(\sigma) = \frac{1}{\sigma - 1} + \mathcal{O}(1),$$

also

$$-\mathrm{Re}\Big(\frac{L'}{L}(\sigma + i\gamma_0, \chi)\Big) \le \frac{-1}{\sigma - \beta_0} + \mathcal{O}\big(\log q(2 + |\gamma_0|)\big)$$

and

$$-\mathrm{Re}\Big(\frac{L'}{L}(\sigma+2i\gamma_0,\chi^2)\Big)\le\mathcal{O}\big(\log q(2+|\gamma_0|)\big).$$

This gives again

$$\frac{3}{\sigma-1}-\frac{4}{\sigma-\beta_0}+\mathcal{O}\big(\log q(2+|\gamma_0|)\big)\ge 0$$

and we proceed as in the case of $\zeta(s)$, choosing $\sigma=1+6(1-\beta_0)$, and obtain

$$\beta_0<1-\frac{\alpha}{\log q(2+|\gamma_0|)}.$$

If χ is quadratic and $0<|\gamma_0|<\dfrac{1}{\log q}$, $\dfrac{L'}{L}(s,\chi^2)$ is essentially $\dfrac{\zeta'}{\zeta}(s)$, and so by (3.2)

$$-\mathrm{Re}\Big(\frac{L'}{L}(\sigma+2i\gamma_0,\chi^2)\Big)=\mathrm{Re}\Big(\frac{1}{s-1}\Big)+\mathcal{O}(\log q)$$

$$=\frac{\sigma-1}{(\sigma-1)^2+4\gamma_0^2}+\mathcal{O}(\log q).$$

Also $L(s,\chi)$ has beside the zero $\beta_0+i\gamma_0$ also the zero $\beta_0-i\gamma_0$, this gives us a term $-\dfrac{\sigma-\beta_0}{(\sigma-\beta_0)^2+4\gamma_0^2}$ in (3.4), thus our inequality (3.4) becomes

$$\frac{3}{\sigma-1}-\frac{4}{\sigma-\beta_0}-\frac{4(\sigma-\beta_0)}{(\sigma-\beta_0)^2+4\gamma_0^2}+\frac{\sigma-1}{(\sigma-1)^2+4\gamma_0^2}$$
$$+\mathcal{O}(\log q)\ge 0.$$

If we again choose $\sigma=1+6(1-\beta_0)$, we see that the third term more than cancels out the fourth, and so again

$$\frac{3}{\sigma-1}-\frac{4}{\sigma-\beta_0}+\mathcal{O}(\log q)\ge 0,$$

and

$$\beta_0<1-\frac{\alpha}{\log q}.$$

Finally if χ is quadratic and $\gamma_0=0$, we cannot exclude that there may be a real zero very close to 1, but it could be at most one such, since from

$$\mathrm{Re}\Big(-\frac{\zeta'}{\zeta}(\sigma)-\frac{L'}{L}(\sigma+\chi)\Big)=\sum_{n=1}^{\infty}\frac{\lambda(n)}{n^\sigma}\big(1+\chi(n)\big)\ge 0,$$

the assumption of two real zeros leads the inequality for $\sigma > 1$

$$\frac{1}{\sigma-1} - \frac{1}{\sigma-\beta} - \frac{1}{\sigma-\beta'} + \mathcal{O}(\log q) \geq 0,$$

where β and β' are two zeros. If $\beta' < \beta < 1$, then

$$\frac{1}{\sigma-1} - \frac{2}{\sigma-\beta'} + \mathcal{O}(\log q) \geq 0,$$

choosing $\sigma = 1 + 2(1-\beta')$, we get

$$\beta' < 1 - \frac{\alpha}{\log q},$$

so $L(s,\chi)$ where χ is quadratic has at most one zero (which then has to be real) in the region

$$\sigma > 1 - \frac{\alpha}{\log q(2+|t|)}.$$

Finally, if we have two different primitive quadratic characters χ_1 and χ_2 both belonging to moduli q_1 and $q_2 \leq q$, then $\chi_1\chi_2$ is also a quadratic character belonging to the modulus $[q_1, q_2] < q^2$. We have then

$$\operatorname{Re}\Big(-\frac{\zeta'}{\zeta}(\sigma) - \frac{L'}{L}(\sigma,\chi_1) - \frac{L'}{L}(\sigma,\chi_2) - \frac{L'}{L}(\sigma,\chi_1\chi_2)\Big)$$

$$= \sum_{n=1}^{\infty} \wedge(n)\big(1 + \chi_1(n) + \chi_2(n) + \chi_1\chi_2(n)\big)/n^{\sigma}$$

$$= \sum_{n=1}^{\infty} \wedge(n)\big(1 + \chi_1(n)\big)\big(1 + \chi_2(n)\big)/n^{\sigma} \geq 0$$

for $\sigma > 1$. If both $L(s,\chi_1)$ and $L(s,\chi_2)$ have real zeros β_1 and β_2 respectively, we get the inequality for $\sigma > 1$

$$\frac{1}{\sigma-1} - \frac{1}{\sigma-\beta_1} - \frac{1}{\sigma-\beta_2} + \mathcal{O}(\log q) \geq 0,$$

and so

$$\frac{1}{\sigma-1} - \frac{2}{\sigma-\min(\beta_1,\beta_2)} + \mathcal{O}(\log q) \geq 0.$$

Take $\sigma = 1 + 2\big(1 - \min(\beta_1,\beta_2)\big)$. Then we conclude again that

$$\min(\beta_1,\beta_2) < 1 - \frac{\alpha}{\log q},$$

and we have proved.

Theorem 3.3. *Among the characters to modulus* $\leq q$*, at most one* $L(s,\chi)$ *can have at most one real zero in the region*

$$\beta \geq 1 - \frac{\alpha}{\log q(2+|t|)},$$

where χ *is quadratic.*

The possibility of this exceptional real zero is very annoying, and complicates the statements of a P.N.T. with remainder term for the arithmetic progression. If there is no exceptional real zero β for any of the quadratic characters whose modulus divides q, then similar to (3.3) we get

$$\psi_{q,\ell}(x) = \frac{1}{\psi(q)}x + \mathcal{O}(xe^{-\alpha'\sqrt{\log x}}).$$

However, if we have such an exceptional zero β for the character χ we get

$$\psi_{q,\ell}(x) = \frac{1}{\varphi(q)}x - \frac{\chi(\ell)}{\psi(q)}x^{\beta} + \mathcal{O}(xe^{-\alpha'\sqrt{\log x}}).$$

From these two formulas, we may obtain easily the corresponding formulas for $\pi_{q,\ell}(x)$. Combining with (3.3), we state the main results on P.N.T. with remainder terms as follows:

Theorem 3.4. *We have*

$$\pi(x) = \int_2^x \frac{dt}{\log t} + \mathcal{O}\big(xe^{-\alpha'\sqrt{\log x}}\big),$$

and if there is no exceptional real zero β *for any of the quadratic characters whose modulus divides* q*, then*

$$\pi_{q,\ell}(x) = \frac{1}{\varphi(q)}\int_2^x \frac{dt}{\log t} + \mathcal{O}\big(xe^{-\alpha'\sqrt{\log x}}\big),$$

and if we have such an exceptional zero β *for the character* χ*, then*

$$\pi_{q,\ell}(x) = \frac{1}{\varphi(q)}\int_2^x \frac{dt}{\log t} - \frac{\chi(\ell)}{\varphi(q)}\int_2^{x^{\beta}} \frac{dt}{\log t} + \mathcal{O}\big(xe^{-\alpha'\sqrt{\log x}}\big),$$

where α' *and the constants implicit in* $\mathcal{O}'s$ *are constants not depending on* q*.*

β if it exists, clearly does, but in a way we know very little about. We proved earlier only that $\beta < 1$. A lower bound for $1-\beta$ is hard to come by. If we have a positive lower bound for $L(1,\chi)$ and an upper bound for the derivative $L'(s,\chi)$ in the neighbourhood of 1, we can obviously get a lower bound for $1-\beta$. In fact, for $0<\sigma<1$, we get

$$\begin{aligned}|L(\sigma,\chi)| &= |L(1,\chi) - \int_\sigma^1 L'(u,\chi)du| \\ &\geq L(1,\chi) - (1-\sigma) \max_{\sigma\leq u\leq 1} |L'(u,\chi)|,\end{aligned} \tag{3.5}$$

and from this we may easily obtain a lower estimation for $1-\beta$.

Write

$$C(\xi) = \sum_{q\leq n\leq \xi} \chi(n).$$

Then

$$\begin{aligned}-L'(\sigma,\chi) &= \sum_{n=1}^{\infty} \frac{\chi(n)}{n^\sigma}\log n \\ &= \sum_{n<q} \frac{\chi(n)}{n^\sigma}\log n + \int_q^\infty \frac{\log\xi}{\xi^\sigma} dC(\xi) \\ &= \sum_{n<q} \frac{\chi(n)}{n^\sigma}\log n - \int_q^\infty C(\xi) d\Big(\frac{\log\xi}{\xi^\sigma}\Big)\end{aligned}$$

for $\sigma \geq 1 - \dfrac{1}{\log q}$, since $|C(\xi)| = \mathcal{O}(q)$. So we have

$$\begin{aligned}\sum_{n<q} \frac{\chi(n)}{n^\sigma}\log n &= \mathcal{O}\Big(\sum_{n<q} n^{-\sigma}\log n\Big) = \mathcal{O}\Big(\sum_{n<q} n^{-1}\log n\Big) \\ &= \mathcal{O}\big(\log^2 q\big)\end{aligned}$$

and

$$\begin{aligned}\int_q^\infty C(\xi) d\frac{\log\xi}{\xi^\sigma} = &= \mathcal{O}\Big(q\int_q^\infty \big(\xi^{-\sigma-1} + \xi^{-\sigma-1}\log\xi\big)\Big) d\xi \\ &= \mathcal{O}\big(\log^2 q\big).\end{aligned}$$

Therefore

$$L'(\sigma,\chi) = \mathcal{O}(\log^2 q) \qquad \Big(\sigma > 1 - \frac{1}{\log q}\Big).$$

Since $\zeta(s)L(s,\chi)$ is the Dedekind zeta function of some quadratic field, the connection of $L(1,\chi)$ with the class number (which has to be at least 1) gives that $L(1,\chi) \geq c/\sqrt{q}$, from (3.5) we see that $|L(\sigma,\chi)| > 0$ for $1-\sigma < \dfrac{c'}{\sqrt{q}\log^2 q}$, and hence $1-\beta \geq \dfrac{c'}{\sqrt{q}\log^2 q}$, with an effective constant c'. (Page's Theorem). There is a theorem by Siegel which shows that for any $\varepsilon > 0$ there is $c(\varepsilon) > 0$ such that

$$1-\beta > c(\varepsilon)q^{-\varepsilon}.$$

But only the existence of $c(\varepsilon)$ is proved. So this is not an effective constant, we have no way of giving bounds for it. The best effective bound for $1-\beta$ is by Goldfeld-Cross-Zagier. If improves the bound of Page's theorem by a power of $\log q$.

We shall prove $1-\beta > c'/\sqrt{q}\log^3 q$ which is slightly weaker than Page's theorem in Lecture IV, and prove Siegel's theorem in Lecture V. Concerning the estimation of $L(1,\chi)$ in the proof of Page's theorem, we refer to E. Hecke "Lectures on the theory of Algebraic Numbers, GTM 77, Springer, 1981".

3.4. Riemann hypothesis

Riemann conjectured in his paper of 1859 that all the zeros of $\xi(s)$ lie on the line $\sigma = \dfrac{1}{2}$. This is the so-called Riemann hypothesis and we write it simply by R.H. This has later been extended to the more general conjecture that for any χ, all the zeros of $\zeta(s,\chi)$ lie on the line $\sigma = 1/2$. We call it the generalized Riemann hypothesis and denote by G.R.H. We shall see what consequences these conjectures for the estimation of $\psi(x), \pi(x)$ and $\psi_{q,\ell}(x)$ and $\pi_{q,\ell}(x)$.

We go back to the formulas

$$\int_0^x \psi(t)dt = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \frac{x^{s+1}}{s(s+1)}\Big(-\frac{\zeta'}{\zeta}(s)\Big)ds$$

(see §2.5) and

$$\frac{\zeta'}{\zeta}(s) = c + \frac{1}{s-1} - \sum_\rho \Big(\frac{1}{s-\rho}+\frac{1}{\rho}\Big) - \sum_{n=1}^\infty \Big(\frac{1}{s+2n}-\frac{1}{2n}\Big)$$

(see (2.6)). Inserting the second formula in the first and integrating term by term, we get by Lemma 2.6

$$\int_1^x \psi(t)dt = \frac{x^2}{2} - \sum_\rho \frac{x^{\rho+1}}{\rho(\rho+1)} + c'x + c'' - \sum_{n=1}^\infty \frac{x^{1-2n}}{2n(2n-1)},$$

where c' and c'' are certain constants. If R.H. is true we may write $\rho = \dfrac{1}{2} + i\gamma$. So we have

$$\int_0^x \psi(t)dt = \frac{x^2}{2} - \sum_\gamma \frac{x^{3/3+i\gamma}}{(\frac{1}{2}+i\gamma)(\frac{3}{2}+i\gamma)} + c'x + \mathcal{O}(1).$$

For $1 < h < x$, we get

$$\begin{aligned} \psi(x) &\leq \frac{1}{h}\int_x^{x+h} \psi(t)dt \\ &= x + \frac{h}{2} - \frac{1}{h}\sum_\gamma \frac{(x+h)^{\frac{3}{2}+i\gamma} - x^{\frac{3}{2}+i\gamma}}{(\frac{1}{2}+i\gamma)(\frac{3}{2}+i\gamma)} + \mathcal{O}(1). \end{aligned}$$

We have also

$$(3.6) \qquad \Big|\frac{(x+h)^{s+1} - x^{s+1}}{h}\Big| \leq \begin{cases} |s+1|\ |x+h|^\sigma, \\ \dfrac{2}{h}|x+h|^{\sigma+1}. \end{cases}$$

In fact,

$$\begin{aligned} \Big|\frac{(x+h)^{s+1} - x^{s+1}}{h}\Big| &= \Big|\frac{1}{h}\int_x^{x+h} dt^{s+1}\Big| \leq \frac{|s+1|}{h}\int_x^{x+h} t^\sigma dt \\ &\leq |s+1|\,|x+h|^\sigma, \end{aligned}$$

and the lower inequality in (3.6) is obvious.

