THE LEGACY OF ABEL IN ALGEBRAIC GEOMETRY

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This paper is based on a talk given at the bicentenary celebration of the birth of Neils Henrik Abel held in Oslo in June, 2002. The objectives of the talk were first to recall Abel's theorem in more or less its original form, secondly to discuss two of the perhaps less well known converses to the theorem, and thirdly to present two (from among the many) interesting issues in modern algebraic geometry that may at least in part be traced to the work of Abel. Finally, in the reprise I will suggest that the arithmetic aspects of Abel's theorem may be a central topic for the 21^{st} century.

This talk was not intended to be a "documentary" but rather to tell the story — from my own perspective — of Abel's marvelous result and its legacy in algebraic geometry. Another talk at the conference by Christian Houzel gave a superb historical presentation and analysis of Abel's works.

In keeping with the informal expository style of this paper (the only proof given is one of Abel's original proofs of his theorem) at the end are appended a few general references that are intended to serve as a guide to the literature, and should not be thought of as a bibliography.

1. Origins of Abel's theorem

During the period before and at the time of Abel there was great interest among mathematicians in *integrals of algebraic functions*, by which we mean expressions

$$\int y(x)dx \tag{1.1}$$

where $y(x)$ is a 'function' that satisfies an equation

$$f(x, y(x)) = 0 \tag{1.2}$$

where $f(x, y) \in \mathbb{C}[x, y]$ is an irreducible polynomial with complex coefficients. Although not formalized until later, it seems to have been understood that (1.1) becomes well-defined upon choosing a particular branch of the solutions to (1.2) along a path of integration in the x-plane that avoids the branch points where there are multiple roots. In more modern terms, one considers the algebraic curve F° in $\mathbb{C}^2$ defined by

$$f(x, y) = 0 \ ,$$

and on F° one considers the rational differential ω defined by the restriction to F° of

$$\omega = ydx \ .$$

On the closure F of F° in the compactification of $\mathbb{C}^2$ given either by the projective plane $\mathbb{P}^2$ or by $\mathbb{P}^1 \times \mathbb{P}^1$ one considers an arc γ avoiding the singularities of F and the poles of ω, and then (1.1) is defined to be

$$\int_\gamma \omega \ . \tag{1.3}$$

Actually, among mathematicians of the time the interest was in more general expressions

$$\int r(x, y(x))dx \tag{1.4}$$

where $r(x, y)$ is a rational function of x and y, and $y(x)$ is as above. The formal definition of (1.4) is as in (1.3) where now ω is the restriction to F of the rational differential 1-form $r(x, y)dx$.

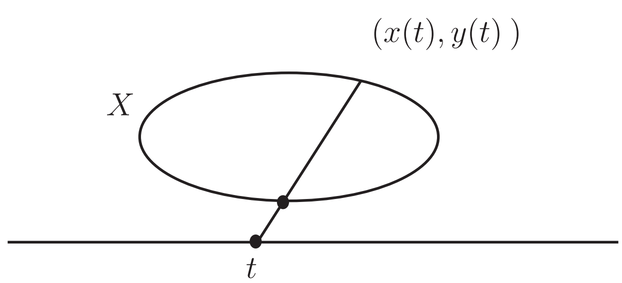
Of special interest were the *hyperelliptic integrals*

$$\int \frac{p(x)dx}{\sqrt{q(x)}} \tag{1.5}$$

where $p(x)$ and $q(x)$ are polynomials with, say,

$$q(x) = x^n + q_1 x^{n-1} + \cdots + q_n$$

of degree n and having distinct roots. When $n = 1, 2$ it was well understood at the time of Abel that these integrals are expressible in terms of the "elementary" — i.e., trigonometric and logarithmic — functions. The geometric reason, which was also well understood, is that any plane curve may be rationally parametrized as expressed by the picture



Plugging the rational functions $x(t)$ and $y(t)$ into (1.5) gives an integral

$$\int r(t)dt \ ,$$

where $r(t)$ is a rational function, and this expression may be evaluated by the partial fraction expansion of $r(t)$.

