**Notes for a lecture at the Mostow conference, October 2013.**

Robert Langlands

**Three parts of the theory of automorphic forms.** I am inclined to divide the modern theory of automorphic forms into three parts:
(i) the theory over a number field, $F$, thus for functions on $G(F)\backslash G(\mathbb{A}_F)$;
(ii) the theory for functions on $G(F)\backslash G(\mathbb{A})$, where $F$ is the field of rational functions on a Riemann surface, thus closed nonsingular curve, $X$;
(iii) the relation between automorphic forms in the sense of (i) and diophantine equations in the sense of Grothendieck's conjectural notion of motives. This I refer to as reciprocity. It is a difficult subject, hardly broached, although Wiles's proof of the Shimura-Taniyama conjecture is a major contribution to it.

The first part is well-developed but far from complete. The major problem may be taken to be **functoriality**. This part may also be taken to include the theory for function fields over a finite field, which of course is not unrelated to (ii). This first part was examined in "A prologue to Functoriality and Reciprocity," which appeared in a volume dedicated to Rogawski. The major work remains to be done.

The theory of automorphic forms, in the sense of (i), is the study of functions on the quotient $G(F)\backslash G(\mathbb{A})$, and of the action of $G(\mathbb{A})$ on them. Of central importance is the spectral decomposition of $L^2(G(F)\backslash G(\mathbb{A}))$ with respect to this action and the examination of the irreducible representations of $G(\mathbb{A})$ that appear. The principal notion of this analytic theory is "functoriality" which is by no means simply a formal property and has not yet been established in any generality. It appears, certainly to me, and, I believe, to some of the major practitioners of the subject that the proof of functoriality will demand a combination of the trace formula, as introduced by Selberg and developed as a fully general theory by Arthur, and methods from the analytic theory of $L$-functions as used, for example, in the prime number theorem.

The notion of functoriality in this context, although not the designation, was introduced in my letter to Weil. It demands associating to the group $G$ a second group ${}^LG$, the $L$-group no longer a group over a number field, but just a group over $\mathbb{C}$. It is also best to develop a theory of endoscopy, which reduces the general theory to the theory for quasi-split groups. Endoscopy itself is a fascinating topic, pursued by a relatively small number of mathematicians and not well-developed, even though it recently acquired considerable renown thanks to Ngô's proof of the *fundamental lemma*, fundamental to endoscopy. Many mathematicians will have heard of the fundamental lemma, although very few will have inquired what endoscopy is. Ali Altuğ in his recent thesis at Princeton is one of the first to prepare the way for the use of methods from analytic number theory in the context of the trace formula and functoriality.

The essence of functoriality, which appears not only in a global form but also in an adjunct local form, is that it associates to an automorphic representation of a group $H$ and a (holomorphic) homomorphism of ${}^LH$ to the $L$-group ${}^LG$ an automorphic representation of $G$. The theories, both local and global, exist at best in a nascent form. I am persuaded that the methods described in my Rogawski paper will eventually yield them, but that will be a major achievement, indeed, several major achievements.

I add that functoriality in the above sense over a number field will contain, if it is ever demonstrated, a non-abelian class field theory. The initial hope to create such a theory was abandoned by Artin, pretty much explicitly, at the Princeton University bi-centennial conference in 1956. It was renewed, I believe, in my letter to Weil in 1967.

The geometric theory is much different from the arithmetic theory, but it appears that functoriality is an important common element, with a different but similar content and, no doubt, a different proof. I have spent a good deal of time reflecting on the geometric theory, especially as presented in the expositions of E. Frenkel, but I am still dissatisfied with my grasp of it and with the state of the present understanding of the theory. My understanding or the understanding of others aside, the topic is a fascinating mixture of the theory of systems of ordinary differential equations, classical mechanics, and both differential and algebraic geometry. Unfortunately, my reflections have not reached the maturity of those on the arithmetic theory that appear in the Rogawski paper. I add that reciprocity in the sense of (iii) — which promises to be deeper and more difficult than (i) and (ii) and which was not a topic of that paper and is not one in the present lecture — does not, so far as I can see, appear in the geometric theory.

There is, on the other hand, an entirely different issue in the geometric theory, raised ten years after my letter to Weil, and so far as I know independently of it, not by me but by Montonen and Olive, apparently influenced by Goddard, Nuyts, and Olive. It may be best to refer to it as *duality*. For physicists and for geometers who interest themselves in these matters, it may be the principal issue. In my letter to Weil and in all my later reflections, the groups ${}^LG$ and $G$ play quite different roles. The algebraic group $G$ is defined over a number field or some variant thereof, such as a $p$-adic field; the group ${}^LG$ is defined over $\mathbb{C}$ and there is no question of changing their roles either in functoriality, not even in the geometric theory over an algebraic curve, or in reciprocity, which was only proposed in the arithmetic context and which, indeed, has, so far as I understand, no meaning in a geometric theory. In the problems posed by Montonen and Olive, $G$ and ${}^LG$ play, I believe, symmetric roles. That is, indeed, the point of their suggestions. So it is somewhat ironic that the phrase *Langlands program* has come to represent not only the problems posed by me but also those posed by these physicists. Indeed, for most mathematicians and perhaps for all physicists — in so far as such questions are of any concern to them at all — it refers principally and, so far as I know, referred initially to the dual physical role of $G$ and ${}^LG$ and not to functoriality and reciprocity. My intention was to discuss both in the Prologue, and I still hope to do so. My goal over the past few months has been more restrictive: to see to what extent I can understand geometric functoriality.