Now using the upper inequality for $|\gamma| \leq \dfrac{2x}{h}$ and the lower when $|\gamma| > \dfrac{2x}{h}$, we get

$$\begin{aligned} \psi(x) &\leq x + \frac{h}{2} + \sqrt{2x}\sum_{0<|\gamma|\leq\frac{2x}{h}} \frac{1}{\gamma} + \frac{2(2x)^{3/2}}{h}\sum_{|\gamma|\geq\frac{2x}{h}} \frac{1}{|\gamma|^2} \\ &\quad +\mathcal{O}(1). \end{aligned}$$

From theorem 2.3, we obtain

$$\begin{aligned} \sum_{0<|\gamma|\leq T} \frac{1}{|\gamma|^2} &= 2\sum_{0<\gamma<T}\frac{1}{\gamma} < A\sum_{1\leq n\leq T}\frac{\log n}{n} \\ &= \mathcal{O}\Big(\log^2 T\Big) \end{aligned}$$

and

$$\begin{aligned} \sum_{|\gamma|>T} \frac{1}{|\gamma|^2} &= 2\sum_{\gamma>T}\frac{1}{\gamma^2} < A\sum_{n\geq T}\frac{\log n}{n^2} \\ &= \mathcal{O}\Big(\frac{\log T}{T}\Big). \end{aligned}$$

Inserting this, with $T = 2x/h$, we get

$$\psi(x) \le x + \frac{h}{2} + \mathcal{O}\Big(x^{1/2}\log^2\frac{2x}{h}\Big) + \mathcal{O}\Big(x^{1/2}\log\frac{2x}{h}\Big).$$

Choosing, say $h = 2\sqrt{x}$, we get

$$\psi(x) \le x + \mathcal{O}\big(x^{1/2}\log^2 x\big).$$

From,

$$\psi(x) \ge \frac{1}{h}\int_{x-h}^{x}\psi(t)\,dt,$$

we get in a similar way

$$\psi(x) \ge x + \mathcal{O}\big(x^{1/2}\log^2 x\big).$$

Thus

$$\psi(x) = x + \mathcal{O}\big(x^{1/2}\log^2 x\big) \tag{3.7}$$

and so

$$\mathcal{J}(x) = x + \mathcal{O}\big(x^{1/2}\log^2 x\big)$$

and finally by (3.1)

$$\pi(x) = \int_2^x \frac{dt}{\log t} + \mathcal{O}\big(x^{1/2}\log x\big).$$

In the same way one gets, assuming the G.R.H.; that

$$\psi_{q,\ell}(x) = \frac{1}{\psi(q)}x + \mathcal{O}\big(x^{1/2}\log^2 x\big)$$

and the corresponding formulas for $\mathcal{J}_{q,\ell}(x)$ and $\pi_{q,\ell}(x)$. Now we state our main results as follows:

Theorem 3.5. *Under the assumption of R.H., and G.R.H., we have*

$$\pi(x) = \int_2^x \frac{dt}{\log t} + \mathcal{O}\big(x^{1/2}\log x\big),$$

and

$$\pi_{q,\ell}(x) = \frac{1}{\varphi(q)}\int_2^x \frac{dt}{\log t} + \mathcal{O}\big(x^{1/2}\log x\big)$$

respectively.

The Riemann hypothesis is today supported by very massive numerical evidence. A proof seems still far off.

3.5. Riemann hypothesis and the remainder term of Prime Number Theorem.

Now we proceed to show the following theorem which may be regarded as the inverse of (3.7).

Theorem 3.6. *Let*

$$\psi(x) = x + R(x).$$

If there is a $\theta > 0$ such that for all large x

$$R(x) > -Ax^{\theta}, \quad \theta > 0, \tag{3.8}$$

then $\zeta(s)$ has no zero in $\sigma > \theta$. The same conclusion holds if (3.8) is changed by

$$R(x) < Ax^{\theta}$$

for all large x.

Proof. We shall prove the theorem under the assumption of (3.8).

We consider for $\sigma > 1$

$$\int_1^{\infty} \frac{R(x) + Ax^{\theta} + \beta}{x^{s+1}} dx,$$

where by (3.8) the constant is chosen large enough that

$$R(x) + Ax^{\theta} + \beta > 0$$

for all $x \geq 1$. Then

$$\begin{aligned} \Big| \int_1^{\infty} \frac{R(x) + Ax^{\theta} + B}{x^{s+1}} dx \Big| &\leq \int_0^{\infty} \Big| \frac{R(x) + Ax^{\theta} + B}{x^{s+1}} \Big| \, dx \\ &= \int_1^{\infty} \frac{R(x) + Ax^{\theta} + B}{x^{s+1}} \, dx. \end{aligned} \tag{3.9}$$

We have

$$\begin{aligned}\int_1^\infty \frac{R(x)}{x^{s+1}}dx &= \int_1^\infty \frac{\psi(x)}{x^{s+1}}dx - \int_1^\infty \frac{dx}{x^s} \\ &= \sum_{n=1}^\infty \wedge(n)\int_n^\infty \frac{dx}{x^{s+1}} - \frac{1}{s-1} \\ &= \frac{1}{s}\sum_{n=1}^\infty \frac{\wedge(n)}{n^s} - \frac{1}{s-1} \\ &= \frac{1}{s}\frac{\zeta'}{\zeta}(s) - \frac{1}{s-1},\end{aligned}$$

and

$$\int_1^\infty \frac{Ax^\theta}{x^{s+1}}dx = \frac{A}{s-\theta}, \int_1^\infty \frac{B}{x^{s+1}}\,dx = \frac{B}{s}.$$

So

$$\int_1^\infty \frac{R(x)+Ax^\theta+B}{x^{s+1}}\,dx$$

$$-\frac{1}{s}\frac{\zeta'}{\zeta}(s) - \frac{1}{s-1} + \frac{1}{s-\theta} + \frac{B}{s}.$$

By (3.9), it follows that this function is analytic up to the nearest singularity on the real axis. This is $s = \theta$, that is; the function is analytic in $\sigma > \theta$. So $\frac{\zeta'}{\zeta}(s)$ can have no singularity in $\sigma > \theta$, which proves the theorem.

Lecture IV

In this lecture we shall give the lower estimation for the remainder term in P.N.T; and the estimation for real zero of $L(s,\chi)$ with χ a quadratic character.

4.1. Zero-free region for $\zeta(s)$ and the estimation for $R(x)$

If $\zeta(s)$ has no zero in $\sigma > \theta$, we can show

$$R(x) = \mathcal{O}(x^{\theta}\log^2 x),$$

in a way quite similar to the one used assuming R.H. (see (3.7)). If $\theta > \frac{1}{2}$ then the stronger conclusion

$$R(x) = \mathcal{O}(x^{\theta})$$

can be shown to hold, and if $\zeta(s)$ has no zeros in $\sigma \geq \theta > \frac{1}{2}$, then

$$R(x) = o(x^{\theta})$$

holds. (We omit the proofs here). We see that if $R(x) = \mathcal{O}\big(x^{1/2+\varepsilon}\big)$ for all $\varepsilon > x_0$, or even if for all $\varepsilon > 0$

$$R(x) > -A(\varepsilon)x^{1/2+\varepsilon} \qquad \text{for all } x > x_0(\varepsilon),$$

where $A(\varepsilon) > 0$, then from Theorem 3.6, it yields that $\zeta(s)$ has no zero in $\sigma > \frac{1}{2} + \varepsilon$. Since ε is arbitrary, the R.H. must hold and so:

$$R(x) = \mathcal{O}(x^{1/2}\log^2 x).$$

4.2. Lower bound for $R(x)$

We shall now try to see what lower bound we can find for the maximal order of $R(x)$. On the R.H. we have

$$\int_0^x \psi(t)dt = \frac{x^2}{2} - x^{3/2}\sum_{\gamma}\frac{x^{i\gamma}}{\big(\frac{1}{2}+i\gamma\big)\big(\frac{3}{2}+i\gamma\big)} + cx + \mathcal{O}(1)$$

(see §3.4). From this we see that

$$\int_1^x R(t)dt = -x^{3/2}\sum_{\gamma}\frac{x^{i\gamma}}{\big(\frac{1}{2}+i\gamma\big)\big(\frac{3}{2}+i\gamma\big)} + cx + \mathcal{O}(1). \tag{4.1}$$

From this it is easy to see that we cannot have

$$\int_1^x R(t)dt = o\big(x^{3/2}\big)$$

and also

$$R(x) = o\big(x^{1/2}\big)$$

must be false. Thus the order of $R(x)$ is at least $x^{1/2}$.

To get further, we write $x = e^u$ and $t = e^v$ in (4.1), and get

$$\int_v^u e^v R(e^v)dv = -e^{3/2}u \sum_\gamma \frac{e^{i\gamma u}}{\big(\frac{1}{2}+i\gamma\big)\,\big(\frac{3}{2}+i\gamma\big)} + ce^u + \mathcal{O}(1).$$

From this, with $e^{-u} < \delta < 1$, we get

$$\begin{aligned}
\frac{1}{2\delta}\int_{u-\delta}^{u+\delta} e^v R(e^v)dv &= -\frac{e^{\frac{3}{2}u}}{2\delta}\sum_\gamma \frac{e^{i\gamma u}\big(e^{(\frac{3}{2}+i\gamma)\delta} - e^{-(\frac{3}{2}+i\gamma)\delta}\big)}{\big(\frac{1}{2}+i\gamma\big)\big(\frac{3}{2}+i\gamma\big)} \\
&\qquad + ce^u \frac{e^\delta - e^{-\delta}}{2\delta} + \mathcal{O}\big(\frac{1}{\delta}\big) \\
&= -\frac{e^{\frac{3}{2}u}}{2\delta}\sum_\gamma \frac{e^{i\gamma u}\big(e^{i\gamma\delta} - e^{i\gamma\delta}\big)}{\big(\frac{1}{2}+i\gamma\big)\big(\frac{3}{2}+i\gamma\big)} + \mathcal{O}\big(e^{\frac{3}{2}u}\big).
\end{aligned}$$

Since

$$\frac{1}{\gamma^2} + \frac{1}{\big(\frac{1}{2}+i\gamma\big)\big(\frac{3}{2}+i\gamma\big)} = \frac{\frac{3}{4}+2i\gamma}{\gamma^2\big(\frac{1}{2}+i\gamma\big)\big(\frac{3}{2}+i\gamma\big)},$$

and by Theorem 2.3

$$\begin{aligned}
\sum_\gamma \left| \frac{\big(\frac{3}{4}+2i\gamma\big)\big(e^{i\gamma\delta} - e^{-i\gamma\delta}\big)}{\gamma^2\big(\frac{1}{2}+i\gamma\big)\big(\frac{3}{2}+i\gamma\big)} \right| &= \sum_{|\gamma|<\frac{1}{\delta}} + \sum_{|\gamma|\ge\frac{1}{\delta}} \\
&= \mathcal{O}\big(\sum_{|\gamma|\le\frac{1}{\delta}} \frac{\delta}{\gamma^2}\big) + \mathcal{O}\big(\sum_{|\gamma|\ge\frac{1}{\delta}} \frac{\delta}{|\gamma|^3}\big) \\
&= \mathcal{O}(\delta) + \mathcal{O}\big(\delta^2 \log\frac{1}{\delta}\big) = \mathcal{O}(\delta),
\end{aligned}$$

we have

$$\begin{aligned}
\frac{1}{2\delta}\int_{u-\delta}^{u+\delta} e^v R(e^v)dv &= \frac{e^{\frac{3}{2}u}}{2\delta}\sum_\gamma \frac{e^{i\gamma u}\big(e^{i\gamma\delta} - e^{-i\gamma\delta}\big)}{\gamma^2} + \mathcal{O}\big(e^{\frac{3}{2}u}\big) \\
&= \frac{e^{\frac{3}{2}u}}{2\delta}\sum_{\gamma>0} \frac{\big(e^{i\gamma u} - e^{-i\gamma u}\big)\big(e^{i\gamma\delta} - e^{-i\gamma\delta}\big)}{\gamma^2} + \mathcal{O}\big(e^{\frac{3}{2}u}\big) \\
&= -\frac{2e^{\frac{3}{2}u}}{\delta}\sum_{\gamma>0} \frac{\sin\gamma u \sin\gamma\delta}{\gamma^2} + \mathcal{O}\big(e^{\frac{3}{2}u}\big).
\end{aligned}$$

From this we get, dividing by $e^{\frac{3}{2}u}$ and writing $u+\theta$ for v, that

$$\frac{1}{2\delta}\int_{-\delta}^{\delta} e^{\frac{3}{2}\theta}\,\frac{R\big(e^{u+\theta}\big)}{e^{\frac{u+\theta}{2}}}\,d\theta = -\frac{2}{\delta}\sum_{\gamma>0}\frac{\sin\gamma u\sin\gamma\delta}{\gamma^2}+\mathcal{O}(1). \tag{4.2}$$

We shall now try to find values of u that make the expression on the right-hand side of (4.2) large positive or large negative. Let $\|x\|$ denote the difference between x and the nearest integer, and in the case of x being an integer plus $1/2$ define $\|x\|=1/2$. Given L and T, for any integer u and real number γ, there exists an integer k satisfying $|k|\le L/2$ such that

$$\frac{k-1}{L}\le\left\|\frac{u\gamma}{2\pi}\right\|<k/L,$$

that is, $\|\dfrac{u\gamma}{2\pi}\|$ falls in a "pigeon hole" of length $\dfrac{1}{L}$. When u is fixed and γ runs over all imaginary parts of zeros $p=\beta+i\gamma$ satisfying $0<\gamma<T$, then we obtain a set of $N(T)$ pigeon holes related to u. The total number of distinct sets of pigeon holes is $L^{N(T)}$.

Now let $u\in[L,L^{N(T)+1}]$. Then the number of u is $L^{N(T)+1}-L+1=L(L^{N(T)}-1)+1>L^{N(T)}$, if we take $L,T>100$. If follows from the so-called Dirichlet principle or pigeon-hole principle that there exists at least one pigeon which corresponds to different U, say u_1 and u_2. We may suppose without loss of generality that $u_1>u_2$. Let $u_0=u_1-u_2$. Then $u_0>0$ and

$$\left\|\left\{\frac{u_0\gamma}{2\pi}\right\}\right\|=\left|\left\|\frac{u_1\gamma}{2\pi}\right\|-\left\|\frac{u_2\gamma}{2\pi}\right\|\right|\le\frac{1}{L}$$

for γ with $0<\gamma<T$.