There was particular interest in the hyperelliptic integrals (1.5) when $n = 3, 4$ and important fragments were understood through the works of Euler, Legendre and others. They go under the general term of *elliptic integrals*, for the following reason: Just as the resolution of the arc length on a circle leads to the trigonometric functions as expressed by

$$\int \sqrt{dx^2 + dy^2} = \int \frac{dx}{\sqrt{1-x^2}} \ , \qquad x^2 + y^2 = 1 \ , \tag{1.6}$$

there was great interest in the functions that arise in the resolution of the arc length of all ellipse. Thus, through the substitution

$$t = \arcsin\left(\frac{x}{a}\right)$$

the arc length on the ellipse

$$\int \sqrt{dx^2 + dy^2}, \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad a > b$$

becomes the elliptic integral

$$a \int \frac{(1 - k^2x^2)dt}{\sqrt{(1 - t^2)(1 - k^2t^2)}}\,, \qquad k^2 = (a^2 - b^2)/a^2 \tag{1.7}$$

in Legendre form.

Of special interest were integrals (1.4) that possessed what was thought to be the very special property of having *functional equations* or *addition theorems*. For example, using the obvious synthetic geometric construction of doubling the length of an arc on the circle applied to the integral (1.6), one recovers the well known formulas for $\sin 2\theta$ and $\cos 2\theta$ expressed in terms of $\sin\theta$ and $\cos\theta$. More generally one may derive expressions for $\sin(\theta + \theta')$ etc. which are expressed as addition theorems for the integral (1.6). In the 18^{th} century the Italian Count Fagnano discovered a synthetic construction for doubling the arc length on an ellipse, and when applied to (1.7) this construction leads to addition theorems for the "elliptic integral" (1.7). As alluded to above this was thought to be a very special feature, one that was the subject of intensive study in the late 18^{th} and early 19^{th} centuries.

2. Abel's theorem and some consequences

In Abel's work on integrals of algebraic functions there are two main general ideas

- abelian sums
- inversion

Together these led Abel to very general forms of

- functional equations

for the integrals. We will now explain these ideas.

Turning first to what are now called abelian sums, the integrals (1.1) and more generally (1.4) are highly transcendental functions of the upper limit of integration and consequently are generally difficult to study directly.[1] Abel's idea was to consider the sum of integrals to the variable points of intersection of $F = \{f(x,y) = 0\}$ with a family of curves $G_t = \{g(x,y,t) = 0\}$ depending rationally on a parameter t. Thus letting

$$F \cap G_t = \sum_i (x_i(t), y_i(t))$$

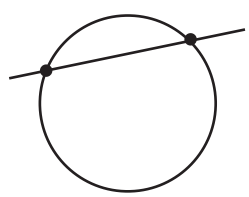
be the set of solutions to

$$\begin{cases} f(x,y) = 0 \\ g(x,y,t) = 0 \end{cases}$$

written additively using the notation of algebraic cycles, the *abelian sum* associated to (1.4) is defined to be

$$u(t) = \sum_i \int_{x_0}^{x_i(t)} r(x, y(x))dx \ . \tag{2.1}$$

Below we will amplify on just how this expression is to be understood. A particularly important example is given by taking the G_t to be a family of lines as illustrated by the figures

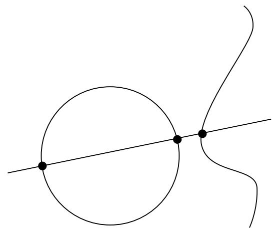
(i)  $F = \{x^2 + y^2 = 1\}$

[1]The term "highly transcendental" needs care in interpretation — cf. the reprise below. Again Abel, in a paper published in 1826, showed the existence of polynomials R, F such that

$$\int \frac{F\,dx}{\sqrt{R}} = \ln\left(\frac{P + \sqrt{R}Q}{R - \sqrt{R}Q}\right)$$

has solutions for relatively prime polynomials P, Q. Here, R is a polynomial of degree $2n$ with distinct roots and F is a polynomial of degree $n-1$, so that the integrand is a differential of the 3$^{\text{rd}}$ kind. This is an "exceptional" case where the integral is transcendental but expressible in terms of elementary functions.