I had best make it absolutely clear that, although I have made some progress, I have a long way to go. This lecture is an account, with few pretensions, of my first attempts to understand the lay of the land. As I have several occasions in the lecture to observe, the geometric theory is, or will probably be, a blending of techniques and notions from a number of domains with which, in spite of my years as a mathematician, I am not very familiar. The lecture meanders and, the structure of the theory sought being so complex and so little understood, returns more than once to the same questions.

My starting point is the space $G(F)\backslash G(\mathbb{A})$ of the geometric theory. If $F$ is the field of rational functions on $X$, let $F_x$ be the local form of $F$, thus formal Laurent series, at the point $x$ and $\mathcal{O}_x$ the ring of formal power series at the same point. The affine grassmannian $\mathrm{gr}_x$ at $x$ is the quotient $G(F_x)/G(\mathcal{O}_x)$. It is an infinite-dimensional algebraic variety, that is, an injective limit of finite-dimensional varieties. For example, for $GL(2)$, it is the limit of the union over the spaces $D^{m,n}/G(\mathcal{O}_x)$, $M \geq m \geq n$, $M$ fixed but growing, where $D_{m,n}$ is the space of $2 \times 2$ matrices with entries from $F_x$ and elementary divisors $x^{-m}, x^{-n}$. For a general $G$, the pair $(m, n)$ is replaced by an element $\delta$ in the closed positive cone in the lattice of coweights of a Cartan subalgebra $T$ and $D^{m,n}$ by $D^\delta$. Let $\Delta^\delta$ be the union of $D^{\delta'}$, $\delta' \leq \delta$ These varieties may have singularities. If we are only considering unramified automorphic forms, we take functions on the restricted direct product

$$\prod_x G(F_x)/G(\mathcal{O}_x) = \prod_x \mathrm{gr}_x,$$

or, better, on

$$\prod_x \Delta^\tau(F_x)/G(\mathcal{O}_x) = \prod_x \mathrm{gr}_x^\tau,$$

where $\tau$ is an element in the closed positive cone. If we were to consider ramified forms, we would replace a finite number of the groups $G(\mathcal{O}_x)$ by congruence subgroups, but that is hardly necessary or appropriate here.

So the space to study is the quotient space

$$G(F)\backslash \prod_x G(F_x)/G(\mathcal{O}_x). \tag{A}$$

Since $G(F)$ is itself an infinite-dimensional group, this could be a disagreeable object. It is, effectively, a stack, but I am not a fan of stacks. Being principally an analyst, I prefer sets. More importantly, I find that the better I understand the notion of stacks the less useful for my purposes it seems. Nevertheless, the difficulties they are used to express and the insights incorporated in their definition appear to be real and important. The quotient (A) is, apparently, not so fearsome as it appears. It is a finite-dimensional object, although it may have an infinite number of components.

To understand why it is finite-dimensional, we take $X$ to be the projective line $\mathbb{P}^1$, and recall Grothedieck's proof of a theorem to the effect that, for this particular curve, (A) is discrete. The theorem is largely an application of the Riemann-Roch formula. In the simple but typical case of $GL(2)$, he would first prove that every element of the quotient (A) is represented by a product $n(u)t$, with

$$\begin{aligned} &t = \mathrm{diag}(a, b), \quad a = z_\infty^{-\mu}, b = z_\infty^{-\nu}, \quad \mu, \nu \in \mathbb{Z}, \mu \geq \nu. \\ &u \in \mathbb{A}_F, \quad n(u) = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}. \end{aligned} \tag{B}$$

The element $z_\infty \in F_\infty$ is a local coordinate at $\infty$. Then he would observe that there is an element $v \in F$ such that $v - u$ is integral except at infinity. Since $n(v) \in G(F)$, he can replace $u$ by $v$ and then remove $n(v)$. So the elements of (A) are represented by the discrete set of $t$ in (B), in spite of its initial appearance as a union of sets of various dimensions or perhaps codimensions. In particular, the set $\mathrm{gr}_\infty$ is the union of the orbits of the elements $t$ in (B) under the group $G(R)$ of matrices with polynomial entries and determinants in $\mathbb{C}^\times$.

My understanding of the construction for general curves is still limited. There are diverse expositions of it, in many of which large chunks of (A) are removed, usually the bundles referred to as unstable. They are perhaps less interesting from a geometrical viewpoint thanks to the Harder-Narasimhan filtration in which the quotients are semi-stable, but from an analytic viewpoint, they are essential, especially — as I suppose — for a description of the continuous (Eisenstein) spectrum, but I have not yet seriously reflected on this. Nevertheless it would be imprudent to remove these chunks.