We then have for $\gamma<T$

$$|\sin\gamma u_0|<\frac{2\pi}{L}\ \text{and}\ \cos\gamma u_0>1-\frac{20}{L^2},$$

so for $u=u_0+\delta$, we have for $\gamma<T$ that

$$\sin\gamma u=\sin\gamma\delta+\mathcal{O}\big(\frac{1}{L}\big)$$

and for $\gamma\ge T$ that

$$\sin\gamma u=\sin\gamma\delta+\mathcal{O}(1).$$

For $u = u_0 + \delta$ we then get by Theorem 2.3 and (4.2)

$$\begin{aligned}
&\frac{1}{2\delta}\int_{-\delta}^{\delta} e^{\frac{3}{2}\theta}\frac{R(e^{u+\theta})}{e^{\frac{1}{2}(u+\theta)}}\,d\theta \\
= \;& -\frac{2}{\delta}\sum_{\gamma}\frac{\sin^2\gamma\delta}{\gamma^2} + \mathcal{O}\Big(\frac{1}{L}\sum_{0<\gamma<T}\frac{1}{\gamma} + \frac{1}{\delta}\sum_{\gamma>T}\frac{1}{\gamma^2} + 1\Big) \\
= \;& -\frac{2}{\delta}\sum_{\gamma}\frac{\sin^2\gamma\delta}{\gamma^2} + \mathcal{O}\Big(\frac{\log^2 T}{L} + \frac{\log T}{\delta T} + 1\Big).
\end{aligned}$$

If we choose $L = [\log^2 T]$ and $\delta = \dfrac{\log T}{T}$, then

$$\frac{1}{2\delta}\int_{-\delta}^{\delta} e^{\frac{3}{2}\theta}\frac{R\big(e^{u+\theta}\big)}{e^{\frac{1}{2}(u+\theta)}}\,d\theta = -\frac{2}{\delta}\sum_{\gamma}\frac{\sin^2\gamma\delta}{\gamma^2} + \mathcal{O}(1). \tag{4.3}$$

Furthermore, by Theorem 2.3 we may write

$$\frac{2}{\delta}\sum_{\gamma}\frac{\sin^2\gamma\delta}{\gamma^2} = \frac{2}{\delta}\int_{20}^{\infty}\frac{\sin^2 t\delta}{t^2}\,dN(t) + \mathcal{O}(1),$$

where

$$N(t) = \frac{t}{2\pi}\Big(\log\frac{t}{2\pi} - 1\Big) + \gamma(t) \text{ and } \gamma(t) = \mathcal{O}(\log t).$$

This gives

$$\frac{2}{\delta}\int_{20}^{\infty}\frac{\sin^2 t\delta}{t^2}\,dN(t) \;=\; \frac{1}{\delta\pi}\int_{20}^{\infty}\frac{\sin^2 t\delta}{t^2}\log\frac{t}{2\pi}\,dt + \frac{2}{\delta}\int_{20}^{\infty}\frac{\sin^2 t\delta}{t^2}\,d\gamma(t). \tag{4.4}$$

First we proceed to estimate the later integral in the right-hand side. Using integration by parts, we get

$$\begin{aligned}
\frac{2}{\delta}\int_{20}^{\infty}\frac{\sin^2 t\delta}{t^2}\,d\gamma(t) \;&=\; \frac{2}{\delta}\left(\gamma(t)\frac{\sin^2 t\delta}{t^2}\right)_{20}^{\infty} \\
+\frac{4}{\delta}\int_{20}^{\infty}\gamma(t)\frac{\sin^2\delta t}{t^3}\,dt \;&-\; 4\int_{20}^{\infty}\gamma(t)\frac{\sin\delta t\cos\delta t}{t^2}\,dt.
\end{aligned}$$

Since

$$\begin{aligned}
\int_{20}^{\infty}\gamma(t)\frac{\sin^2\delta t}{t^3}dt &= \Big(\int_{20}^{1/\delta}+\int_{1/\delta}^{\infty}\Big)\gamma(t)\frac{\sin^2\delta t}{t^3}\,dt,\\
\int_{20}^{1/\delta}\gamma(t)\frac{\sin^2\delta t}{t^3}dt &= \mathcal{O}\Big(\delta^2\int_{20}^{1/\delta}\frac{\log t}{t}\,dt\Big)\\
&= \mathcal{O}\Big(\delta^2\log^2\frac{1}{\delta}\Big)=\mathcal{O}(\delta^{3/2})\\
\int_{1/\delta}^{\infty}\gamma(t)\frac{\sin^2\delta t}{t^3}dt &= \mathcal{O}\Big(\int_{1/\delta}^{\infty}\frac{\log t}{t^3}\,dt\Big)\\
&= \mathcal{O}\Big(\delta^2\log^2\frac{1}{\delta}\Big)=\mathcal{O}(\delta^{3/2}),
\end{aligned}$$

and similarly

$$\int_{20}^{\infty}T(t)\frac{\sin\delta t\cos\delta t}{t^2}dt=\Big(\int_{20}^{1/\delta}+\int_{1/\delta}^{\infty}\Big)\gamma(t)\,\frac{\sin 2\delta t}{2t^2}\,dt=\mathcal{O}(\delta^{1/2}),$$

we have

$$\frac{2}{\delta}\int_{20}^{\infty}\frac{\sin^2 t\delta}{t^2}\,d\gamma(t)=\mathcal{O}(\delta^{1/2}). \tag{4.5}$$

Also writing $t\delta=v$.

$$\frac{1}{\delta\pi}\int_{20}^{\infty}\frac{\sin^2\delta t}{t^2}\log\frac{t}{2\pi}\,dt=\frac{1}{\pi}\int_{20\delta}^{\infty}\frac{\sin^2 v}{v^2}\log\frac{v}{2\pi\delta}\,dv$$

$$=\frac{\log\frac{1}{\delta}}{\pi}\int_{20\delta}^{\infty}\frac{\sin^2 v}{v^2}\,dv+\frac{1}{\pi}\int_{20\delta}^{\infty}\frac{\sin^2 v}{v^2}\log\frac{v}{2\pi}\,dv.$$

Since

$$\int_0^{\infty}\frac{\sin v}{v}\,dv=\frac{\pi}{2},$$

we obtain

$$\begin{aligned}
\int_{20\delta}^{\infty}\frac{\sin^2 v}{v^2}\,dv &= -\frac{\sin^2 v}{v}\Big|_{20\delta}^{\infty}+\int_{20\delta}^{\infty}\frac{2\sin v\cos v}{v}\,dv\\
&= \mathcal{O}(\delta)+\int_0^{\infty}\frac{\sin^2 v}{v}\,dv-\int_0^{20\delta}\frac{\sin^2 v}{v}\,dv\\
&= \frac{\pi}{2}+\mathcal{O}(\delta).
\end{aligned}$$

We have also

$$\int_{20\delta}^{\infty} \frac{\sin^2 v}{v^2} \log \frac{v}{2\pi}\, dv = \Big(\int_{1}^{\infty} - \int_{20\delta}^{1} \Big) \frac{\sin^2 v}{v^2} \log \frac{v}{2\pi}\, dv$$

$$= \mathcal{O}\Big(\int_{20\delta}^{1} \log v\, dv \Big) + \mathcal{O}(1) = \mathcal{O}(1).$$

Therefore

$$\frac{1}{\delta\pi} \int_{20}^{\infty} \frac{\sin^2 \delta t}{t^2} \log \frac{t}{2\pi}\, dt = \frac{\log \frac{1}{\delta}}{\pi} \Big(\frac{\pi}{2} + \mathcal{O}(\delta) \Big) + \mathcal{O}(1) = \frac{1}{2} \log \frac{1}{\delta} + \mathcal{O}(1). \tag{4.6}$$

Substituting (4.5) and (4.6) into (4.4), we get

$$\frac{2}{\delta} \int_{20}^{\infty} \frac{\sin^2 \delta t}{t^2}\, dN(t) = \frac{1}{2} \log \frac{1}{\delta} + \mathcal{O}(1),$$

and so by (4.3) we have for $u = u_0 + \delta$

$$\frac{1}{2\delta} \int_{-\delta}^{\delta} e^{\frac{3}{2}\theta} \frac{R\big(e^{u+\theta}\big)}{e^{\frac{1}{2}(u+\theta)}}\, d\theta = -\frac{1}{2} \log \frac{1}{\delta} + \mathcal{O}(1).$$

In the same way we find for $u = u_0 - \delta$, that

$$\frac{1}{2\delta} \int_{-\delta}^{\delta} e^{\frac{3}{2}\theta} \frac{R\big(e^{u+\theta}\big)}{e^{\frac{1}{2}(u+\theta)}}\, d\theta = \frac{1}{2} \log \frac{1}{\delta} + \mathcal{O}(1).$$

This shows that there is a u' in the interval $u_0 \le u' \le u_0 + 2\delta$ for which

$$e^{-\frac{3}{2}\delta} \frac{R\big(e^{u'}\big)}{e^{\frac{1}{2}u'}} \le -\frac{1}{2} \log \frac{1}{\delta} + \mathcal{O}(1)$$

or

$$\frac{R\big(e^{u'}\big)}{e^{\frac{1}{2}u'}} \le -\frac{1}{2} \log \frac{1}{\delta} + \mathcal{O}(1).$$

Also there is a u'' in the interval $u_0 - 2\delta \le u'' \le u_0$ for which

$$\frac{e^{\frac{3}{2}\delta} R\big(e^{u''}\big)}{e^{\frac{1}{2}u''}} \ge \frac{1}{2} \log \frac{1}{\delta} - \mathcal{O}(1),$$

or

$$\frac{R\big(e^{u''}\big)}{e^{\frac{1}{2}u''}} \ge \frac{1}{2} \log \frac{1}{\delta} - \mathcal{O}(1).$$

We have

$$L \le u_0 \le L^{N(T)+1}, \text{ and } L = [\log^2 T],\ \delta = \frac{\log T}{T}.$$

Since by Theorem 2.3

$$N(T) < \frac{T}{6.1}\log T,$$

we get

$$u_0 + 2\delta < e^{\frac{T}{3}\log T \log\log T}$$

and

$$\log\frac{1}{\delta} = \log T - \log\log T.$$

From this

$$\log\frac{1}{\delta} \geq \log\log(u_0+2\delta) - \frac{5}{2}\log\log\log(u_0+2\delta).$$

Thus if we write $x_1 = e^{u'},\ x = e^{u''}$, we get for T sufficiently large,

$$\frac{R(x_1)}{\sqrt{x_1}} \leq -\frac{1}{2}\log\log\log x_1 + 3\log\log\log\log x_1$$

and

$$\frac{R(x_2)}{x_2} \geq \frac{1}{2}\log\log\log x_2 - 3\log\log\log\log x_2.$$

Thus there is a sequence of $x \to \infty$ for which

$$\psi(x) - x \geq \frac{x^{1/2}}{2}\log\log\log x - 3x^{1/2}\log\log\log\log x$$

and

$$\psi(x) - x \leq -\frac{x^{1/2}}{2}\log\log\log x + 3x^{1/2}\log\log\log\log x.$$

From this we can easily derive corresponding results for $\mathcal{J}(x)$ and finally for $\pi(x)$. In fact, since $\mathcal{J}(x) = x + \rho(x)$, we have $\rho(x) = R(x) - \sqrt{x} + \mathcal{O}(x^{1/3})$ and by (3.1)

$$\begin{aligned}\pi(x) - \int_2^x \frac{dt}{\log t} &= \frac{\rho(x)}{\log x} + \int_2^x \frac{\rho(t)}{t\log^2 t}dt \\ &= \frac{\rho(x)}{\log x} + \int_2^x \frac{R(t) - \sqrt{t} + \mathcal{O}(t^{1/3})}{t\log^2 t}\, dt.\end{aligned}$$

We have

$$\int_2^x \frac{1}{\sqrt{t}\log^2 t}\, dt = \mathcal{O}\Big(\big(\int_{\frac{\sqrt{x}}{\log^2 x}}^x t^{-\frac{1}{2}}dt\big)\log^{-2} x\Big) + \mathcal{O}\big(\int_2^{\frac{\sqrt{x}}{\log^2 x}} dt\big) = \mathcal{O}\big(\frac{\sqrt{x}}{\log^2 x}\big).$$

Write

$$R_1(x) = \int_2^x R(t)dt.$$

Then by (4.1) and Theorem 2.3, we get

$$R_1(x) = \mathcal{O}\big(x^{3/2}\big),$$

and

$$\int_2^x \frac{R(t)}{t\log^2 t}\,dt = \frac{R_1(t)}{t\log^2 t}\Big|_2^x + \mathcal{O}\Big(\int_2^x \frac{dt}{\sqrt{t}\log^2 t}\Big) = \mathcal{O}\Big(\frac{\sqrt{x}}{\log^2 x}\Big).$$

Therefore

$$\pi(x) - \int_2^x \frac{dt}{\log t} = \frac{\rho(x)}{\log x} + \mathcal{O}\Big(\frac{\sqrt{x}}{\log^2 x}\Big),$$

and we have the following

Theorem 4.1. *There exists a sequence of $x \to \infty$ such that*

$$\pi(x) - \int_2^x \frac{dt}{\log t} > \frac{1}{2}\cdot\frac{\sqrt{x}}{\log x}\log\log\log x - 4\frac{\sqrt{x}}{\log x}\log\log\log\log x$$

and a sequence of $x \to \infty$ such that

$$\pi(x) - \int_2^x \frac{dt}{\log t} < -\frac{1}{2}\cdot\frac{\sqrt{x}}{\log x}\log\log\log x + 4\frac{\sqrt{x}}{\log x}\log\log\log\log x.$$

This result is essentially due to J.E. Littlewood. In particular we see from Theorem 4.1 that

$$\pi(x) - \int_2^x \frac{dt}{\log t}$$

changes sign infinitely often. However no sign change has ever been numerically verified. The first such is extremely far out.

If one looks at the case of arithmetic progressions one finds that only for the case of the progression $qn+1$ can one prove the analogous result, not for the general $qn+\ell$ with $(\ell, q) = 1$.

4.3. Estimation for the exceptional zero

We return to the possibility of an exceptional zero for $L(s,\chi)$ if χ is a primitive quadratic character modulo q.

In addition to the bounds (1.8) and (1.9) we have established for $\zeta(s)$ and $L(s,\chi)$ for $\sigma > 0$. We need still to bounds for $\sigma > 0$

(4.7) $$\zeta(s)\frac{s}{s-1} + \mathcal{O}((1+|t|)^{1-\sigma}\log(2+|t|))$$

and

$$L(s,\chi) = \mathcal{O}\big((1+|t|)q\big)^{1-\sigma}\log q(1+|t|)\big) + \mathcal{O}(1), \tag{4.8}$$

where the constants implied by $\mathcal{O}$'s are depending only on σ.

To prove these estimations we need the following lemma.

Lemma 4.2 (Euler summation formula). *Suppose that a, b are integers satisfying $b \geq a+1$, and $\varphi(x)$ has continuous derivatives. Then*

$$\sum_{a<n\leq b}\varphi(n) = \int_a^b \varphi(x)dx + \int_a^b \big(x-[x]-\frac{1}{2}\big)\varphi'(x)dx + \frac{1}{2}\varphi(b) - \frac{1}{2}\varphi(a).$$

Proof. We have

$$\begin{aligned}
\int_a^b [x]\varphi'(x)\,dx &= \sum_{n=a}^{b-1}\int_n^{n+1}[x]\varphi'(x)\,dx \\
&= \sum_{n=a}^{b-1} n\int_n^{n+1}\varphi'(x)\,dx = \sum_{n=a}^{b-1} n\big(\varphi(n+1)-\varphi(n)\big) \\
&= \sum_{n=a+1}^{b}\varphi(n)(n-1-n) + b\varphi(b) - a\varphi(a) \\
&= -\sum_{n=a+1}^{b}\varphi(n) + b\varphi(b) - a\varphi(a)
\end{aligned}$$

and

$$\int_a^b (x-\frac{1}{2})\varphi'(x)\,dx = (b-\frac{1}{2})\varphi(b) - (a-\frac{1}{2})\varphi(a) - \int_a^b \varphi(x)\,dx.$$

Taking difference of these two formulas, the lemma follows.