(ii)



$$F = \{y^2 = x^2 + ax + b\}$$

In both cases we take $\omega = dx/y$ and the integrals (1.4) are respectively

$$(2.2) \qquad \begin{cases} \text{(i)} & \int \frac{dx}{\sqrt{1-x^2}} \\ \text{(ii)} & \int \frac{dx}{\sqrt{x^3+ax+b}} \end{cases}$$

Even though the individual terms in the abelian sum are in general highly transcendental functions, *Abel's theorem* expresses the abelian sum as an elementary function:

Theorem: *The abelian sum (2.1) is given by*

$$(2.3) \qquad u(t) = r(t) + \sum_{\lambda} a_\lambda \log(t - t_\lambda)$$

where $r(t)$ is a rational function of t.

One of the proofs given by Abel is as follows:

Proof: For reasons to appear shortly we define the rational function

$$q(x, y) = r(x, y) f_y(x, y) ,$$

so that the integrand in the integrals appearing in the abelian sum is the restriction to the curve F of

$$\frac{q(x, y)dx}{f_y(x, y)} .$$

Then by calculus

$$u'(t) = \sum_i \frac{q(x_i(t), y_i(t))x_i'(t)}{f_y(x_i(t), y_i(t))} .$$

From

$$\begin{cases} f(x_i(t), y_i(t)) & = 0 \\ g(x_i(t), y_i(t), t) & = 0 \end{cases}$$

we have

$$x_i'(t) = \left(\frac{g_t f_y}{f_x g_y - f_y g_x} \right) (x_i(t), y_i(t))$$

so that

$$u'(t) = \sum_i s(x_i(t), y_i(t)) \tag{2.4}$$

where $s(x,y)$ is the rational function given by

$$s(x,y) = \left(\frac{qg_t}{f_x g_y - f_y g_x}\right)(x,y)\ .$$

(The non-vanishing of the rational function in the denominator is a consequence of assuming that the curves F and G_t have no common component). Abel now observes that the right hand side of (2.4) is a rational function of t — from a complex analysis perspective this is clear, since $u'(t)$ is a single-valued and meromorphic function of t for $t \in \mathbb{P}^1$. Integration of the partial fraction expansion of $u'(t)$ gives the result.

In his Paris memoiré, and also in subsequent writings on the subject in special cases, Abel gave quite explicit expressions for the right hand side of (2.4), and therefore for the terms in the formula for $u(t)$ in his theorem. For example, when the curves G_t are lines the Lagrange interpolation formula gives the explicit expression for $u'(t)$.

We shall now give applications of Abel's theorem to the two integrals in (2.2). Both are based on the second of Abel's ideas mentioned above, namely to *invert* the integral (1.4) by defining the coordinates $x(u)$, $y(u)$ on the curve F as *single-valued* functions of the variable u by setting

$$u = \int_{(x_0,y_0)}^{x(u),y(u))} \omega \tag{2.5}$$

where ω is the restriction to the curve F of $r(x,y)dx$. For example, for the integral (i) in (2.2) we obviously have

$$u = \int_{(0,1)}^{(\sin u, \cos u)} \omega\ .$$

The right hand side in (2.3) may be evaluated using the Lagrange interpolation formula and this leads to the relation

$$\int_0^{x_1} \frac{dx}{\sqrt{1-x^2}} + \int_0^{x_2} \frac{dx}{\sqrt{1-x^2}} = \int_0^{x_1 y_2 + x_2 y_1} \frac{dx}{\sqrt{1-x^2}}$$

which we recognize as the addition formula for the sin function.