It seems to me that what is important to understand is why and how, when no pieces are removed, the set (A) is a union of pieces perhaps of different dimension that combine to form a whole variety of uniform dimension, although perhaps singular. As far as I can infer from the book of Le Potier, the dimension of (A) is, according to the genus $g$ of $F$, the rank $r$ of $G$, thus the dimension of a Cartan subgroup, and the dimensions of the center and the semisimple part of $\mathfrak{g}$, given by

$$\begin{aligned}
&\dim \mathrm{Bun}_G = 0, \quad g = 0,\\
&\dim \mathrm{Bun}_G = \operatorname{rank} \mathfrak{g}, \quad g = 1,\\
&\dim \mathrm{Bun}_G = (g-1)\dim_{\mathrm{ss}} \mathfrak{g} + g \dim_{\mathrm{cent}} \mathfrak{g}, \quad g > 1.
\end{aligned}$$

My efforts to persuade myself of this are, so far, not entirely convincing. They have been largely a matter of combining the customary decomposition

$$G(\mathbb{A}_F) = N(\mathbb{A}_F)A(\mathbb{A}_F)\prod_x G(\mathcal{O}_x)$$

with the Riemann-Roch formula and with the theory of the Picard variety. There is no reason not to think that the approach is correct. I have simply not spent enough time with it.

In any case an initial step would be to represent the quotient (A) as the union over a family of restricted products $\prod_{x\in S} \mathrm{gr}_x$ divided by an equivalence relation

(C) $$g \sim g' = \gamma g,$$

where the left side is defined with respect to $S$, the right side with respect to $S'$ and the equation means equality in $\prod_{x\in X} G(F_x)$ modulo $\prod_{x\in X} G(\mathcal{O}_x)$. The quotient determined by the equivalence relation would then be constructed in two steps: first an unwieldy quotient of infinite-dimensional varieties by infinite-dimensional groups that has only formal meaning; then a union, but perhaps over a family of continuous parameters, of quotients of finite-dimensional varieties by finite-dimensional groups. The infinite-dimensional variety is, more precisely, an injective limit of finite-dimensional varieties

(D) $$\mathfrak{S}_n = \bigcup_S \prod_{x\in S} \mathrm{gr}_x^{\tau_x}$$

where $S$ runs over subsets with $n$ elements. It suffices to take $n = g+1$ in order to obtain all points up to equivalence, but it is best to allow $\tau_x$ to grow because, as already observed, $\mathrm{Bun}_G$ may have an infinite number of components, as it does already for $G = GL(1)$. There is, however, no reason not to take $\tau_x = \tau$ uniform, thus independent of $x$. It is, indeed, better to do so, because the set of points $S$ is varying continuously. If it is necessary to draw attention to the presence of $\tau$, I replace the symbol $\mathfrak{S}_n$ by $\mathfrak{S}_n^\tau$.

An example of a different nature for the group $G = GL(2)$ or — with a little imagination — for any group, and for an arbitrary $X$ and an arbitrary point $p$ on it is to take $g_x = 1$ for $x \neq p$ and

$$g_p = \begin{pmatrix} z_p^m & 0 \\ 0 & z_p^n \end{pmatrix}, \quad m >> n,$$

where $z_p$ is the local coordinate at $p$. I doubt that we can embrace all these points with the elements (D) and a given $\tau$. This is, however, an aside. Rather than a single point $p$, we could take a finite number of variable points $p_1, \dots, p_g$, the integer $g$ being typically, as the notation suggests, the genus. The element $g_p$ is replaced by $\prod_{i=1}^g g_{p_i}$. Since the discrete parameters $m, n$ remain, it appears that we have a variety with many components, many of which have a "tail" of dimension $g$. This "tail" may of course be imbedded and not recognizable as such, since the varieties are of a uniform dimension. This property, namely uniform dimension of the quotient, is the essence of the notion of stack.

What is important is to start with a given sufficiently large $n$, and a given point $\mathfrak{s}$ in $\mathfrak{S}_n^\tau$, and to verify that if we increase $\tau$ sufficiently we will be able with some choice of the $\tau$ to reach all points in the connected component $\mathbb{B}_\mathfrak{s}$ of $\mathrm{Bun}_G$ that contains the image of $\mathfrak{s}$. I denote the inverse image of this component by $\mathbb{S}_n^\tau$ without specifying $\mathfrak{s}$. To obtain the fibration

$$\mathbb{S}_n^\tau \to \mathbb{B}_\mathfrak{s},$$

the set $\mathbb{S}_n^\tau$ will be rent into tatters, subvarieties or subsets of varying dimensions and shapes with little in common except that the result will be an algebraic variety all of whose components are of the same dimension. This, if I am not mistaken, is the essence of a stack. It is a remarkable feature of the geometric theory and, almost entirely so far as I know, a consequence of the Riemann-Roch theorem.