If $\sigma > 1$, we take $\varphi(t) = t^{-s}, a = x$ a positive integer and $b = \infty$. Then

$$\begin{aligned}
\zeta(s) &= \sum_{n\leq x} n^{-s} + \sum_{n>x} n^{-s} \\
&= \sum_{n\leq x} n^{-s} + \int_x^\infty t^{-s}dt - s\int_x^\infty (t-[t]-\frac{1}{2})\,t^{-s-1}\,dt - \frac{x^{-s}}{2} \\
&= \frac{s}{s-1} + \sum_{n\leq x} n^{-s} + \frac{x^{1-s}-s}{s-1} - \frac{x^{-s}}{2} - s\int_x^\infty (t-[t]-\frac{1}{2})\,t^{-s-1}\,dt.
\end{aligned}$$

Excepting $s = 1$, the right-hand side is analytic in $\sigma \geq 0$. If $\sigma \geq 1$, we take $x = |[t]|$, then

$$\zeta(s) = \frac{s}{s-1} + \mathcal{O}\big(\log(2 + |t|)\big).$$

If $0 < \sigma < 1$, then

$$\zeta(s) = \frac{s}{s-1} + \mathcal{O}\big((2 + |t|)^{1-\sigma}\big),$$

where the constants implied by $\mathcal{O}'s$ depends only on σ. (4.7) is proved. The proof of (4.8) is similar. In particular for $s = 1 + it$ we get

$$\zeta(s) = \frac{s}{s-1} + \mathcal{O}\big(\log(2 + |t|)\big)$$

and

$$L(s, \chi) = \mathcal{O}(\log q(1 + |t|)).$$

We shall need the better bounds on $\sigma = 0; s = it$. By the functional equation of $\xi(s)$ (see Theorem 1.1) and Stirling's formula (see Lemma 2.2), we obtain

$$\begin{aligned} \zeta(it) &= \mathcal{O}\left(\left|\frac{\Gamma(\frac{1+it}{2})}{\Gamma(\frac{-it}{2})}\right| \big|\zeta(1+it)\big|\right) \\ &= \mathcal{O}\big((|t| + 1)^{1/2} \log(2 + |t|)\big). \qquad (4.9) \end{aligned}$$

Similarly

$$L(it,\ \chi) = \mathcal{O}\bigg(\big(q(1 + |t|)\big)^{1/2} \log q(2 + |t|)\bigg). \qquad (4.10)$$

Lemma 4.3. *We have*

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \frac{x^{s+2}}{s(s+1)(s+2)}\, ds = \begin{cases} 0 & \text{for } 0 < x \leq 1, \\ \frac{1}{2}(x-1)^2 & \text{for } x \geq 1. \end{cases}$$

The proof is similar to that of Lemma 2.6. Let

$$f(s) = \zeta(s)L(s, \chi) = \sum_{n=1}^{\infty} \frac{c_n}{n^s},$$

here all $c_n \ge 0$ and $c_{n^2} \ge 1$ (see §2.6). This gives for $x \ge 2$,

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \frac{x^{s+2} f(s-\frac{1}{2})}{s(s+1)(s+2)}\, ds \\
= &\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \frac{x^{s+2}}{s(s+1)(s+2)} \sum_{n=1}^{\infty} \frac{c_n}{n^{s-1/2}}\, ds \\
= &\sum_{n=1}^{\infty} \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \frac{c_n(\frac{x}{n})^{s+2} n^{5/2}}{s(s+1)(s+2)}\, ds \\
= &\frac{1}{2}\sum_{n<x} (\frac{x}{n}-1)^2 n^{5/2} c_n = \frac{1}{2}\sum_{n<x}(x-n)^2\sqrt{n}\, c_n \\
\ge &\frac{1}{2}\sum_{n^2<x}(x-n^2)^2 n \ge \frac{1}{2}\sum_{n^2<x/2}(x-x/2)^2 n \\
\ge &\frac{1}{2}\cdot\frac{1}{4}x^2 \sum_{n<\sqrt{\frac{x}{2}}} n > \frac{1}{20}x^3. \qquad (4.11)
\end{aligned}$$

Take contour C as shown by Figure 4.1. Then it follows by Cauchy Theorem that

$$\frac{1}{2\pi i}\int_C \frac{x^{s+2} f(s-\frac{1}{2})}{s(s+1)(s+2)}\, ds = \frac{x^{\frac{7}{2}}}{\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}}\, L(1,\chi)$$

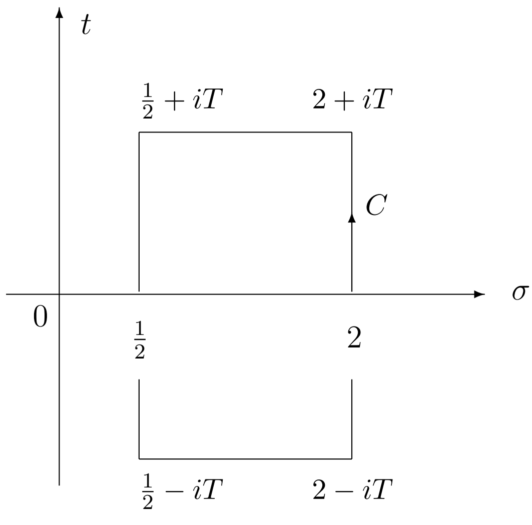

Figure 4.1

Since by (4.9) and (4.10), we have

$$\int_{2+iT}^{\frac{1}{2}+iT} \frac{x^{s+2} f(s-\frac{1}{2})}{s(s+1)(s+2)}\, ds = \mathcal{O}\big(\frac{x^4 qT \log^2(q\phi T)}{T^3}\big) = o(1) \qquad (\text{as } T \to \infty)$$

and

$$\int_{2-iT}^{2-iT} \frac{x^{s+2} f(s-\frac{1}{2})}{s(s+1)(s+2)}\, ds = 0(1) \qquad (\text{as } T \to \infty),$$

If follows by (4.9) and (4.10) again

$$\begin{aligned}
& \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \frac{x^{s+2} f(s-\frac{1}{2})}{s(s+1)(s+2)}\, ds \\
= \;& \frac{8}{105}\, x^{\frac{7}{2}} L(1,\chi) + \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \frac{x^{s+2}\zeta(s-\frac{1}{2})L(s-\frac{1}{2},\chi)}{s(s+1)(s+2)}\, ds \\
= \;& \frac{8}{105}\, x^{\frac{7}{2}} L(1,\chi) + \mathcal{O}\big(x^{\frac{5}{2}}\int_{-\infty}^{\infty} \frac{|t| q^{1/2}\log(|t|+2)\log q(|t|+1)}{(|t|+1)^3}\, dt\big) \\
(4.12) \qquad = \;& \frac{8}{105}\, x^{\frac{7}{2}} L(1,\chi) + \mathcal{O}\big(x^{\frac{5}{2}} q^{1/2}\log q\big).
\end{aligned}$$

Thus by (4.11) and (4.12) we obtain for some constant $A > 0$

$$\frac{x^3}{20} < \frac{8}{105}\, x^{\frac{7}{2}} L(1,\chi) + A x^{\frac{5}{2}} q^{1/2}\log q.$$

Choosing

$$\sqrt{x} = 40 A q^{1/2}\log q,$$

we get

$$\frac{x^3}{20} - A x^{5/2} q^{1/2}\log q = x^{5/2}\big(\frac{\sqrt{x}}{20} - A q^{1/2}\log q\big) = x^{5/2} A q^{1/2}\log q = \frac{x^3}{40},$$

and so

$$\frac{x^3}{40} < \frac{8}{105} x^{\frac{7}{2}} L(1,\chi),$$

or

$$L(1,\chi) > \frac{1}{4\sqrt{x}} = \frac{1}{160 A q^{1/2}\log q} = \frac{\alpha}{q^{1/2}\log q}.$$

Since the estimation

$$L'(\sigma,\chi) = \mathcal{O}(\log^2 q)$$

holds for $\sigma > 1 - \dfrac{1}{\log q}$ (see §3.3), we obtain the following

Theorem 4.4. *Let χ be the primitive quadratic character modulo q. Then the real zero β of $L(s,\chi)$ satisfies*

$$1 - \beta \geq \frac{\alpha'}{q^{1/2}\log^3 q},$$

where α' is a computable absolute constant.

Lecture V

In this lecture we shall prove Siegel's Theorem on the exceptional zero of L-function with primitive quadratic character modulo q.

5.1. Siegel's Theorem

We shall consider further the possibility of the presence of an exceptional zero, a real zero $\beta > 1 - \dfrac{1}{\log q}$ for the primitive quadratic character modulo q. We saw last time that $\beta > 1 - \dfrac{c'}{\sqrt{q}\log^3 q}$, with a constant that could be effectively determined (see Theorem 4.4). It was also mentioned that the connection with the class number gives $\beta > 1 - \dfrac{c'}{\sqrt{q}\log^2 q}$, and that work of Goldfeld, Gross and Zagier gives $\beta > 1 - \dfrac{c'}{\sqrt{q}\log q}$, and I believe this has now been improved to $\beta > 1 - \dfrac{c'}{\sqrt{q}}$. In each case the result is obtained by finding a lower bound for $L(1,\chi)$ and then deriving a lower bound for $1-\beta$ by using that $L'(\sigma,\chi) = \mathcal{O}(\log^2 q)$ for $\sigma > 1 - \dfrac{1}{\log q}$.

We shall now turn to Siegel's Theorem, a result that is in some ways vastly superior in the order of the lower bound for $1-\beta$ in terms of q, but also very deficient for most purposes in that we have no way of estimating the constants that enter as coefficients.

We begin with the inequalities:

$$\zeta(s) = \frac{s}{s-1} + \mathcal{O}\big((1+|t|)^{1-\sigma}\log(2+|t|)\big), \tag{5.1}$$

and

$$L(s,\chi) = 1 + \mathcal{O}\big(q^{1-\sigma}(1+|t|)^{1-\sigma}\log q(2+|t|)\big) \tag{5.2}$$

for $\sigma > 0$ (see (4.7) and (4.8)). We assume we have a primitive quadratic character χ_1 mod q_1 such that $L(s,\chi_1)$ has a real zero $\beta_1 = 1-\delta$ with δ small, and shall see what we can conclude about $L(1,\chi)$ for a primitive quadratic character $\chi \mod q$, $q \neq q_1$.

We consider

$$f(s) = \zeta(s)L(s,\chi)L(s,\chi_1)L(s,\chi\chi_1),$$

where $\sigma > 1$. We have

$$f(s) = \prod_p (1-p^{-s})^{-1}(1-\chi(p)p^{-s})^{-1}(1-\chi_1(p)p^{-s})^{-1}(1-\chi(p)\chi_1(p)p^{-s})^{-1}$$

where p runs over all prime number, and so

$$\begin{aligned}\log f(s) &= -\sum_p \big(\log(1-p^{-s}) + \log(1-\chi(p)p^{-s})\big) \\ &\quad + \log\big(1-\chi_1(p)p^{-s} + \log(1-\chi(p)\chi_1(p)p^{-s})\big) \\ &= \sum_p \sum_{m=1}^{\infty} \frac{1}{mp^{ms}}\big(1+\chi(p)^m + \chi_1(p)^m + \chi(p)^m\chi_1(p)^m\big) \\ &= \sum_p \sum_{m=1}^{\infty} \frac{1}{mp^{ms}}\big(1+\chi(p)^m)(1+\chi_1(p)^m\big)\end{aligned}$$

is a Dirichlet series with non-negative coefficients. Therefore

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s},$$

where $a_1 = 1$ and $a_{\geq} 0$.

By (5.1) and (5.2), we find for $\sigma \geq \frac{4}{5}$ that for $|s-1| \geq \frac{1}{10}$,

$$\begin{aligned}f(s) &= \mathcal{O}\big((1+|t|)^{\frac{4}{5}}(q\,q_1)^{\frac{2}{5}}\log^4 q(2+|t|)\big) \\ &= \mathcal{O}\big((1+|t|)^{\frac{5}{6}}(q\,q_1)^{\frac{2}{5}}\big). \qquad (5.3)\end{aligned}$$

For $x > 2$ we have by Lemma 2.6,

$$\begin{aligned}&\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \frac{x^{s+1}}{s(s+1)} f(s+\beta_1)\,ds \\ &= \sum_{n<x} (x-n)\frac{a_n}{n^{\beta_1}} \geq x-1 > \frac{x}{2}. \qquad (5.4)\end{aligned}$$

Moving the path of integration to $\sigma = \frac{4}{5} - \beta_1 = -\frac{1}{5} + \delta$, we get by (5.3) that the integral above also equals to the residue of integrand at $s = 1 - \beta_1 = \delta$ plus the integral along the line from $\frac{4}{5} - \beta_1 - i\infty$ to $\frac{4}{5} - \beta_1 + i\infty$, that is,

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \frac{x^{s+1}}{s(s+1)}\, f(s+\beta_1)\, ds$$

$$= \frac{x^{1+\delta}L(1,\chi)L(1,\chi_1)L(1,\chi\chi_1)}{\delta(\delta+1)}$$

$$+\frac{1}{2\pi i}\int_{\frac{4}{5}-\beta_1-i\infty}^{\frac{4}{5}-\beta_1+i\infty} \frac{x^{s+1}}{s(s+1)} f(s+\beta_1)\, ds$$

$$= x^{1+\delta}\frac{L(1,\chi)L(1,\chi_1)L(1,\chi\chi_1)}{\delta(1+\delta)} + \mathcal{O}\big(x^{\frac{4}{5}+\delta}(q\, q_1)^{1/2}\big), \tag{5.5}$$

where (5.3) is also used to estimate the last integral.

Comparison of (5.4) and (5.5), we obtain

$$\frac{x}{2} < \frac{x^{1+\delta}L(1,\chi)L(1,\chi_1)L(1,\chi\chi_1)}{\delta(1+\delta)} + Ax^{\frac{5}{4}+\delta}(q\, q_1)^{1/2}$$

where A is an absolute constant. We now choose

$$x > 4Ax^{\frac{4}{5}+\delta}(q\, q_1)^{1/2}$$

or

$$x^{\frac{1}{5}-\delta} > 4A(q\, q_1)^{1/2}.$$

We may assume $\delta < \frac{1}{30}$, and so take

$$x = (4A)^6(q\, q_1)^3.$$

Then we have

$$\begin{aligned} Ax^{\frac{4}{5}+\delta}(q\, q_1)^{1/2} &= x \cdot x^{-\frac{1}{5}+\delta}A(q\, q_1)^{1/2} \\ &< x(4A(q\, q_1)^{1/2})^{-1}A(q\, q_1)^{1/2} = \frac{x}{4}, \end{aligned}$$

and

$$\frac{x}{4} < x^{1+\delta}\frac{L(1,\chi)L(1,\chi_1)L(1,\chi\chi_1)}{\delta}$$

or

$$L(1,\chi)L(1,\chi_1)L(1,\chi\chi_1) > \frac{\delta}{4}x^{-\delta} = \delta A'(q\, q_1)^{-3\delta}.$$

Since

$$L(1,\chi_1) = \mathcal{O}(\log q_1) \text{ and } L(1,\chi\chi_1) = \mathcal{O}(\log q\, q_1)$$

by (5.2), we get

$$L(1,\chi) > c(\delta)(q\, q_1)^{-4\delta}.$$

From

$$L'(\sigma,\chi) = \mathcal{O}(\log^2 q)$$

for $\sigma > 1 - \dfrac{1}{\log q}$, we then get

$$1 - \beta > c'(\delta) q^{-5\delta}.$$

Now, either the β have an absolute upper bound $\theta < 1$, or we can find q_1 with arbitrary close one. In the second case, we may choose χ_1 such that $1 - \beta_1 < \varepsilon/5$ for any given $\varepsilon > 0$. We then get

$$1 - \beta > c(\varepsilon) q^{-\varepsilon}.$$

While in the first case we of course have $1 - \beta \geq 1 - \theta > 0$. We could of course also phrase the result as

$$1 - \beta > c(\varepsilon) q^{-\varepsilon}, \qquad \text{for } q > q_0(\varepsilon),$$

where ε can be chosen positive and arbitrary small. In either formulation, the constants $c(\varepsilon)$ and $q_0(\varepsilon)$ exist, but we can not give any bound for their size. Now we restate our result as follows:

Theorem 5.1. *For any given $\varepsilon > 0$, and for any primitive quadratic character χ* mod q, *there exists a constant $c'(\varepsilon)$ such that $L(s,\chi) \neq 0$ for*

$$\sigma \geq 1 - c'(\varepsilon) q^{-\varepsilon}.$$

If on the other hand we could actually find a $L(s,\chi_1)$ with a β_1 very close to 1, we could obtain an effective estimate (and quite a bit better than indicated by our proof of Siegel's result, since we did not try to obtain the best factor in front of δ in the exponent of q).