Before turning to the second integral in (2.2), we remark that already in his Paris memoiré Abel singled out a "remarkable" class of abelian integrals (1.4), *now called integrals of the* 1^{st} *kind*, by the condition that the right hand side of (2.3) reduce to a constant — this is evidently equivalent to the abelian integral (1.4) being locally a bounded function of the upper limit of integration. Abel explicitly determined the integrals of the 1^{st} kind for a large number of examples. For instance for the hyperelliptic curves

$$y^2 = p(x)$$

where $p(x)$ is a polynomial of degree $n+1$ with distinct roots, Abel showed that the integrals of the 1^{st} kind are

$$\begin{cases} \omega = \frac{g(x)dx}{y} \\ \deg g(x) \leqq \left[\frac{n}{2}\right] \end{cases} .$$

In particular, assuming that the cubic x^3+ax+b has distinct roots, the expression (ii) in (2.2) is an integral of the 1^{st} kind. Abel's theorem for the family of lines meeting the cubic may be expressed by the relation

$$u_1 + u_2 + u_3 = c \tag{2.6}$$

where c is a constant and

$$u = \int_{(x_0,y_0)}^{(x(u),y(u))} \frac{dx}{y} \tag{2.7}$$

with $u = u_i$ plugged into (2.7) for $i = 1,2,3$ in (2.6). Differentiation of (2.7) gives

$$1 = \frac{x'(u)}{y(u)}$$

so that

$$y'(u) = x(y) . \tag{2.8}$$

Choosing (x_0, y_0) appropriately (specifically the flex $[0,0,1]$ on the intersection of F with the line at infinity in $\mathbb{P}^2$) we will have

$$\begin{cases} c = 0 \\ x(-u) = x(u) \end{cases}$$

and (2.6) becomes the famous addition theorem for the elliptic integral

$$x(u_1 + u_2) = R(x(u_1), x'(u_1), x(u_2), x'(u_2)) \tag{2.9}$$

where R is a rational function that expresses the x-coordinates of the third point of intersection of a line with F as a rational function of the coordinates of the other two points.

Of course, $x(u)$ is the well-known Weierstrass p-function and the above discussion gives the functional equation (2.9) and differential equation

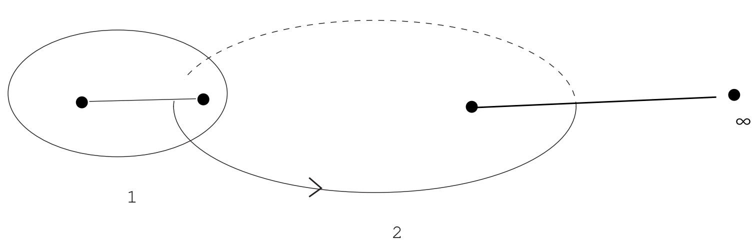
$$x'(u)^2 = x(u)^3 + ax(u) + b$$

satisfied by the p-function. We give two remarks amplifying this discussion.

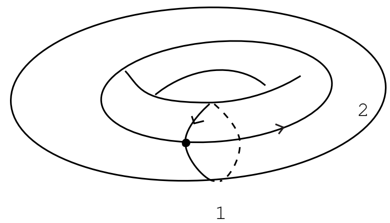
The first is that in order to *define* the integral

$$\int \frac{dx}{\sqrt{x^3 + ax + b}} \tag{2.10}$$

one cuts the x-plane, including the point at infinity, along slits connecting two of the roots of $x^3 + ax + b$ and connecting the third root to $x = \infty$



Then $\sqrt{x^3 + ax + b}$ is single-valued on the slit plane, and one may envision the algebraic curve F as a 2-sheeted covering of the x-plane where crossing a slit takes one to the "other sheet" — i.e., F is the Riemann surface associated to the algebraic function $\sqrt{x^3 + ax + b}$. The topological picture of F is the familiar torus



The integral (2.10) is then interpreted as an integral along a path on the Riemann surface. The choice of path is only well-defined up to linear combinations of δ_1 and δ_2. In particular, from (2.7) we infer that

$$\begin{cases} x(u+\lambda_i) = x(\lambda_i) \\ y(u+\lambda_i) = y(\lambda_i) \end{cases} \tag{2.11}$$

where

$$\lambda_i = \oint_{\delta_i} \frac{dx}{y}$$

are the *periods* of dx/y. Letting Λ be the lattice in the complex plane generated by λ_1 and λ_2, we have the familiar parametrizaton

$$\begin{array}{ccc} \mathbb{C}/\Lambda & \longrightarrow & F \\ \downarrow & & \Cup \\ u & \longrightarrow & (x(u), x'(u)) \end{array}$$

of the cubic curve by the p-function and its derivative.