We have two intimately linked spaces (D), or — without the truncation — the union of the sets $\prod_{x\in S} \mathrm{gr}_x$, and the quotient $\mathrm{Bun}_G$. The Hecke operators in the geometrical context are customarily defined in a sheaf-theoretic manner, thus on (D). I am proposing that they also be defined, as in the arithmetic/analytic theory, in the spirit initiated and developed by, among others, Hecke and Maaß, as integral operators on a Hilbert space of functions on $\mathrm{Bun}_G$. The transition is not evident. In the sheaf-theoretic version, one is dealing with sheaves on $\mathfrak{S}_n$ or on $\mathfrak{S} = \mathfrak{S}_n^\tau$. The distinction is not too important. The point is that the Hecke operators, as integral operators or as sheaf-theoretic operations are defined by a process of smearing, so that we need to know the sheaf on a larger $\mathfrak{S}_{n'}^{\tau'}$ in order to calculate its image under a Hecke operator on $\mathfrak{S}_n^\tau$. That is not the principal issue here. The principal issue is the distinction between sheaves and their cohomology and functions and their integrals.

It is not the only issue. There are, in fact, a good number of mathematical domains — algebro-geometric, differential geometric, sheaf-theoretic, functional analytic — of whose techniques a solid technical understanding is required in the geometric theory. I will not try to hide my current lack of the necessary understanding. Indeed, I run a great danger of committing an egregious blunder, even several. Dealing with the geometric theory is in fact like watching a three-ring circus. There is more happening than one can follow and it is always happening in a different place.

One difficulty is that moduli spaces like $\mathrm{Bun}_G$ have been studied largely be algebraic geometers who have specific preferences. In particular, they prefer finite-dimensional varieties with one or at least only finitely many components. This cannot be achieved in the present context without sacrificing the spectral theory, in particular, the continuous part represented by Eisenstein series. It is not, however, the principal issue. That is to replace sheaves by functions. None the less, from the point of view that I am advocating, the theory as created by the geometers, who discard all but the "stable" bundles, represents an arbitrary and, ultimately, disfiguring, puzzling, and even occasionally confusing aesthetic choice.

It may, however, be asserted to be no more than aesthetic. This is not so for the principal difficulty, that arising from the definition of the Hecke operators, where fundamentally different methodological, even philosophical differences appear. The stance I prefer is set-theoretic, not categorical, and one issue to be resolved is whether a categorical context, represented, it appears, by the Spring School in Jerusalem, is necessary or desirable. In a purely set-theoretical context, we need, I believe, a Kähler metric on $\mathrm{Bun}_G$ and the associated measures. My hope is that the metric can be introduced with the methods employed by Atiyah and Bott in the paper *The Yang-Mills equations over Riemann surfaces* which employs, in particular, methods from classical mechanics. I do not yet have the necessary command of them.

In, for example, Frenkel's notes entitled *Lectures on the Langlands program and conformal field theory*, the Hecke operators are introduced as operators on sheaves on $\prod_x \mathrm{gr}_x$. Strongly influenced by my experience with the theory of Eisenstein series, I would like to take as a central element of the geometric theory a spectral theory of the Hecke operators as operators on functions on $\mathrm{Bun}_G$. The sheaf-theoretic form of the theory has a kind of elegance, but, in my view, the connections with other domains, apart from category theory, are richer in the set-theoretic context.

The possibilities I still see only through a dense fog. The full moduli space $\mathrm{Bun}_G$, thus with no constraints of stability or otherwise on the bundles, appears to be an algebraic variety with an infinite number of connected components with a complicated structure of unstable bundles, which form "cusps" as in the arithmetic theory and which appear to be the principal support of the Eisenstein series. The stable bundles on the other hand, on which the geometers have focused our attention, may support the theory of "cuspidal" automorphic forms, although the geometric theory is still so immature that there is no true spectral theory. One can hope that this can, with a certain amount of diligence, be created.

We have observed already that $\mathrm{Bun}_G$ is a very complicated quotient of $\prod \mathrm{gr}_x$. That the fibers are infinite-dimensional is not the principal difficulty. It seems relatively easy, at least locally, to reduce to quotients with finite-dimensional fibers. The difficulty is that the fibers are a tangle of varieties, whose exact nature is not yet clear to me, of varying dimension. So we have, as in the theory of intersection homology or cohomology, but in a more sophisticated form, some kind of stratification and some kind of perverse sheaves, whose generating elements are the inverse images of points in $\mathrm{Bun}_G$. Since $\mathrm{Bun}_G$ is not discrete, although it may have discrete elements, it will be necessary not only to add such objects but to integrate them over the base. It is for this that we need a measure on $\mathrm{Bun}_G$ and that I need to continue my study of the Atiyah-Bott paper. All being well, once it has been correctly understood, we shall be able to introduce the Hecke operators as convolution operators just as by various generalized forms of the Gauss-Bonnet theorem (or of the index theorem) we can calculate Euler-Poincaré characteristics as integrals.

The measure on $\mathrm{Bun}_G$ defined and the Hecke operators defined as integrals, they will form a commuting family of operators on $L^2(\mathrm{Bun}_G)$ for which it may be possible in the usual functional-analytic context to construct a simultaneous spectral decomposition. The eigenfunctions and the simultaneous eigenvalues and various properties of the two would be the core of an analytic geometric theory, which would presumably ultimately include a local theory with endoscopy.