5.2. Remarks

This essentially concludes the material in analytic prime number theory that I had planned to talk about. I have not touched on developments like:

Better (that is: wider) zero-free regions along $\sigma = 1$. These can be obtained using the theory of exponential sums of the type

$$\sum_{N}^{N'} e^{it}, \qquad \text{or} \sum_{N}^{N'} \chi(n) n^{it}.$$

Such estimates lead for instance for $\zeta(s)$ to a zero-free region of the type

$$\sigma > 1 - \frac{\alpha}{(\log(2+|t|))^{\theta}}, \qquad \alpha > 0,$$

with a $0 < \theta < 1$. This gives us the remainder terms

$$\psi(x) = x + \mathcal{O}\big(xe^{-\alpha'(\log x)^{\frac{1}{1+\theta}}}\big),$$

and

$$\pi(x) = \int_2^x \frac{dt}{\log t} + \mathcal{O}\big(xe^{-\alpha'(\log x)^{\frac{1}{1+\theta}}}\big).$$

Another development of great importance for many applications are the so-called “density theorems”.

Lecture VI

In this lecture, we shall introduce the concept of Beurling's generalized integers, and use the elementary method to establish the P.N.T. of Beurling's integers. The proof is based on the Selberg's asymptotic formula.

6.1. Beurling's generalized integers

We shall begin by describing the most general context in which the elementary approach to the prime number theorem works (at least at present). We consider the case of Beurling's generalized integers. Let us have a set of real numbers $p_i,\ i = 1, 2, 3, \cdots$,

$$1 < p_1 \le p_2 \le \cdots \le p_i \le \cdots,$$

such that $p_i \to \infty$ as $i \to \infty$. We form all possible products $\prod {p_i}^{\alpha_i}$ and order them according to magnitude or size:

$$n_1 = 1,\ n_2 = p_1,\ \cdots,\ n_i \le n_{i+1},\ \cdots.$$

We denote by $N(x)$ the number of $n_i \le x$ and assume we have an asymptotic law

$$N(x) = Ax + R(x), \tag{6.1}$$

where we shall assume $A = 1$ and

$$R(x) = o\Big(\frac{x}{\log^2 x}\Big). \tag{6.2}$$

and shall try to establish the P.N.T. that is if we denote by $\pi(x)$ the number of $p_i \le x$ we shall show that

$$\pi(x) = \frac{x}{\log x} + o\Big(\frac{x}{\log x}\Big),$$

in the form

$$\mathcal{J}(x) = x + o(x),$$

where

$$\mathcal{J}(x) = \sum_{p_i \le x} \log p_i.$$

Similarly we use the notation

$$\psi(x) = \sum_{p_i^r \le x} \log p_i,$$

and

$$\wedge(n_i) = \begin{cases} \log p_i, & \text{if } n_i = {p_i}^{\alpha_i},\ \alpha > 0, \\ 0, & \text{otherwise.} \end{cases}$$

For simplicity, we shall drop the indices and write n, m or d for the generalized integers, p, q and r for generalized primes, and use greek letters μ, ν to denote ordinary integers, and x, y, t, u, v to denote real numbers. Constants will be denoted by capital latin letters.

We define $\mu(n) = (-1)^\nu$ if n is the product of ν distinct primes p and otherwise $\mu(n) = 0$. We also write

$$d|n \text{ if } n = \prod {p_i}^{\alpha_i},\ d = \prod {p_i}^{\beta_i} \text{ and } \alpha_i \geq \beta_i \text{ for all } i.$$

We have then

$$\sum_{d|n} \mu(d) = \begin{cases} 0 & \text{for } n \neq 1, \\ 1 & \text{for } n = 1. \end{cases} \tag{6.3}$$

In fact, we may confine that d runs over all products of distinct prime factors of n. Let the prime divisors of n be $p_1, \cdots, p_s$. Then

$$\sum_{d|n} \mu(d) = \prod_{i=1}^{s} \bigl(1 - \mu(p_i)\bigr),$$

and (6.3) follows.

We need some preliminary lemmas.

Lemma 6.1. *We have*

$$\sum_{d \leq x} \frac{\mu(d)}{d} = \mathcal{O}(1).$$

Proof. For $x \geq 1$, we have by (6.1), (6.2) and (6.3)

$$\begin{aligned} 1 &= \sum_{n \leq x} \sum_{d|n} \mu(d) = \sum_{d \leq x} \mu(d) N\bigl(\frac{x}{d}\bigr) \\ &= x \sum_{d \leq x} \frac{\mu(d)}{d} + \mathcal{O}\Bigl(\sum_{d \leq x} \bigl| R\bigl(\frac{x}{d}\bigr)\bigr| \Bigr). \end{aligned}$$

Here

$$\sum_{d\le x}\left|R\big(\frac{x}{d}\big)\right| = x\sum_{d\le x}\frac{\varepsilon(\frac{x}{d})}{d(1+\log\frac{x}{d})^2},$$

where we here and in the future denoted by $\varepsilon(z)$ a function that tends to zero as $z\to\infty$. Dividing the interval $[1,x]$ into subintervals $xe^{-\nu}\le d< xe^{1-\nu}$ for $\nu = 2,\cdots,[\log x]$ plus two intervals $1\le d< xe^{-[\log x]}$ and $xe^{-1}\le d\le x$. However it is no influence on our conclusions if we write the subintervals by $xe^{-\nu}\le d< e^{1-\nu}$ for $\nu = 1,2,\cdots,\log x$. We get

$$\begin{aligned}\sum_{d\le x}\frac{\varepsilon(\frac{x}{d})}{d(1+\log\frac{x}{d})^2} &\le \sum_{1\le\nu\le\log x}\frac{\varepsilon(e^{\nu-1})N(\frac{x}{e^{1-\nu}})}{xe^{-\nu}(1+\nu-1)^2}\\ &< A\sum_{\nu\ge1}\frac{\varepsilon(e^{\nu-1})}{\nu^2} = \mathcal{O}(1),\end{aligned}$$

where we may assume that $\varepsilon(t)$ is decreasing in $xe^{-\nu}\le t< xe^{1-\nu}$. Thus

$$1 = x\sum_{d\le x}\frac{\mu(d)}{d}+\mathcal{O}(x),$$

and the lemma is proved.

Lemma 6.2. *We have*

$$\sum_{d\le x}\frac{\mu(d)}{d}\log\frac{x}{d} = o(\log\log x).$$

Proof. Consider first

$$\begin{aligned}\sum_{n\le y}\frac{1}{n} &= \int_{1/2}^{y}\frac{1}{t}dN(t) = \frac{N(y)}{y}+\int_1^y\frac{N(t)}{t^2}\,dt\\ &= 1+o\big(\frac{1}{\log^2 y}\big)+\log y+\int_1^y\frac{\varepsilon(t)}{t(1+\log t)^2}\,dt\\ &= \log y+1+o\big(\frac{1}{\log^2 y}\big)+\int_1^\infty\frac{\varepsilon(t)}{t(1+\log t)^2}\,dt\\ &\qquad +o\Big(\int_1^\infty\frac{dt}{t(1+\log t)^2}\Big)\\ &= \log y+C+o\big(\frac{1}{1+\log y}\big).\end{aligned}$$

So we can write

$$\log\frac{x}{d}=\sum_{n\le\frac{x}{d}}\frac{1}{n}-C+\frac{\varepsilon(\frac{x}{d})}{1+\log\frac{x}{d}}, \tag{6.4}$$

and so

$$\sum_{d\le x}\frac{\mu(d)}{d}\log\frac{x}{d}=\sum_{d\le x}\frac{\mu(d)}{d}\sum_{n\le x/d}\frac{1}{n}-C\sum\frac{\mu(d)}{d}$$
$$+\mathcal{O}\big(\sum_{d\le x}\frac{\varepsilon(\frac{x}{d})}{d(1+\log\frac{x}{d})}\big).$$

Here by (6.3)

$$\sum_{d\le x}\frac{\mu(d)}{d}\sum_{n\le x/d}\frac{1}{n}=\sum_{nd\le x}\frac{\mu(d)}{nd}$$
$$=\sum_{m\le x}\frac{1}{m}\sum_{d|m}\mu(d)=1,$$

and we have also

$$\begin{aligned}\sum_{d\le x}\frac{\varepsilon(\frac{x}{d})}{d(1+\log\frac{x}{d})} &< \sum_{1\le\nu\le\log x}\frac{\varepsilon(e^{\nu-1})e^{\nu}N(xe^{1-\nu})}{x(1+\nu-1)}\\ &< A\big(\sum_{1\le\nu\le\sqrt{\log x}}\frac{1}{\nu}+\sum_{\sqrt{\log x}<\nu\le\log x}\frac{\varepsilon(e^{\nu-1})}{\nu}\big)\\ &= o(\log\log x).\end{aligned}$$

Hence by Lemma 6.1 the lemma follows.

Lemma 6.3. *We have*

$$\sum_{d\le x}\frac{\mu(d)}{d}\log^2\frac{x}{d}=\log x+\sum_{n\le x}\frac{\wedge(n)}{n}+o(\log x)$$

$$\log x+\sum_{p\le x}\frac{\log p}{p}+o(\log x).$$

Proof. By (6.4) we obtain

$$\sum_{d\le x}\frac{\mu(d)}{d}\log^2\frac{x}{d}=\sum_{d\le x}\frac{\mu(d)}{d}\log\frac{x}{d}\sum_{n\le\frac{x}{d}}\frac{1}{n}-C\sum_{d\le x}\frac{\mu(d)}{d}\log\frac{x}{d}$$

$$+\mathcal{O}\Big(\sum_{d\le x}\frac{\varepsilon(\frac{x}{d})\log\dfrac{x}{d}}{d(1+\log\frac{x}{d})}\Big).$$

The first term on the right-hand side equals

$$\sum_{nd\le x}\frac{\mu(d)\log\frac{x}{d}}{nd}=\sum_{m\le x}\frac{1}{m}\sum_{d|m}\mu(d)\log\frac{x}{d}.$$

We have

$$\sum_{d|m}\mu(d)\log\frac{x}{d}=\begin{cases}\log x, & \text{for } m=1,\\ \wedge(m), & \text{for } m>1.\end{cases}$$

In fact, the equality is obvious for $m=1$. If $m>1$, then the left-hand side is equal to $-\sum_{d|m}\mu(d)\log d=\sum(-1)^{r-1}\log p_{i_1}\cdots,p_{i_r}$, where $p_{i_1}\cdots,p_{i_r}$ run over all sets of distinct prime factors of m, and so we get the second part of equality. Thus

$$\sum_{d\le x}\frac{\mu(d)}{d}\log\frac{x}{d}\sum_{n\le\frac{x}{d}}\frac{1}{n}=\log x+\sum_{n\le x}\frac{\wedge(n)}{n}$$

$$=\log x+\sum_{p\le x}\frac{\log p}{p}+\mathcal{O}(1).$$

We have also

$$\sum_{d\le x}\frac{\varepsilon(\frac{x}{d})\log\frac{x}{d}}{d(1+\log\frac{x}{d})}<\sum_{d\le x}\frac{\varepsilon(\frac{x}{d})}{d}<\sum_{1\le\nu\le\log x}\frac{\varepsilon(e^{\nu-1})e^{\nu}}{x}N(\frac{x}{e^{\nu-1}})$$

$$<A\sum_{1\le\nu\le\log x}\varepsilon(e^{\nu-1})=o(\log x). \tag{6.5}$$

The lemma follows from Lemma 6.2.

Lemma 6.4. *We have for* $n\le x$

$$\sum_{d|n}\mu(d)\log^2\frac{x}{d}=\begin{cases}\log^2 x, & \text{for } n=1,\\ \log\frac{x^2}{p}\log p, & \text{for } n=p^\nu,\ \nu>0,\\ 2\log p\log q, & \text{for } n=p^\nu q^\mu,\ p\ne q\\ 0, & \text{if } n \text{ has 3 or more prime factors.}\end{cases}$$

Proof. Let $n = {p_1}^{\nu_1} \cdots {p_r}^{\nu_r}$, where p_i's are distinct prime divisors of n. Then

$$\sum_{d|n} \mu(d) \log^2 \frac{x}{d} = \log^2 x - \sum_i \log^2 \frac{x}{p_i} + \sum_{j<i} \log^2 \frac{x}{p_i p_j}$$
$$- + \cdots + (-1)^r \log^2 \frac{x}{p_1 \cdots p_r},$$

the coefficient of $\log^2 x$ equals to $1 - \binom{r}{1} + \binom{r}{2} - \cdots + (-1)^r \binom{r}{r}$; the coefficient of $\log p_i \log x$ equals to $2\big(1 - \binom{r-1}{1} + \binom{r-2}{2} - \cdots + (-1)^{r-1}\binom{r-1}{r-1}\big)$; the coefficient of $\log^2 p_i$ equals to $-1 + \binom{r-1}{1} - \cdots + (-1)^{r-1}\binom{r-1}{r-1}$; and the coefficient of $\log p_j \log p_i$ equals to $2\big(1 - \binom{r-2}{1} + \cdots + (-1)^{r-2}\binom{r-2}{r-2}\big)$. The lemma follows immediately.

We may give another proof as follows. Let

$$f(y) = x^y \sum_{d|n} \mu(d) d^{-y} = x^y \prod_{p|n} (1 - p^{-y}).$$

Then

$$\begin{aligned} f'(y) &= x^y \log x \sum_{d|n} \mu(d) d^{-y} - x^y \sum_{d|n} \mu(d) \log d \cdot d^{-y}, \\ f''(y) &= x^y \log^2 x \sum_{d|n} \mu(d) d^{-y} - 2x^y \log x \sum_{d|n} \mu(d) \log d \cdot d^{-y}, \\ &\quad + x^y \sum_{d|n} \mu(d) \log^2 d \cdot d^{-y}, \end{aligned}$$

and so

$$\begin{aligned} f''(0) &= \log^2 x \sum_{d|n} \mu(d) - 2 \log x \sum_{d|n} \mu(d) \log d \\ &\quad + \sum_{d|n} \mu(d) \log^2 d = \sum_{d|n} \mu(d) \log^2 \frac{x}{d}. \end{aligned}$$

On the other hand, by the derivatives of $f(y)$ for the product form, we see that if n has at most 2 distinct prime factors, then $f''(0)$ have the stated expressions. If n has 3 or more prime factors, then every term in $f''(y)$ must have a factor of $(1 - p^{-y})$ with $p|n$. So $f''(0) = 0$, and the lemma is proved.