In his Paris memoiré, Abel gave in generality the essential analytic properties of elliptic functions, defined as those functions that arise by inversion of the integral of the first kind on curves having one such integral.

Remark that the dimension of the space of integrals of the first kind is one definition of the *genus* of the algebraic curve F (or *arithmetic genus*, in case F is singular). Following Abel's pioneering work, the extension of the above story to curves of arbitrary genus was carried out by Jacobi, Riemann and other 19^{th}-century mathematicians.

A second remark is that the functions $x(u), y(u)$ in (2.7) may be defined *locally* with (2.8) holding, and the *functional equation* (2.9) is valid where defined. But then this functional equation may be used to extend $x(u)$ and $y(u)$ to entire meromorphic functions — e.g., if $x(u)$ is defined for $|u| < \epsilon$, then from (2.9) we may define $x(2u)$ and continuing in this way proceed to define $x(u)$ for $|u| < 2\epsilon$, $|x| < 3\epsilon, \ldots$. The principle that *a functional equation may be used to propogate a local object into a global one* is a central consequence of Abel's theorem, one that will be discussed further below.

In concluding this section we mention two direct consequences of Abel's theorem in algebraic geometry:

(i) the first beginnings of Hodge theory
(ii) the use of correspondences.

Under (i) we mean that Abel isolated what we now call the space of regular differentials $H^0(\Omega^1_F)$ as a basic invariant of an algebraic curve. He also computed $h^0(\Omega^1_F) = \dim H^0(\Omega^1_F)$ in a number of examples, which may be interpreted as taking the first steps toward identifying $h^0(\Omega^1_F)$ with the algebro-geometrically defined arithemetic genus. The further intepretation of $h^0(\Omega^1_F)$ as one-half the first Betti number — which marks the real beginning of Hodge theory — was to await Riemann.

Regarding (ii), the proof given above of Abel's theorem may be summarized by the diagram

$$\begin{array}{ccc} & I \subset F \times \mathbb{P}^1 & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ F & & \mathbb{P}^1 \end{array}$$

where

$$I = \{(x, y, t) : f(x, y) = g(x, y, t) = 0\}$$

is the incidence correspondence, and where the map

$$\omega \to d\left(\sum_i \int_{x_0}^{x_i(t)} \omega\right)$$

in the proof is, in modern terms, the *trace*

$$\omega \to (\pi_2)_* \left(\pi_1^* \omega\right)$$

taking rational 1-forms on F to rational 1-forms on $\mathbb{P}^1$.

3. Converses to Abel's theorem

In addition to recalling the usual global version of Abel's theorem and its converse, we will in this section give two less well known *local* converses to the result. These will illustrate the aforementioned principle that a local object having a functional equation may be propogated into a global one.

The usual version, found in textbooks, of Abel's theorem and its converse deals with the following question:

> On a compact Riemann surface X we ask when a divisor
>
> $$D = \sum_i n_i p_i \tag{3.1}$$
>
> is the divisor of a meromorphic function; i.e., what is the *test* to determine if
>
> $$D = \sum_{p \in X} \nu_p(f) p \tag{3.2}$$
>
> for some function $f \in \mathbb{C}(X)^*$.?

The answer is the following: For a regular 1-form $\omega \in H^0(\Omega^1_X)$ we recall that a period is defined to be the integral

$$\int_\delta \omega$$

where $\delta \in H_1(X, \mathbb{Z})$. Then there are two conditions that (3.2) hold. The first is that the degree of the divisor D

$$\deg D =: \sum_i n_i = 0 \ . \tag{3.3(i)}$$

If this is satisfied, then we may write $D = \delta\gamma$ for a 1-chain γ and then the second condition is that

$$\int_\gamma \omega \equiv 0 \quad \text{mod (periods)} \tag{3.3(ii)}$$

for all $\omega \in H^0(\Omega^1_X)$.