**Complete family of eigenfunctions in $L^2(\mathrm{Bun}_G)$.** Since we are dealing with a spectrum that is likely to be continuous, it is a little uncertain what complete means, even if it exists. We want the corresponding eigenfunctions to be closed in the set of all eigenfunctions. So some care is necessary. As with the arithmetic theory, we expect the spectrum to have except under unusual circumstances — when the genus of the curve is 0 — both discrete and continuous parts.

**TWO CENTRAL POINTS** We are, I think, discussing here a central difficulty of a possible analytic form of the geometric theory. There is a second central point that appears upon reflection: the difference between the analytic theory and the sheaf-theoretic theory proposed in Frenkel's notes.

What appears is that there may be two parallel theories: one for eigensheaves and this is the theory of the Russian émigré school of Drinfeld, Beilinson and others, and a parallel theory for eigenfunctions. The existence of two theories does not necessarily imply methodological differences. I do not exclude the possibility of establishing the existence and properties of the eigensheaves as a consequence of the existence and properties of eigenfunctions. The passage from one to the other will be achieved — I suppose — by incorporating the equivalence between flat bundles on the one hand and harmonic bundles on the other. This is an equivalence described by Carlos Simpson in his report, *Nonabelian Hodge theory*, to the ICM in 1970. I have still to understand it, but I do my best here to explain what I think is happening. I begin with the conjectures — or better the problems — in the context of eigenfunctions.

**Conjectures.** *Each eigenfunction yields a homomorphism of the Hecke algebra into* $\mathbb{C}$. *The known structure of the Hecke algebra implies that each eigenfunction yields for each* $x \in X$ *a conjugacy class* $\Theta(x)$ *of semi-simple elements of* ${}^L G$. *Let* ${}^L\mathrm{Un} \subset {}^L G$ *be a given unitary form of* ${}^L G$. *For a given eigenfunction, there are two independent questions: (i) is there a function* $\theta(\cdot)$*, say differentiable, from* $X$ *to semi-simple elements in* ${}^L G$ *such* *that $\theta(x) \in \Theta(x)$ for all $x$?; (ii) does the class $\Theta(x)$, $x \in X$, meet ${}^L\mathrm{Un}$ for all $x$? A third question that assumes a positive answer to the first two is: can $\theta$ be taken to have values in ${}^L\mathrm{Un}$? The second conjecture would be, I suppose, the geometric form of the Ramanujan conjecture. I have not yet tested it. I have not, I should confess, even exhibited in a reasonable, namely concrete, fashion the spectrum and the eigenfunctions for $X = \mathbb{P}^1$ equal to the projective line. This should be an exercise; perhaps it has been carried out somewhere. If the third conjecture is valid, the function $x \to \theta_x$ is the integral of a unitary connection $\theta^{-1}d\theta$ on $X$.*

*A fourth question is whether every unitary connection, up to equivalence, is associated to an automorphic representation occuring in $L^2(\mathrm{Bun}_G)$. A fifth is the multiplicity. The latter question has been little explored in the arithmetic theory and, so far as I know, hardly at all in the geometric theory.*

Positive answers to these questions settle to some extent the questions of geometric functoriality. If we have a homomorphism $\varphi : {}^LH \to {}^LG$, then a unitary integral $\theta_H(x)$ of a connection for ${}^LH$ can be composed with $\varphi$ to give $\theta_G = \varphi \circ \theta_H$, a connection for $H$, to which a Hecke eigenfunction for $G$ is associated when the answers to the above questions are positive.

**Correction/refinement.** *The conditions are, in some respects, too restrictive. In the abelian case, thus when the group is $GL(1)$, we did not demand that the eigenfunctions (or eigenvalues, almost the same thing for $GL(1)$) be single-valued. We allowed them to transform by a character of the fundamental group of $GL(1, \mathbb{C})$. In the present context, we could allow $\theta$, to change, as we move along a closed curve, by a nontrivial element of the fundamental group of ${}^LG$, thus to define a homomorphism of the fundamental group of $X$ into the fundamental group of ${}^LG$. Since this fundamental group is almost abelian — the quotient by a finite-subgroup is abelian — this would introduce only a minor technical or notational inconvenience into the discussion. Since the fundamental group may not be abelian, it would, however, be necessary to introduce not just characters but also finite-dimensional representations. For our present purposes, the refinement need not be considered.*

There is a conjecture of a different kind in Frenkel's notes. He is concerned not with unitary connections but with holomorphic connections defining flat, thus locally trivial, ${}^LG$-bundles. They are referred to as local systems. He excludes certain complications by assuming that he has in hand a Hecke eigensheaf, because he works with sheaves not functions. The sheaf will be a complex sheaf the sheaf-theoretic notion of an eigensheaf with the local system $E$ as eigenvalue appears, at first glance, to be quite different than the notion of an eigenfunction I have just explained. The relation between them, which is very close, has to be explained.