Now by Lemma 6.3 and (6.5) we obtain

$$\begin{aligned} \sum_{n \le x} \sum_{d|n} \mu(d) \log^2 \frac{x}{d} &= \sum_{d \le x} \mu(d) \log^2 \frac{x}{d} N(\frac{x}{d}) \\ &= x \sum_{d \le x} \frac{\mu(d)}{d} \log^2 \frac{x}{d} + \mathcal{O}\big(x \sum \frac{\varepsilon(\frac{x}{d}) \log^2 \frac{x}{d}}{d(1 + \log \frac{x}{d})^2}\big) \\ &= x \log x + x \sum_{p \le x} \frac{\log p}{p} + o(x \log x), \end{aligned}$$

and by Lemma 6.4,

$$\sum_{n\le x}\sum_{d|n}\mu(d)\log^2\frac{x}{d} = \log^2 x + \sum_{p^\nu\le x}\log\frac{x^2}{p}\log p + \sum_{\substack{p^\nu q^\mu\le x\\ p\ne q}}\log p\log q.$$

Comparing the above two formulas we get

$$\sum_{p^\nu\le x}\log\frac{x^2}{p}\log p+\sum_{\substack{p^\nu q^\mu\le x\\ p\ne q}}\log p\log q$$

$$= x\log x + x\sum_{p\le x}\frac{\log p}{p}+o(x\log x). \tag{6.6}$$

We still need to estimate the sum

$$\sum_{p\le x}\frac{\log p}{p}.$$

We consider the sum

$$\begin{aligned}\sum_{n\le x}\log n &= \int_1^x \log t\, d\,N(t)\\ &= N(x)\log x-\int_1^x\frac{N(t)}{t}\,dt\\ &= x(\log x-1)+o\big(\frac{x}{\log x}\big). \qquad (6.7)\end{aligned}$$

Also since

$$\log n=\sum_{d|n}\wedge(d)$$

by the definition of $\wedge(n)$, we have, using $N(x)>\frac{1}{A}x$ for $x\ge 1$, that

$$\begin{aligned}\sum_{n\le x}\log n &= \sum_{n\le x}\sum_{d|n}\wedge(d)=\sum_{d\le x}\wedge(d)N\big(\frac{x}{d}\big)\\ &> \frac{1}{A}x\sum_{d\le x}\frac{\wedge(d)}{d},\end{aligned}$$

and so

$$\sum_{d\le x}\frac{\wedge(d)}{d}<Ax^{-1}\sum_{n\le x}\log n<Ax^{-1}\log xN(x)=\mathcal{O}(\log x).$$

Consequently

$$\sum_{p\le x}\frac{\log p}{p}=\mathcal{O}(\log x).$$

Inserting this bound into (6.6), we get

$$\log x\psi(x)=\sum_{p^\nu\le x}\log x\log p\le\sum_{p^\nu\le x}\log\frac{x^2}{p}\log p=\mathcal{O}(x\log x),$$

and so

$$\psi(x)=\sum_{d\le x}\wedge(d)=\mathcal{O}(x),$$

and

$$\mathcal{J}(x)=\sum_{p\le x}\log p=\mathcal{O}(x).$$

From these formulas we derive

$$\sum_{\substack{p^\nu\le x\\ \nu\ge 2}}\log\frac{x^2}{p}\log p=\mathcal{O}\big(\log^2 x\sum_{p\le\sqrt{x}}1\big)=\mathcal{O}(\sqrt{x}\log^2 x)$$

and

$$\sum_{\substack{p^\nu q^\mu\le x\\ \nu\ge 2}}\log p\log q=\sum_{\substack{p^\nu\le x\\ \nu\ge 2}}\log p\,\psi\big(\frac{x}{p^\nu}\big)=\mathcal{O}\big(x\sum_{\substack{p^\nu\le x\\ \nu\ge 2}}\frac{\log p}{p^\nu}\big)$$

$$=\mathcal{O}(x).$$

Inserting into (6.6), we get

$$\sum_{p\le x}\log\frac{x^2}{p}\log p+\sum_{pq\le x}\log p\log q=x\log x+x\sum_{p\le x}\frac{\log p}{p}$$

$$+\,o(x\log x) \tag{6.8}$$

6.2. Selberg's asymptotic formula

We begin by giving a more precise estimation for

$$\sum_{d\le x}\wedge(d)N\big(\frac{x}{d}\big).$$

We have

$$\sum_{d\le x}\wedge(d)N\big(\frac{x}{d}\big)=x\sum_{d\le x}\frac{\wedge(d)}{d}+x\sum_{d\le x}\frac{\wedge(d)}{d}\frac{\varepsilon(\frac{x}{d})}{(1+\log\frac{x}{d})^2},$$

where

$$x\sum_{d\le x}\frac{\wedge(d)}{d}\cdot\frac{\varepsilon(\frac{x}{d})}{(1+\log\frac{x}{d})^2}\le\sum_{1\le\nu\le\log x}\psi(xe^{1-\nu})\frac{e^{\nu}\varepsilon(e^{\nu-1})}{\nu^2}$$

$$< Ax\sum_{\nu\ge 1}\frac{\varepsilon(e^{\nu-1})}{\nu^2}=\mathcal{O}(x).$$

Hence

$$\sum_{d\le x}\wedge(d)N\big(\frac{x}{d}\big)=x\sum_{d\le x}\frac{\wedge(d)}{d}+\mathcal{O}(x).$$

On the other hand, it follows from (6.7) that

$$\sum_{d\le x}\wedge(d)N\big(\frac{x}{d}\big)=\sum_{n\le x}\sum_{d|n}\wedge(d)=\sum_{n\le x}\log n$$

$$=x(\log x-1)+o\big(\frac{x}{\log x}\big).$$

Combining these we obtain

Lemma 6.5. *We have*

$$\sum_{d\le x}\frac{\wedge(d)}{d}=\log x+\mathcal{O}(1)$$

or

$$\sum_{p\le x}\frac{\log p}{p}=\log x+\mathcal{O}(1).$$

From Lemma 6.5 and (6.8) we get

$$\sum_{p\le x}\log\frac{x^2}{p}\log p+\sum_{pq\le x}\log p\log q=2x\log x+o(x\log x).$$

Since

$$\sum_{p\le x}\log\frac{x}{p}\log p=\int_1^x\log\frac{x}{t}d\mathcal{J}(t)=\int_1^x\frac{\mathcal{J}(t)}{t}\,dt=\mathcal{O}(x),$$

we have

$$\begin{aligned}\sum_{p\le x}\log\frac{x^2}{p}\log p &= \log x\mathcal{J}(x)+\sum_{p\le x}\log\frac{x}{p}\log p\\ &= \log x\mathcal{J}(x)+\mathcal{O}(x),\end{aligned}$$

and so

$$\log x\mathcal{J}(x)-\sum_{p\le x}\log^2 p=\mathcal{O}(x).$$

Thus we have proved the following

Theorem 6.6 (Selberg's asymptotic formula). *We have*

$$\sum_{p\le x}\log^2 p+\sum_{pq\le x}\log p\log q=2x\log x+o(x\log x)$$

or

$$\log x\mathcal{J}(x)+\sum_{p\le x}\log p\mathcal{J}\big(\frac{x}{p}\big)=2x\log x+o(x\log x).$$

6.3. Consequences of Selberg's asymptotic formula

By Lemma 6.5, we get

$$\begin{aligned}\mathfrak{S}(n) &= \sum_{pq\le n}\frac{\log p\log q}{\log pq}\\ &= \Big(\sum_{pq\le\sqrt{n}}+\sum_{\sqrt{n}<pq\le n}\Big)\frac{\log p\log q}{\log pq}\\ &= \mathcal{O}\Big(\sum_{p\le\sqrt{n}}\log p\sum_{q\le\sqrt{n}/p}1\Big)+\mathcal{O}\Big(\frac{1}{\log n}\sum_{p\le n}\log p\sum_{q\le n/p}\log q\Big)\\ &= \mathcal{O}\Big(\sqrt{n}\sum_{p\le\sqrt{n}}\frac{\log p}{p}\Big)+\mathcal{O}\Big(\frac{n}{\log n}\sum_{p\le n}\frac{\log p}{p}\Big)=\mathcal{O}(n),\end{aligned}$$

and so

$$\begin{aligned}\sum_{pq\le x}\log p\log q &= \sum_{n\le x}\big(\mathfrak{S}(n)-\mathfrak{S}(n-1)\big)\log n\\ &= \sum_{n\le x}\mathfrak{S}(n)\big(\log n-\log(n+1)\big)+\mathfrak{S}([x])\log([x]+1)\\ &= \sum_{n\le x}\mathfrak{S}(n)\log\frac{n}{n+1}+\mathfrak{S}([x])\log([x]+1)\\ &= \mathfrak{S}(x)\log x+\mathcal{O}(x).\end{aligned}$$

Hence by Theorem 6.6 we have

$$\sum_{p\le x}\log p+\sum_{pq\le x}\frac{\log p\log q}{\log pq}=2x+o(x), \tag{6.9}$$

and so

$$\sum_{pq\le x}\log p\log q=\sum_{p\le x}\log p\sum_{q\le x/p}\log q$$

$$= 2x\sum_{p\le x}\frac{\log p}{p} - \sum_{p\le x}\log p\sum_{qr\le x/p}\frac{\log q\log r}{\log qr} + \mathcal{O}\big(x\sum_{p\le x}\varepsilon\big(\frac{x}{p}\big)\frac{\log p}{p}\big)$$

$$= 2x\log x - \sum_{qr\le x}\frac{\log q\log r}{\log qr}\mathcal{J}\big(\frac{x}{qr}\big) + o(x\log x),$$

where the error term is estimated by

$$x\sum_{p\le x}\varepsilon\big(\frac{x}{p}\big)\frac{\log p}{p} = x\big(\sum_{p\le\frac{x}{\log x}} + \sum_{\frac{x}{\log x}<p\le x}\big)\varepsilon\big(\frac{x}{p}\big)\frac{\log p}{p}$$

$$= o(x\log x).$$

Thus we get

$$\log x\mathcal{J}(x) = \sum_{pq\le x}\frac{\log p\log q}{\log pq}\mathcal{J}\big(\frac{x}{pq}\big) + o(x\log x). \tag{6.10}$$

If we write $\mathcal{J}(x) = x + \rho(x)$ then by Theorem 6.6 we get

$$x\log x + \log x\cdot\rho(x) + x\sum_{p\le x}\frac{\log p}{p} + \sum_{p\le x}\log p\rho\big(\frac{x}{p}\big)$$

$$= 2x\log x + o(x\log x),$$

and so by Lemma 6.5,

$$\log x\cdot\rho(x) = -\sum_{p\le x}\log p\cdot\rho\big(\frac{x}{p}\big) + o(x\log x). \tag{6.11}$$

Set $A(n) = \sum_{pq\le n}\frac{\log p\log q}{pq}$. Then by Lemma 6.5, we obtain

$$\begin{aligned} A(x) &= \sum_{p\le x}\frac{\log p}{p}\sum_{q\le x/p}\frac{\log q}{q} \\ &= \sum_{p\le x}\frac{\log p}{p}\big(\log\frac{x}{p} + \mathcal{O}(1)\big) \\ &= \log x\sum_{p\le x}\frac{\log p}{p} - \sum_{p\le x}\frac{\log^2 p}{p} + \mathcal{O}(\log x) \\ &= \log^2 x - \sum_{p\le x}\frac{\log^2 p}{p} + \mathcal{O}(\log x). \end{aligned}$$

Set $B(n) = \sum\limits_{p \le n} \dfrac{\log p}{p}$. Then

$$\begin{aligned}
\sum_{p \le x} \frac{\log^2 p}{p} &= \sum_{n \le x} \big(B(n) - B(n-1)\big) \log n \\
&= \sum_{n \le x} B(n)\big(\log n - \log(n+1)\big) + B\big([x]\big) \log\big([x]+1\big) \\
&= -\sum_{n \le x} \frac{\log n}{n} + \log^2 x + \mathcal{O}(\log x) \\
&= \frac{1}{2} \log^2 x + \mathcal{O}(\log x),
\end{aligned}$$

and so

$$A(x) = \frac{1}{2} \log^2 x + \mathcal{O}(\log x).$$

Therefore

$$\begin{aligned}
\sum_{pq \le x} \frac{\log p \log q}{pq \log pq} &= \sum_{n \le x} \big(A(n) - A(n-1)\big) \frac{1}{\log n} \\
&= \sum_{n \le x} A(n)\Big(\frac{1}{\log n} - \frac{1}{\log(n+1)}\Big) + A([x])/\log([x]+1) \\
&= \frac{1}{2} \sum_{n \le x} \frac{\log^2 n}{n \log^2 n} + \frac{1}{2} \log x + o(\log x) \\
&= \frac{1}{2} \log x + \frac{1}{2} \log x + o(\log x) \\
&= \log x + o(\log x),
\end{aligned}$$

and by (6.10) we get

$$\begin{aligned}
x \log x + \log x \cdot \rho(x) &= x \sum_{pq \le x} \frac{\log p \log q}{pq \log pq} + \sum_{pq \le x} \frac{\log p \log q}{\log pq} \rho\Big(\frac{x}{pq}\Big) + o(x \log x) \\
&= x \log x + \sum_{pq \le x} \frac{\log p \log q}{\log pq} \rho\Big(\frac{x}{pq}\Big) + o(x \log x),
\end{aligned}$$

that is,

$$\log x \cdot \rho(x) = \sum_{pq \le x} \frac{\log p \log q}{\log pq} \rho\Big(\frac{x}{pq}\Big) + o(x \log x). \tag{6.12}$$

From (6.11) and (6.12), we get

$$2\log x\cdot|\rho(x)|\le\sum_{p\le x}\log p\cdot\big|\rho\big(\frac{x}{p}\big)\big|+\sum_{pq\le x}\frac{\log p\log q}{\log pq}\big|\rho\big(\frac{x}{pq}\big)\big|+o(x\log x).$$

If we write (6.8) by

$$s(y)=\sum_{p\le y}\log p+\sum_{pq\le y}\frac{\log p\log q}{\log pq}=2y+o(y),$$

we may rewrite the above inequality by

$$2\log x\cdot|\rho(x)|\le\int_1^x\big|\rho\big(\frac{x}{t}\big)\big|\,ds(t)+o(x\log x).$$

If $y>y'$, then it yields from (6.8) that

$$0\le\mathcal{J}(y)-\mathcal{J}(y')\le 2(y-y')+o(y),$$

and so

$$\begin{aligned}0\ \ &\le\ \ \rho(y)-\rho(y')+(y-y')\le 2(y-y')+o(y),\\ &-(y-y')\le\rho(y)-\rho(y')\le(y-y')+o(y)\end{aligned}$$

or

$$|\rho(y)-\rho(y')|\le y-y'+o(y). \tag{6.13}$$

Since $\big||a|-|b|\big|\le|a-b|$, we have by (6.13) that

$$\begin{aligned}\sum_{n\le x}n\,\big||\rho(\frac{x}{n})|-|\rho(\frac{x}{n+1})|\big|&=\mathcal{O}\big(\sum_{n\le x}n\,\big|\rho(\frac{x}{n})-\rho(\frac{x}{n+1})\big|\big)\\&=\mathcal{O}\big(x\sum_{n\le x}\frac{1}{n}\big)=\mathcal{O}(x\log x).\end{aligned}$$

Therefore

$$\begin{aligned}2\log x\cdot|\rho(x)|\ \ &\le\ \ \sum_{n\le x}\big(s(n)-s(n-1)\big)\big|\rho(\frac{x}{n})\big|+o(x\log x)\\ &=\ \ \sum_{n\le x}s(n)\big(\big|\rho(\frac{x}{n})\big|-\big|\rho(\frac{x}{n+1})\big|\big)+s([x])\rho\big(\frac{x}{[x]+1}\big)\\ &\qquad+o\big(\sum_{n\le x}n\,\big||\rho(\frac{x}{n})|-|\rho(\frac{x}{n+1})|\big|\big)+o(x\log x)\\ &=\ \ 2\sum_{n\le x}n\,\big(|\rho(\frac{x}{n})|-|\rho(\frac{x}{n+1})|\big)+o(x\log x)\\ &=\ \ 2\sum_{n\le x}|\rho(\frac{x}{n})|\big(n-(n-1)\big)+o(x\log x)\\ &=\ \ 2\int_1^x|\rho(\frac{x}{t})|\,dt+o(x\log x).\end{aligned}$$