The necessity of (3.3(i)) is a consequence of the residue theorem

$$\sum_{p \in X} \text{Res}_p \left(\frac{df}{f} \right) = \sum_{p \in X} \nu_p(f) = 0 \ , \qquad f \in \mathbb{C}(X)^* \ .$$

The necessity of (3.3(ii)) is essentially Abel's theorem as given above: Setting for $t \in \mathbb{P}^1$

$$f^{-1}(t) = \sum_i p_i(t) =: D_t$$

the configuration of points $\sum_i p_i(t)$ moves with a rational parameter and $D = D_0 - D_\infty$. Since ω is a regular differential the abelian sum

$$\sum_i \int_{p_i(0)}^{p_i(t)} \omega$$

is constant, and since $\int_\gamma \omega$ is only well-defined modulo periods the assertion (3.3(ii)) follows. Alternatively, for

$$I \subset X \times \mathbb{P}^2$$

the incidence correspondence defined by

$$I = \{(p,t) : f(p) = t\}$$

we have as before that

$$d\left(\sum_i \int_{p_i(0)}^{p_i(t)} \omega\right) = (\pi_2)_* \left(\pi_1^* \omega\right)$$

is a regular 1-form on $\mathbb{P}^1$, hence equal to zero.

The usual global converse to Abel's theorem is that the conditions (3.3(i)) and (3.3(ii)) are sufficient that (3.2) hold. This may be formulated by the statement that the map

$$\mathrm{Div}^0(X) \to J(X)$$

form the group of divisors of degree zero into the Jacobian variety

$$J(X) =: H^0\left(\Omega^1_X\right)^* / H_1(X, \mathbb{Z}) \ ,$$

given by the above construction

$$\langle D, \omega\rangle =: \int_\gamma \omega \qquad \text{mod periods}$$

where $D \in \mathrm{Div}^0(X)$, $\partial\gamma = D$ and $\omega \in H^0(\Omega^1_X)$, should be injective.

The first local converse deals with what we shall call *Abel's differential equations*. These simply state the conditions that a configuration of points $p_i \in X$ together with tangent vectors $\tau_i \in T_{p_i}X$ should satisfy the infinitesimal form

$$\sum_i \langle \omega(p_i), \tau_i \rangle = 0, \qquad \omega \in H^0(\Omega^1_X) \tag{3.4}$$

of Abel's theorem. We may re-express (3.4) as follows: We consider $\sum_i p_i$ as a point in the symmetric product $X^{(d)}$. Each regular 1-form ω on X induces a 1-form $\operatorname{Tr}\omega$ on $X^{(d)}$ by

$$(\operatorname{Tr}\omega)(p_1 + \cdots + p_d) = \omega(p_1) + \cdots + \omega(p_d) \ .$$

Then (3.4) is equivalent to the differential system

$$\operatorname{Tr}\omega = 0 \ , \qquad \omega \in H^0\left(\Omega^1_X\right) \tag{3.5}$$

on $X^{(d)}$. From a differential equations perspective the remarkable fact is that *the maximal local integral manifolds of (3.5) are open sets in a global integral manifold* $\mathbb{P}^r \subset X^{(d)}$. Thus the DE's (3.5) truly do represent the condition for infinitesimal rational motion of divisors. The more precise statement is:

> *In each tangent space $T_zX^{(d)}$, $z = p_1 + \cdots + p_d$, the equations (3.5) define a subspace V with the properties (i) V is tangent to a local integral manifold of (3.5); and (ii) these local integral manifolds may be propogated to a global integral manifold isomorphic to $\mathbb{P}^r$.*

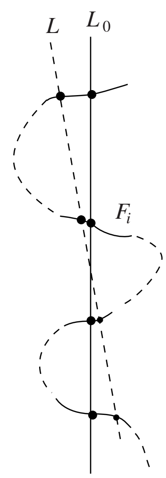
The property (i) is not automatic — it requires the *involutivity* of the exterior differential system (3.5), which imposes conditions beyond $d(\operatorname{Tr}\omega) = 0$ in neighborhoods where the rank of the equations (3.5) jumps. Property (ii) reflects the functional equation aspect of Abel's theorem discussed above.