This is a matter of simultaneously explaining or assuming several theories, none of which I have yet fully understood. The listener — these are the notes for a lecture — will have to forgive me. These theories can moreover function at the level of arbitrary reductive groups or at the level of vector bundles. The first is generally implicit in the second, which is that discussed by Simpson. So, for the moment, I continue in the context of vector bundles, thus

$G = GL(n)$. Vector bundles are implicitly present for all $G$. I will not attempt precision. His first theorem, the one of principal concern to us, is the equivalence between harmonic bundles and semisimple flat bundles. These flat bundles are holomorphic and form the stuff of the sheaf-theoretic theory explained by Frenkel. The harmonic bundles are, for me at least, extremely complicated objects. They are provided with a hermitian metric and a unitary connection. This is a little less than what is offered in our conjectures, where what is offered is the integral $\theta$ of a connection.

My guess is that the conjectures offered by the Russian school and taken by me from Frenkel's lecture and repeated in the Prologue are, in contrast to the "harmonic-bundle" form described in the first conjectures of these lectures, the "flat-bundle" form of the conjectures. The distinction is, in some sense, the distinction between a holomorphic function $f(z)$ and its unitary part $f(z)/|f(z)|$ or, after passage to the logarithm between a meromorphic or holomorphic function and its imaginary part. One can be single-valued even when the other is not.

What is the passage from the conditions on the eigenfunctions to the conditions on the eigensheaves? For the eigenfunctions, the condition, namely the conjugacy class $\Theta(x)$, varies from point to point of $X$, thus one condition for each of the different Hecke algebras. For the eigensheaves, it comes in sheaf-theoretic dress. I recall the formulas from the Prologue, themselves taken directly from §6.1 of Frenkel's notes.

The Hecke correspondence is expressed diagramatically as

(E)
$$\begin{array}{ccc} & \mathit{Hecke} & \\ {}^{\mathfrak{h}^{\leftarrow}}\swarrow & & \searrow^{\mathfrak{h}^{\rightarrow}} \\ \mathrm{Bun}_G & & X \times \mathrm{Bun}_G \end{array}$$

The sheaf-theoretic condition is expressed as

(F)
$$H_\lambda(\mathcal{F}) = \mathfrak{h}^{\rightarrow}_{*}(\mathfrak{h}^{*\leftarrow}\mathcal{F} \otimes \mathrm{IC}_\lambda); \quad \iota_\lambda : H_\lambda(\mathcal{F}) \simeq V^E_\lambda \boxtimes \mathcal{F}, \qquad \lambda \in P_+.$$

We have to persuade ourselves that these equations are little more than a way to recover the classes $\Theta(x)$ from the sheaf. The difference is that for the sheaf-theoretic connection we do not want to recover — in the language of Simpson's report — the eigenvalues of the unitary matrices arising from a harmonic bundle, we want to recover the eigenvalues of the complex matrices appearing in the associated Higgs bundle. So the transition from sheaf-theoretic eigenvalues to function-theoretic eigenvalues takes place in the context of the nonabelian Hodge theory.

I refer to the Prologue for a full description of the diagram (E). One function of the left-hand arrow is to make the description of $\mathrm{Bun}_G$ as a quotient of $\prod \mathrm{gr}_x$ explicit, thus to allow one to treat sheaves on $\mathrm{Bun}_G$ as sheaves on $\prod_x \mathrm{gr}_x$. This done we can examine their restriction to each $\mathrm{gr}_x$. There are two questions implicit in (F): the parameters $\Theta(x)$ at each point in $X$; the nature of their variation, local and global, local referring principally to questions of continuity, analyticity and so on, global referring principally to monodromy. The sheaves in question are often perverse sheaves. This introduces another impediment, not the least, to easy understanding of the relevant constructions. It is not an impediment that I have overcome.

The geometric Hecke algebra is similar to the classical Hecke algebra, but with a much more sophisticated definition. Its action at the point $x$ is given by the restriction to the fiber at $x$ of the direct image of $\mathfrak{h}_*^{\rightarrow}$ in (F). This is a sheaf on the affine grassmannian. So it is a sheaf on which the spherical Hecke algebra at $x$ acts. This is an algebra which is very well understood. The pertinent facts are explained in §5 of Frenkel's lectures; I am only now beginning to appreciate their import. I had long overlooked the significance of the theorem that, under the Satake isomorphism, the representation with highest weight $\lambda$ corresponds, up to an explicitly given sign, to the Goresky-MacPherson sheaf on the closure of the double coset in $G(\mathcal{O}_x)\backslash G(F_x)/G(\mathcal{O}_x)$.