Thus we have

Lemma 6.7. *We have*

$$\begin{aligned}|\rho(x)| &\leq \frac{1}{\log x}\int_1^x \left|\rho(\frac{x}{t})\right| dt + o(x)\\ &= \frac{x}{\log x}\int_1^x \frac{|\rho(t)|}{t^2}\, dt + o(x).\end{aligned}$$

6.4. Proof of Prime Number Theorem of the Beurling's integers

By Theorem 6.6, we may assume that $|\rho(x)| \leq \alpha x$ for $x > x_0$, where α is a constant with $0 \leq \alpha < 2$. We may assume also $\alpha > 0$. Since

$$\int_1^x \frac{\mathcal{J}(t)}{t^2}\, dt = -\int_1^x \mathcal{J}(t) d\big(\frac{1}{t}\big) = -\frac{\mathcal{J}(x)}{x} + \int_1^x \frac{d\mathcal{J}(t)}{t}$$

$$= \sum_{p\leq x} \frac{\log p}{p} + \mathcal{O}(1) = \log x + \mathcal{O}(1),$$

we have

$$\int_1^x \frac{\rho(t)}{t^2}\, dt = \int_1^x \frac{\mathcal{J}(t)}{t^2}\, dt - \int_1^x \frac{dt}{t} = \mathcal{O}(1).$$

and so

$$\left|\int_y^{ye^\lambda} \frac{\rho(t)}{t^2}\, dt\right| \leq A$$

for any $\lambda > 0$ and $y > 1$, where A is a constant. Therefore either $p(t)$ changes sign in the interval $[y, ye^\lambda]$, that is, there is a $t_0 \in [y, ye^\lambda]$ with $|p(t_0)| < \log t_0$, or $\rho(t)$ does not change sign in $[y, ye^\lambda]$, that is, we have

$$\int_y^{ye^\lambda} \frac{|\rho(t)|}{t^2}\, dt \leq A.$$

In the second case, choosing λ so large that $\lambda\alpha = 2A$, we have

$$\int_y^{ye^\lambda} \frac{|\rho(t)|}{t^2}\, dt \leq \frac{1}{2}\alpha \int_y^{ye^\lambda} \frac{dt}{t}.$$

In the first case, taking $t \in J_{t_0} = [t_0, t_0 + \frac{\alpha}{3}t_0]$ (or $J_{t_0} = [t_0 - \frac{\alpha}{3}t_0, t_0]$), we see by (6.13) that

$$\begin{aligned}|\rho(t)| &\leq |\rho(t_0)| + |t - t_0| + \varepsilon t\\ &\leq \log t_0 + |t - t_0| + \varepsilon t < \frac{\alpha}{2}t.\end{aligned}$$

Dividing $[1, x]$ as the union of $[1, x_0]$ and $I_\nu = [xe^{-\nu\lambda},\ xe^{-(\nu-1)\lambda}]$, where $1 \leq \nu \leq \frac{1}{\lambda}\log\frac{x}{x_0}$, we obtain from Lemma 6.7 that

$$|\rho(x)| \leq \frac{x}{\log x}\Big(\int_1^{x_0}\frac{|\rho(t)|}{t^2}\,dt + \sum_\nu \int_{I_\nu}\frac{|\rho(t)|}{t^2}\,dt\Big)$$
$$= \frac{x}{\log x}\Big(C_{x_0} + \sum_\nu \frac{\alpha}{2}\int_{J_{t_0}}\frac{dt}{t} + \sum_\nu \alpha\int_\sigma \frac{dt}{t}\Big),$$

where σ is the complementary set of $\bigcup_\nu J_{t_0}$ with respect to $[x_0, x]$ and C_{x_0} is a constant depending on x_0 but not the same in different occurrences. Therefore

$$\begin{aligned}
|\rho(x)| &\leq \frac{x}{\log x}\Big(C_{x_0} - \sum_\nu \frac{\alpha}{2}\int_{J_{t_0}}\frac{dt}{t} + \alpha\int_{x_0}^{x}\frac{dt}{t}\Big)\\
&\leq \frac{x}{\log x}\Big(C_{x_0} - \frac{\alpha}{2}\sum_\nu \log\frac{t_0 + \frac{\alpha}{3}t_0}{t_0} + \alpha\log\frac{x}{x_0}\Big)\\
&\leq \frac{x}{\log x}\Big(C_{x_0} - \frac{\alpha}{2}\sum_\nu \log\big(1 + \frac{\alpha}{3}\big) + \alpha\log x\Big)\\
&\leq \frac{x}{\log x_0}\Big(C_{x_0} + \big(1 - \frac{\alpha}{8\lambda}\big)\alpha\log x\Big)\\
&\leq \big(1 - \frac{\alpha}{10\lambda}\big)\alpha x = \alpha_1 x
\end{aligned}$$

for $x > x_1$, where

$$\alpha_1 = \big(1 - \frac{\alpha}{10\lambda}\big)\alpha.$$

Continuing the above steps, we get

$$|\rho(x)| < \alpha_n x$$

for $x > x_n$, where

$$\alpha_n = \big(1 - \frac{\alpha_{n-1}}{10\lambda}\big)\alpha_{n-1},\ \alpha_0 = \alpha,\ n = 1, 2, \cdots. \tag{6.14}$$

Since $0 < \alpha < 2$, $\{\alpha_n\}$ is a positive decreasing sequence, so that it has a limit a. Let $n \to \infty$. Then it follows from (6.14) that

$$a = \big(1 - \frac{a}{10\lambda}\big)a,$$

and thus

$$a = 0.$$

We have proved the following

Theorem 6.8. *We have*

$$\mathcal{J}(x) \sim x.$$

Remark. Since the Riemann zeta-function $\zeta(s)$ and the estimation for its zero-free region haven't used in the proof of Theorem 6.8, the method in Lectures II and III cannot be applied to the problem for estimation of $\pi(x)$ or $\mathcal{J}(x)$ of the Beurling integers.

The Prime Number Theorem

—— a historical overview of the first hundred years

Antiquity: Euclid established that there exist infinitely many prime numbers in his "Elements". Eratosthenes proposed a so-called "sieve" for finding out all prime numbers in a given interval $1 < n \leq N$.

L. Euler notes that for $s > 1$,

$$\prod_p (1 - p^{-s})^{-1} = \sum_n n^{-s}, \tag{0.1}$$

where p runs over all prime numbers and n all positive integers. Since the series is the right hand side of $(0,1)$ is divergent as $s \to \iota+$ the series

$$\sum \frac{1}{p}$$

divergent, and consequently, there are infinitely many prime numbers.

C.F. Gauss in 1792 or 1793 empirically arrives at

$$\pi(x) \sim \int_2^x \frac{dt}{\log t} = \ell i\, x,$$

where $\pi(x)$ denotes the number of primes not exceeding x. He continues throughout his life as new more extensive prime number tables appears. (Correspondence with Bessel in 1810, and letter to Encke in 1849).

A.M. Legendre in his "Essai sur la theorie des nombres" 1st edition 1798 states that $\pi(x)$ probably can be approximated by an expression

$$\frac{x}{A \log x + B},$$

where A and B are constants. In the second edition 1808 he gives the formula

$$\pi(x) \sim \frac{x}{\log x - 1.08366}.$$

In a letter to Holmboe in 1824, N.H. Abel states about the distribution of prime numbers in an arithmetic progression, and P.G.L. Dirichlet in 1939 proved that there are infinitely many prime numbers in the arithmetic progression $kx + \ell$, where $(k, \ell) = 1$.

On May 24, 1848, P.L. Chebyshev read a paper before the St. Petersburg Academy where he proved:

If a very good simple approximation function to $\pi(x)$ exists, it has to be $\ell i\, x$. More precisely he showed

$$\sum_p \frac{\log^k p}{p^s} - \sum_n \frac{\log^{k-1} n}{n^s} = O(1) \tag{0.2}$$

as $s \to 1+$, also

$$\int_2^\infty \frac{\pi(x) - \ell i\, x}{x^{s+1}} \cdot \log^k x\, dx = O(1) \tag{0.3}$$

as $s \to 1+$.

From this he concludes: for any given $\alpha > 0$ and N, we have

$$|\pi(x) - \ell i\, x| < \frac{\alpha x}{\log^N x}, \tag{0.4}$$

for a sequence of x that tends to ∞.

Chebyshev first to utilize

$$\zeta(s) = \sum_n n^{-s} = \prod_p (1 - p^{-s})^{-1}$$

for real $s > 1$ in this context. His proof depends on the identity

$$\int_2^\infty \frac{\pi(x) - \ell i\, x}{x^{s+1}}\, dx = \frac{1}{s} \sum_p p^{-s} - \frac{1}{s} \int_2^\infty t^{-s} \frac{dt}{\log t}. \tag{0.5}$$

In fact, the left hand side of (0.5) is equal to

$$-\frac{1}{s} \cdot \frac{\pi(x) - \ell i\, x}{x^s} \Big|_2^\infty + \frac{1}{s} \int_2^\infty \frac{d(\pi(x) - \ell i\, x)}{x^s}$$
$$= \frac{1}{s} \sum_p p^{-s} - \frac{1}{s} \int_2^\infty x^{-s} \frac{dx}{\log x}.$$

We have also

$$\begin{aligned}
\frac{1}{s}\log\zeta(s) &= -\frac{1}{s}\sum_p \log(1-p^{-s}) \\
&= \frac{1}{s}\sum_p p^{-s} + \frac{1}{s}\sum_{m=2}^{\infty}\sum_p \frac{1}{m}p^{-ms} \\
&= \frac{1}{s}\sum_p p^{-s} + g_1(s), \qquad (0.6)
\end{aligned}$$

where $g_1(s)$ is regular at $s = 1$. Set

$$F(s) = \frac{1}{s}\sum_{n=2}^{\infty}\frac{1}{n^2\log n}.$$

Then

$$\begin{aligned}
&\left|F(s) - \frac{1}{s}\int_2^{\infty} x^{-s}\frac{dx}{\log x}\right| \\
&\le \frac{1}{s}\sum_{n=2}^{\infty}\int_n^{n+1}\left|\frac{1}{n^s\log n} - \frac{1}{x^s\log x}\right| dx \qquad (0.7) \\
&\le \frac{1}{s}\sum_{n=2}^{\infty}\left(\frac{1}{n^s\log n} - \frac{1}{(n+1)^s\log(n+1)}\right) = \frac{1}{s2^s\log 2},
\end{aligned}$$

and we have

$$\frac{1}{s}\int_2^{\infty} x^{-s}\frac{dx}{\log x} = F(s) + g_2(s), \qquad (0.8)$$

where $g_2(s)$ is regular at $s = 1$. Since

$$\frac{dsF(s)}{ds} = -\zeta(s),$$

we have

$$F(s) = -\frac{1}{s}\log(s-1) + g_3(s), \qquad (0.9)$$

where $g_3(s)$ is regular at $s = 1$. Substituting (0.6), (0.7) and (0.8) into (0.5), we have

$$\int_2^{\infty}\frac{\pi(x) - \ell i\, x}{x^{s+1}}\, dx = \frac{1}{s}\log\{(s-1)\zeta(s)\} + g(s), \qquad (0.10)$$

where $g(s)$ is regular at $s=1$. Differentiating k times with respect to s in (0.5) and (0.9) and using $F(s)$ instead of $\frac{1}{s}\int_2^\infty t^{-s}\frac{dt}{\log t}$, we have (0.2) and (0.3). Now we proceed to prove (0.4). If there is a sequence x_n that tends to ∞ such that $\pi(x) - \ell i\, x$ change signs at x_n, that is; $\pi(x_n) - \ell i\, x_n = 0$, then (0.4) is obviously true. Otherwise suppose that $\pi(x) - \ell i\, x$ is always > 0 (or < 0) for sufficiently large x. Then from (0.3), we have

$$\int_2^\infty \frac{|\pi(x) - \ell i\, x|}{x^{s+1}} \log^k x\, dx = O(1). \tag{0.11}$$

If (0.4) does not hold, that is;

$$|\pi(x) - \ell i\, x| \geq \frac{\alpha x}{\log^N x} \qquad (x > x_0),$$

then we take $k = N$, and by (0.10), we deduce that

$$\int_{x_0}^\infty \frac{|\pi(x) - \ell i\, x|}{x^{s+1}} \log^k x\, dx \geq \alpha \int_{x_0}^\infty \frac{1}{x^s} dx \to \infty$$

as $s \to \iota +$ which leads to a contradiction with (0.10) and (0.4) follows.

In a second paper presented in 1850, Chebyshev obtains the first good bounds for $\pi(x)$.

Writing: $\mathcal{J}(x) = \sum\limits_{p \leq x} \log p$, $\Psi(x) = \sum\limits_{p^r \leq x} \log p$, so that

$$\Psi(x) = \mathcal{J}(x) + \mathcal{J}(x^{1/2}) + \cdots + \mathcal{J}(x^{\frac{1}{n}}) + \cdots,$$

Chebyshev considered

$$\begin{aligned} T(x) &= \sum_m \Psi\left(\frac{x}{m}\right) = \sum_m \sum_{p^r \leq \frac{x}{m}} \log p \\ &= \sum_{p^r m \leq x} \log p = \sum_{m' \leq x} \sum_{p^r | m'} \log p \\ &= \sum_{m' \leq x} \log m' = \log([x]!) \\ &= x(\log x - 1) + O(\log x), \end{aligned} \tag{0.12}$$

where $[x]$ denotes the integral part of x. He formed the linear combination.

$$\begin{aligned} U(x) &= T(x) - T\left(\frac{x}{2}\right) - T\left(\frac{x}{3}\right) - T\left(\frac{x}{5}\right) + T\left(\frac{x}{30}\right) \\ &= Ax + O(\log x), \end{aligned}$$

where

$$A = \frac{\log 2}{2} + \frac{\log 3}{3} + \frac{\log 5}{5} - \frac{\log 30}{30}$$
$$= 0.92129202294....$$

Inserting the expression for T by Ψ one gets

$$\begin{aligned} U(x) &= \Psi(x) - \Psi\left(\frac{x}{6}\right) + \Psi\left(\frac{x}{7}\right) - \Psi\left(\frac{x}{10}\right) \\ &+\Psi\left(\frac{x}{11}\right) - \Psi\left(\frac{x}{12}\right) + \Psi\left(\frac{x}{13}\right) - \Psi\left(\frac{x}{15}\right) + \Psi\left(\frac{x}{17}\right) \\ &-\Psi\left(\frac{x}{18}\right) + \Psi\left(\frac{x}{19}\right) - \Psi\left(\frac{x}{20}\right) + \Psi\left(\frac{x}{23}\right) - \Psi\left(\frac{x}{24}\right) \\ &+\Psi\left(\frac{x}{29}\right) - \Psi\left(\frac{x}{30}\right) + \Psi\left(\frac{x}{31}\right) - \cdots \\ &= \sum_{n\le x} \Sigma_n \Psi\left(\frac{x}{n}\right), \qquad (0.13) \end{aligned}$$

where

$$\Sigma_n = \begin{cases} 1, & \text{if } (n, 3O) = 1; \\ O, & \text{if there is only one of 2, 3, 5 dividing } n; \\ -1, & \text{if there are at least two of 2, 3, 5 dividing } n; \\ \Sigma_{n'}, & \text{if } n \equiv n' \pmod{30}. \end{cases}$$

If follows from (0.12) that $U(x)$ is an alternating sum and he absolute values of the terms are monotonic decreasing, so we see that

$$\Psi(x) - \Psi\left(\frac{x}{6}\right) < U(x) < \Psi(x),$$

from which

$$\begin{aligned} U(x) &< \Psi(x) < U(x) + \Psi\left(\frac{x}{6}\right) \\ &< U(x) + U\left(\frac{x}{6}\right) + \Psi\left(\frac{x}{6^2}\right) \\ &< \cdots < U(x) + U\left(\frac{x}{6}\right) + \cdots + U\left(\frac{x}{6^n}\right) + \cdots \end{aligned}$$

and so:

$$Ax - O(\log x) < \Psi(x) < \frac{6}{5}Ax + O(\log^2 x) = A'x + O(\log^2 x),$$

where $A' = 1.1055504275$. Since

$$\mathcal{J}(x) = \Psi(x) + \mathcal{O}(\sqrt{x}),$$

and

$$\begin{aligned} \pi(x) &= \int_2^x \frac{d\mathcal{J}(t)}{\log t} \\ &= \frac{\mathcal{J}(x)}{\log x} + \int_2^x \frac{\mathcal{J}(t)dt}{t\log^2 t}, \end{aligned}$$

where

$$\int_2^x \frac{\mathcal{J}(t)dt}{t\log^2 t} = \mathcal{O}\left(\frac{x}{\log^2 x}\right),$$

we have similar bounds for $\mathcal{J}(x)$ and $\pi(x)$.