The second converse to Abel's theorem was first formulated and proved by Abel's fellow countryman Soplus Lie. We shall state it in a special case based on the picture below. Here we are given local analytic arcs F_i in the plane and on each F_i a non-zero regular differential ω_i. For L in a neighborhood U of L_0 in the space of lines in the plane, we may define a mapping

$$L \to F_1 \times \cdots \times F_n \tag{3.6}$$

by

$$L \to (p_1(L), \ldots, p_n(L))$$



where $p_i(L) = L \cdot F_i$. In this situation Abel's relation is

$$\sum_i \omega_i(p_i(L)) = 0\ ; \tag{3.7}$$

i.e., the pullback of $(\omega_1, \cdots \omega_n)$ on $F_1 \times \cdots \times F_n$ to U under the mapping (3.6) should be zero. Lie's result is:

Under the condition (3.7) there is a global algebraic curve F and regular differential ω on F such that

$$\begin{cases} F_i \subset F \\ \omega | F = \omega_i\ . \end{cases}$$

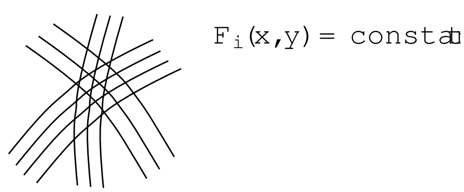
Again, using the functional equation (3.7) the local data (F_i, ω_i) may be propogated to give a global (F, ω).

4. Some legacies of Abel's theorem

Of course probably the main "legacy" of the theorem is the string of developments — by many mathematicians and continuing to modern times — leading to our understanding of the *Picard variety* or *divisor class group* of an algebraic variety. Abel's influence is reflected by the fact that the identity component of the Picard variety is an *abelian variety*, and at least in the complex case the functions on it are termed *abelian functions*. Rather than recount these developments in any detail, however, I will discuss briefly two other legacies. One is the interesting but less well known subject of *webs* and the other is based on recent joint work with Mark Green.

4.1. **Webs.** We will restrict to plane webs — however, the subject is of interest in any dimension and codimension. Also, although the definition may be given globally on manifolds, thus far the main interest is in the local geometry and so we shall work in an open set in $\mathbb{R}^2$.

Definition: *An n-web $\mathcal{W}(n)$ is given by n foliations in general position.*



The leaves of the i^{th} web are given by the level sets of a function $F_i(x, y)$; general position means that the tangent lines to the leaves through a point are distinct. It is sometimes convenient to give these tangent lines by a Pfaffian equation

$$\omega_i = 0$$

where

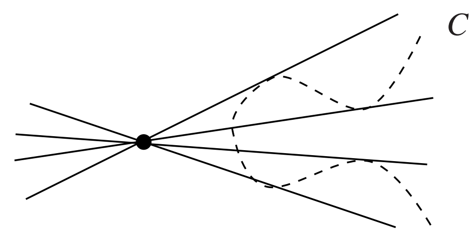
$$\omega_i = \lambda_i dF_i$$

for some non-zero function λ_i.

The subject of web geometry was initiated by Blaschke and his colleagues in Hamburg in the 1920's. A central problem was to find invariants of a web; in particular to find sufficient conditions that the web be *linearizeable*; i.e., after a diffeomorphism the leaves of the web should become lines in the plane.

From the beginning it was understood that webs were related to algebraic geometry. For example, the following picture gives a linear web

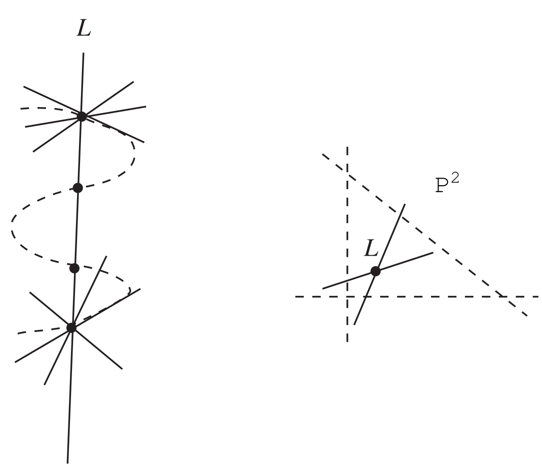
(i)



Here, C is an algebraic curve in the plane, and through a general outside point we draw the tangents to C (here, and similarly in what follows, for the purpose of illustration we assume that all the tangents are real). The *degree* n of the web in the usual algebro-geometric degree of the dual curve.