As a consequence, the function of $\mathrm{IC}_\lambda$ in the formula (F) is to convert the ${}^LG$-local system $\mathfrak{h}^{*\leftarrow}\mathcal{F}$, as well as its image under $\mathfrak{h}_*^{\rightarrow}$ to a flat vector bundle. The assumption is that this is identical to the tensor product of the perverse sheaf $\mathcal{F}$ and the local system defined by the ${}^LG$-local system and the representation of ${}^LG$ with highest weight $\lambda$. To identify the eigenvalue — better the eigencharacter, because it is a matter of a homomorphism of the Hecke algebra at $x \in X$ to $\mathbb{C}$ — at a point $x \in X$, we pull back, as explained in the correspondence $\mathrm{Bun}_G \leftrightarrow X \times \mathrm{Bun}_G$ of (E) to $\mathrm{Bun}_G \leftrightarrow x \times \mathrm{Bun}_G$ or, effectively, to the fibre $\mathrm{gr}_x$ with the action of the Hecke algebra on the right. There are two subtleties in (F). The first is the presence of the sheaf $\mathcal{F}$, whose fibers may not be one-dimensional. This is a kind of icing on the cake. The second, more difficult to grasp is that the ${}^LG$-local system is, in another terminology, a bundle — for Frenkel a Higgs bundle, thus a holomorphic bundle, for us a harmonic bundle, thus a bundle with flat connection and no holomorphic structure. As observed, these are, according to the report of Simpson and the articles he cites, the same thing, but to understand how and why is no easy matter. I am still hanging on by the skin of my teeth.

What seems to me the central difficulty? It is to establish a spectral theory for the commuting algebra of Hecke operators the $L^2$-space. This entails, in particular, a definition of the Hecke operators as operators as functions, a difficulty that I have treated all too briefly with cursory references to stratifications and intersection cohomology, thus implicitly to the technical difficulties that lead to the introduction of stacks.

**The heart of the matter.** *We can assume – or hope – that, the operators defined, their spectral theory can be constructed with the techniques from functional analysis presently available. We have then to establish that the (classes of) unitary connections serve as the parameters of the eigenfunctions. We have already seen why we can expect to pass not from connections but from their integrals, when these are admissible, thus have no periods, to conjugacy classes of semi-simple elements in* $G(F)$. *The passage from conjugacy classes to eigenfunctions is less clear. The conjugacy class and the subgroups* $G(\mathcal{O}_x)$ *given for each* $x$, *there is no ambiguity about the associated function on* $G(\mathbb{A}_F)$. *The difficulty is to establish that it is a function on the quotient* $G(F)\backslash G(\mathbb{A}_F)$. *For* $GL(1)$ *this is the argument — presumably of Weil — with contour integrals and residues reproduced in the Rogawski paper. I have not had time to reflect on how to extend it to a general group, or rather to reflect successfully!*

There is an interesting and familiar question that arises in the geometric theory. The answer is not exactly the same as in the arithmetic theory. It is a form of the Ramanujan conjecture, which is as one knows intimately connected to functoriality. It was remarked many years ago in the lecture

http://publications.ias.edu/rpl/paper/47

dedicated to Solomon Bocher that functoriality and a theorem of Landau imply Ramanujan's conjecture. The remark was in fact suggested by the estimates of Rankin and Selberg. Since functoriality is an integral part of the approach to the geometric theory suggested here and the spherical functions in the geometric theory are well understood, we can hope that the Ramanujan conjecture in an appropriate form will be a consequence. One suspects — it is indeed implicit in the preceding suggestions, but I have not seriously reflected on the matter — that, for the geometric theory, the unitary spectrum is tempered, in the sense that unitary conjugacy classes suffice for the construction of the $L^2$-theory.

I have observed that there are alternative approaches to the geometric theory, in particular, approaches in which the principal tool is no longer expected to be analysis. I prefer, in part because my formation and taste are, as already suggested, that of my time, to work in a set-theoretic context formed by mid-twentieth century analysis, algebra and geometry. There are other, competing and very clever, ways to proceed. I draw the reader's attention to an announcement received in April of this year from Dennis Gaitsgory. It reflects a different stance.

**Spring School.** *We are writing this email to announce a Spring School "Towards the proof of the geometric Langlands conjecture," to be held at IAS, Hebrew University of Jerusalem, on March 16-21, 2014.*
*The idea is to explain some recent progress in this field. Much of this progress became possible due to an influx of methods from derived algebraic geometry and higher category theory. We intend to supply enough background to make the talks accessible to people who are not familiar with these new techniques (however some familiarity with the language of infinity categories will be very helpful).*
*The Spring School will consist of several mini-courses.*
*1) Singular support of coherent sheaves*
*2) Chiral categories and chiral homology*
*3) Localization of Kac-Moody representations*
*4) D-modules on infinite-dimensional spaces*
*5) Local and global Whittaker models*

## Supplementary comments.

An element of the gauge group for ${}^LG$ as introduced in §2 of [AB] defines a map from $M$ to conjugacy classes.

**Question 1.** *A first question for any satisfactory theory is whether the family of eigenvalues of the (geometric) Hecke operators, thus a collection of conjugacy classes in ${}^LG$, one attached to each point, can be realized by an element of the gauge group.*

The theory of [AB] is a unitary theory. Their gauge transformations are unitary.

An element of the gauge group is, if differentiable, the integral of a connection. We can let the gauge group act on the the connection or on its integral. The integral is better because it incorporates a parameter represented by the initial condition — at some arbitrary but given point. The question arises as to how many different conjugacy classes in the gauge group represent the same family of eigenvalues for the Hecke operators.