J.J. Sylvester gives improvements on the bounds A and A' in 1891 and 1892, and H. Poincare obtains analogue for "Gaussian integers" in 1891.

G.F.B. Riemann's note to Prussian Academy of science in Berlin (of which he had just been elected a corresponding member) in 1859 finally brings is $\zeta(s)$ as a function of a complex variable.

The motivation is inversion of the relation

$$\frac{1}{s}\log\zeta(s) = \int_2^\infty \frac{f(x)}{x^{s+1}}\,dx, \tag{0.14}$$

where

$$f(x) = \pi(x) + \frac{1}{2}\pi(x^{1/2}) + \cdots + \frac{1}{n}\pi(x^{\frac{1}{n}}) + \cdots .$$

In fact, if $s > 1$, the right hand side of (0.13) is equal to

$$-\frac{1}{s}\cdot\frac{f(s)}{x^s}\int_2^\infty + \frac{1}{s}\int_2^\infty x^{-x}d\,f(x)$$

$$= \frac{1}{s}\left(\sum_p p^{-s} + \frac{1}{2}\sum_p p^{-2s} + \cdots\right) = \frac{1}{s}\log\zeta(s).$$

which is essence is already present in Chebyshev's work. (see (0.9)).

Considering the right-hand expression of (0.13) as a Fourier integral (writing $x = e^u$; $s = a + it$) he finds,

$$f(x) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{x^s}{s} \log \zeta(s) ds$$

for $a > 1$. (This also follows from Mellin's invention formula) Riemann writes $s = \frac{1}{z} + it$ (where t may be complex) and

$$\xi(t) = \frac{1}{2} s(s-1) \Gamma\left(\frac{s}{2}\right) \pi^{-s/2} \zeta(s),$$

and shows that ξ is an integral function of t^2, all of whose zeros have their imaginary parts between $-i/2$ and $i/2$. From growth considerations he concludes that

$$\xi(t) = \xi(0) \prod_{\alpha} \left(1 - \frac{t^2}{\alpha^2}\right),$$

where α runs through the zeros of ξ, with position real part. He states that

$$N(T) = \frac{T}{2\pi}\left(\log \frac{T}{2\pi} - 1\right) + \mathcal{O}(\log T)$$

if $N(T)$ denotes the number of zeros with real part in the interval $(0, T)$, and that there seems to be about that many real zeros there, so he conjectures that all zeros of $\xi(t)$ are real (or all non-trivial zeros of $\zeta(s)$ on the line $\sigma = 1/2$).

Using a rather reckless procedure of integrating term wise (after having expressed $\log \zeta(s)$ in terms of $\log \zeta\left(\frac{s-1/2}{i}\right)$ and simple terms, and integrating first by parts) he arrives at the formula

$$\begin{aligned} f(x) &= \ell i\, x - \sum_{\alpha} \left(\ell i(x^{\frac{1}{2}+\alpha i}) + \ell i(x^{\frac{1}{2}-\alpha i})\right) \\ &\quad + \int_x^{\infty} \frac{1}{t^2-1} \cdot \frac{dt}{t \log t} - \log 2. \end{aligned}$$

It is clearly a preliminary note, and might not have been written if L. Kronecher had not urged him to write up something about this work (see letter to Weierstrass, Oct. 26, 1859). It is clear there are holes that need to be filled in, but also clear that he had a lot more material than is in the note.

What also seems clear: Riemann is not interested in an asymptotic formula, not in the prime number theorem, what he is after is an exact formula!

In his introduction Riemann mentions Gauss and Dirichlet it is known (letter from Schwalfuss) that he had read Legendre. He had undoubtedly also seen the work of Chebyshev which had been published in French.

It is quite possible that is was Chebyshev's first paper referred to earlier, which inspired him to consider the zeta function. I am convinced that Riemann knew that $\zeta(s)$ has no zeros on the line $\sigma = 1$. If there were one it would have to be a simple zero since

$$\begin{aligned}\zeta(\sigma)\big|\zeta(\sigma+it)\big| &= \prod_p (1-p^{-\sigma})^{-1}\left|\prod_p \left(1-\frac{1}{p^{\sigma+it}}\right)^{-1}\right| \\ &\geq \prod_p (1-p^{-\sigma})^{-1}(1+p^{-\sigma})^{-1} = \prod_p (1-p^{-2\sigma})^{-1} \\ &> 1\end{aligned}$$

for $\sigma > 1$. If there were one say $1+it_0$, one gets by looking at the k^{th} derivative of

$$\frac{\zeta'}{\zeta}(s) + \frac{\zeta'}{\zeta}(s+it_0)$$

as $s \to 1+$, and taking the real part that

$$\sum_p \frac{\log^{k+1} p}{p^\sigma}(1+\cos(t^0 \log p)) = \mathcal{O}(1),$$

at $\sigma + 1+$ (This is because the principal points of $\frac{\zeta'}{\zeta}(s)$ and $\frac{\zeta'}{\zeta}(s+it_0)$ at $s=1$ cancel each other). This means that

$$\sum_p \frac{\log^{k+1} p}{p} \cos^2\left(\frac{t_0 \log p}{2}\right) < \infty.$$

Since $\sum_p \frac{1}{p}$ diverges, this leads to a contradiction. Had Riemann's goal been the prime number theorem, he would probably have considered $\Psi(x)$ instead of his $f(x)$, and used a smoothened expression like $\int_2^x \Psi(t)dt$ or $\int_2^x \frac{\Psi(t)}{t}dt$, leading to factors $\frac{x^{s+1}}{s(s+1)}$ or $\frac{x^s}{s^2}$ in his integrals instead of $\frac{x^s}{s}$. It is very likely that he would have succeeded had he tried.

Some asymptotic relations involving primes were established in the following decades by F. Mertens who in 1874 proved

$$\sum_{p<x} \frac{\log p}{p} = \log x + \mathcal{O}(1),$$

and

$$\sum_{p<x}\frac{1}{p} = \log\log x + c + \mathcal{O}\left(\frac{1}{\log x}\right).$$

Mertens also conjectured based on empiric evidence that

$$\left|\sum_{n<x}\mu(n)\right|\sqrt{x}.$$

Mertens first formula probably was known to Chebyshev since

$$T(x) = \sum_{p^r<x}\log p\left[\frac{x}{p^r}\right] = x\log x + \mathcal{O}(x)$$

by (0.11), and

$$\begin{aligned} T(x) &= x\sum_{p<x}\frac{\log p}{p} + x\sum_{\substack{p^r<x\\ r\geq 2}}\frac{\log p}{p^r} + \mathcal{O}(\Psi(x)) \\ &= x\sum_{p<x}\frac{\log p}{p} + \mathcal{O}(x). \end{aligned}$$

T.J. Stieltjes in two C.R. notes 185 claimed to have shown that the series

$$\sum_{n}\frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}$$

is convergent for $\sigma > 1/2$, (which would clearly imply Riemann's statement about the zeros of $\zeta(s)$ being on the line $\sigma = 1/2$); from this he concluded

$$\Psi(x) = x + \mathcal{O}\left(x^{\frac{3}{4}+\varepsilon}\right)$$

for any $\varepsilon > 0$.

G. Halphen in a C.R. note from 1883 states that $\mathcal{J}(x) \sim x$ as $x \to \infty$. By Some French authors this is later referred to as: la loi asymptotiques d' Halphen! (it surely was conjectured by Chebyshev if not earlier!)

1893 E. Cahen claims to prove $\mathcal{J}(x) \sim x$ "Halphen's law" assuming the Riemann Hypothesis (as "proved" by Stieltjes).

Substantial progress was made when J. Hadamard in 1892 is connection with his work on entire functions proved rigorously Riemann's assertion

$$\xi(t) = \xi(0)\prod_{\alpha}\left(1 - \frac{t^2}{\alpha^2}\right),$$

he also showed

$$aT\log T < N(T) < AT\log T,$$

with positive constants a and A for $T > 15$.

Finally in 1896 Hadamard rigorously proved $\mathcal{J}(x) \sim x$, "Halphen's law" (from which the prime number theorem follows, but he does not mention this at all!) He based his proof on the formula

$$\sum_{p^r<x} \log p \log^{\mu-1}\frac{x}{p^r} = -\frac{\Gamma(\mu)}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{x^s}{s^\mu}\frac{\zeta'}{\zeta}(s)\,ds \tag{0.15}$$

for integer $\mu > 1$. Using his results from 1892, and that $\zeta(1+it) \neq 0$ for t real, he proves that the left hand side of (0.14) is asymptotic to $\Gamma(\mu)x$, as $x \to \infty$. Taking $\mu = 2$, he gets $\mathcal{J}(x) \sim x$ by a difference argument.

In fact, the left hand side of (0.14) for $\mu = 2$ is equal to

$$\begin{aligned}
&\sum_{p<x}\log p\log\frac{x}{p} + \mathcal{O}\left(\sum_{\substack{p^r<x\\ r\geq 2}}\log p\log\frac{x}{p}\right)\\
&= \int_2^x \log\frac{x}{t}d\mathcal{J}(t) + \mathcal{O}(\sqrt{x}\log x)\\
&= \int_2^x \frac{\mathcal{J}(t)}{t}dt + \mathcal{O}(\sqrt{x}\log^2 x).
\end{aligned}$$

For any $\varepsilon > 0$, we have

$$\mathcal{J}(x)\int_x^{(1+\varepsilon)x}\frac{dt}{t} \leq \int_x^{(1+\varepsilon)x}\frac{\mathcal{J}(t)}{t}dt \leq \mathcal{J}((1+\varepsilon)x)\int_x^{(1+\varepsilon)x}\frac{dt}{t},$$

and so

$$\frac{\mathcal{J}(x)\varepsilon x}{(1+\varepsilon)x} \leq \int_x^{(1+\varepsilon)x}\frac{\mathcal{J}(t)}{t}\,dt \leq \mathcal{J}((1+\varepsilon)x)\frac{\varepsilon x}{x}.$$

Since the left hand side of (0.14) is asymptotic to x, we have

$$\int_x^{(1+\varepsilon)x}\frac{\mathcal{J}(t)}{t}\,dt \sim \varepsilon x$$

and so

$$\frac{\mathcal{J}(x)}{(1+\varepsilon)x} \leq 1 \leq \mathcal{J}((+\varepsilon)x)/x.$$

Since ε is arbitrary, we have $\mathcal{J}(x) \sim x$. Hadamard also sketches a proof of the analogous result for an arithmetic progression.

As for $\zeta(1+it) \neq 0$, his proof is as follows. If $\zeta(1+it_0) = 0$, then $1+it_0$ is a simple zero as shown above. We have

$$\log \zeta(s) = \sum_p \frac{1}{p^s} + f(s)$$

for $\sigma > 1$, where $f(s)$ is analytic for $\mathrm{Re}\, s \geq \frac{1}{2}$. Taking real part, we have

$$\log |\zeta(\sigma + it_0)| = \sum_p \frac{1}{p^\sigma} \cos(t_0 \log p) + \mathfrak{Re}\, f(s) \sim \log(\sigma - 1)$$

as $\sigma \to 1+$, that is;

$$\sum_p \frac{1}{p^\sigma} \cos(t_0 \log p) \sim \log(\sigma - 1)$$

as $\sigma \to 1+$. Roughly speaking, $\cos(t_0 \log p)$ should be close to -1 for a large set of primes, and so $\cos(2t_0 \log p)$ should be close to 1 for a large set of prime numbers, that is;

$$\sum_p \frac{1}{p^\sigma} \cos(2t_0 \log p) \sim -\log(\sigma - 1)$$

as $\sigma \to 1+$. This means that $1 + 2it_0$ is a simple pole of $\zeta(s)$ which is impossible. Hence $\zeta(1+it_0) \neq 0$.

The same year de la Vallee Poussian independently, but building on Hadamard's 1892 paper, gives a proof along somewhat similar lines. He does state the prime number theorem in his paper! His paper treats not only the case of the arithmetic progression but also that of a binary quadratic form.

de la Vallee Poussin's proof of $\zeta(1+it) \neq 0$ is based on the inequality

$$3 + 4\cos\varphi + \cos 2\varphi = 2(1+\cos\varphi)^2 \geq 0.$$

So we have

$$\begin{aligned} &3\log 3(\sigma) + 4\log|\zeta(\sigma+it)| + \log|\zeta(\sigma+2it)| \\ &= \sum_p \sum_{m=1}^{\infty} \frac{1}{mp^{ms}} \big\{3 + 4\cos(mt\log p) + \cos(2mt\log p)\big\} \geq 0, \end{aligned}$$

and

$$|\zeta(\sigma)|^3 \; |\zeta(\sigma+it)|^4 \; |\zeta(\sigma+2it)| \geq 1 \tag{0.16}$$

for $\sigma > 1$. if $\zeta(1+it_0) = 0$, then we take $t = t_0$ in (0.15). Since $\zeta(s)^3$ has a pole of degree 3 at $s = 1$, $\zeta(\sigma + it_0)$ has a zero of degree 4 at $1 + it_0$, and $\zeta(s)$ is analytic at $1 + 2it_0$. The left hand side of (0.15) tends to zero as $\sigma \to -\iota+$ which leas to a contradiction. Hence $\zeta(1 + it_0) \neq 0$. A few years later he develops this idea, now applied to the logarithmic derivative as

$$\mathfrak{Re}\Big\{ -3\frac{\zeta'}{\zeta}(\sigma) - 4\frac{\zeta'}{\zeta}(\sigma + it) - \frac{\zeta'}{\zeta}(\sigma + 2it) \Big\} \geq 0,$$

into an argument that shows $\zeta(s) \neq 0$ for

$$\sigma > 1 - \frac{a}{\log|t|}, \quad |t| > A,$$

where a and A are certain positive constants. From this he concludes

$$\pi(x) = \ell i\, x + \mathcal{O}\big(xe^{-\alpha\sqrt{\log x}}\big)$$

for some constant $\alpha > 0$.

Later progress by J.E. Littlewood and I. Vinogradov and others in the direction of improving the remainder term is entirely based on improving estimates for certain exponential sums. Apart from that it is still de la Vallee Poussin's argument that is used. This can in principle never give us more than a zero free region which lies close to $\sigma = 1$, whose width tends to zero as $|t| \to \infty$.