This is a different question than that arising in §4 of [AB]. The questions suggested there by Atiyah-Bott refer to the index and nullity. Our question refers rather to questions of conjugacy: how does global conjugacy of the integral differ from local conjugacy. There are at several notions of local conjugacy. The most evident is conjugacy over $\mathbb{C}$ at each point; the other two are conjugacy over the field of formal Laurent series (or even convergent Laurent series) at each point and conjugacy over the ring of formal power series at each point (or even convergent Laurent series).

The question that perplexed me for some time, namely whether the set-theoretic object, the collection of eigenvalues, can be converted into a function-theoretic object, is perhaps not so difficult as I thought. For $G = GL(n)$, there is a simple holomorphic structure on the semi-simple conjugacy classes, that given by the characteristic polynomial. A similar structure is available for any reductive group. There is a global problem and a local problem. There is also the difference between the parameters of the conjugacy class and the parameters of the element in the group ${}^LG$. The second is, I think, an algebraic problem, although some attention will have to be paid to "branching". The first is a matter of showing that the Hecke eigenvalues are analytic or, at least, differentiable functions of the local parameters with values in these conjugacy classes. In other words, the eigenvalues of the Hecke algebra define a semisimple conjugacy class in $G(\bar{F})$, where $\bar{F}$ is the algebraic closure of $F$. One of the first questions is whether this conjugacy class can be realized in $G(F)$. This must be a problem that has already been fully discussed in the literature, so that we know when the answer is positive and when not. If it is positive, there will be the difference between conjugacy and stable conjugacy to be discussed. These matters understood, the family of eigenvalues can be treated as a connection, or rather as the integral of a connection.

So the decomposition $z = \rho e^{i\theta z}$ or $g = \rho\mu$, $\rho$ symmetric positive, $\mu$ unitary has to be considered. What does that mean? We have not yet discussed the nature of the eigenvalues, nor the possible distinction between the unitary theory and the holomorphic theory. For the unitary theory, thus the $L^2$-theory, the conjugacy classes are unitary. There is, however, an entirely different theory, the focus of the efforts of the Russo-American school, in which it is holomorphic connections that matter. So the decomposition of an arbitrary invertible matrix $g$ as the product of a unitary matrix and a positive-definite hermitian matrix becomes important.

**The distinction between the holomorphic theory and the $L^2$-theory.** *What is it? We know what it is at the level of a connection. What is it at the level of eigenvalues of the Hecke operators?*

I have explained at some length in the Rogawski volume the meaning of this question and its answer for $G = GL(1)$. It is contained in the theory of Fourier series and integrals and can only be understood in the context of a spectral theory. For abelian $G$ this is familiar to us. We understand that not all eigenfunctions of the pertinent differential operators are needed for the spectral theory in an $L^2$-context. More are necessary in the theory of distributions, in the Paley-Wiener theory, for the Laplace transform and so on. If one focuses on the general theory of distributions, although not on the theory of tempered distributions, these distinctions are not important, but the theory is not so rich. An informed mathematician needs to be aware of all possibilities.

It is natural to expect that the holomorphic theory requires a holomorphic structure on $\mathrm{Bun}_G$. How does it arise? For line bundles, it arises from the structure on the base. I suppose that, in general, the "vertical" complex structure on $\mathrm{gr}_x$ has to be taken into account. The division by $G(F)$ is compatible with that structure. So the complex structure is inherited from that on $\prod_{x\in S} \mathrm{gr}_x$. There are two references for the construction of $\mathrm{Bun}_G$: the first consists of a reference to Weil on p.7 of Frenkel's report *Recent advances in the Langlands program* together with a comment on p. 39 of *Lectures on the Langlands program and conformal field theory* by the same author; the second is given on p. 6 of his report *Ramifications of the geometric Langlands program* and is to two papers, one by Beauville-Laszlo and one by Drinfeld-Simpson. It is not obvious that these various references are compatible. At first glance, the second reference seems to provide a strengthening of the first. The definition (1.1) in the latter paper of the moduli space leads to a clear definition of the bundle as

$$(1)\qquad G_{\mathrm{out}}\backslash G(\mathcal{K}_x) \to G_{\mathrm{out}}\backslash G(\mathcal{K}_x)/G(\mathcal{O}_x), \quad \mathcal{K}_x = F_x,\, G_{\mathrm{out}} = G_S(F),\, S = \{x\}.$$

The notation $\mathcal{K}_x$, $G_{\mathrm{out}}$ is Frenkel's while $F_x$, $G_S(F)$ is mine. What has to be proven is that, for any finite $S$ that contains $x$,

$$(2)\qquad G_{\mathrm{out}}\backslash G(\mathcal{K}_x)/G(\mathcal{O}_x) \to G(F_S)\backslash \prod_{y\in S} G(F_y)/\prod_{y\in S} G(\mathcal{O}_y)$$

is surjective. Then it is bijective and $G_{\mathrm{out}}\backslash G(\mathcal{K}_x) \to G_{\mathrm{out}}\backslash G(\mathcal{K}_x)/G(\mathcal{O}_x)$ defines the relevant $G$-bundle. The second references define $\mathrm{Bun}_G$ as a complex variety.

## References


[AB] *Atiyah-Bott.*