The projective dual of figure (i) associates to an algebraic curve C of degree n in the plane an n-web in an open set U in the dual projective space of lines in the plane, as illustrated by the figure

(ii)



Here, a point in U is given by a line L in the plane. The lines through each of the n points of intersection of C with L give n pencils of lines, and by projective duality each such pencil gives a line in the dual space.

An additional source of examples of web is provided by the solution curves to an ODE

$$P(x, y, y') = (y')^n + P_1(x, y)(y')^{n-1} + \cdots + P_n(x, y) = 0$$

in the plane.

What has turned out to be thus far the most important invariant of a web was defined already by the school of Blaschke:

Definition: *An abelian relation is given by*

$$\sum_i g_i(F_i)dF_i = 0 \tag{4.1}$$

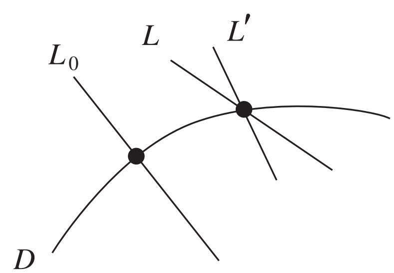
where $\mathbf{g}(\xi) = (g_1(\xi), \cdots g_n(\xi))$ *is a vector of functions of* 1*-variable. We denote by* $A(\mathcal{W})$ *the vector space of functions* $\mathbf{g}(\xi)$ *satisfying (4.1) and define the* **rank** $r(\mathcal{W})$ *of the web by*

$$r(\mathcal{W}) = \dim A(\mathcal{W}) \ .$$

As an example of an abelian relation we let ω be a non-zero differential of the first kind and consider first the local integral

$$I(L) = \int_{L_0 \cdot D}^{L \cdot D} \omega \tag{4.2}$$

as depicted by a local picture



where we restrict attention to an arc D on C and on an open neighborhood U of a line L_0 having one intersection point with the arc. Clearly in the above picture

$$I(L) = I(L') \ ;$$

i.e., $I(L)$ is constant on the pencil of lines through a fixed point on C. Thus the level sets of I define lines in U, and the integral curves of the differential $dI(L)$ are exactly these lines. Referring to figure (ii) and adding up this discussion over arcs around each of the intersection points we see from Abel's theorem that each ω gives an abelian relation. If, for example, C is non-singular and we set $h^0(\Omega^1_C) = \dim H^0(\Omega^1_C)$, we then have for the web $\mathcal{W}_C$ associated to C as in figure (ii)

$$h^0\left(\Omega^1_C\right) \leqq r(\mathcal{W}_C)\ . \tag{4.3}$$

A result from the Blaschke school is that for any n-web

$$r(\mathcal{W}(n)) \leqq (n-1)(n-2)/2\ . \tag{4.4}$$

For $\mathcal{W}(n) = \mathcal{W}_C$ as above it is well known that

$$h^0(\Omega^1_C) = (n-1)(n-2)/2$$

so that equality holds in (4.3). In general we say that a web $\mathcal{W}(n)$ has *maximum rank* if equality holds in (4.4). A central question in the subject is the

(4.5) **Problem:** *Determine all webs of maximum rank.*

Before discussing this problem we mention as another relation between web geometry and algebraic geometry that Sophus Lie's converse to Abel's theorem discussed above has the following consequence

(4.6) *A* **linear** *web with non-zero rank is algebraic*

Here it should be understood that the abelian relation is *complete* in the sense that each g_i is not identically equal to zero. The assertion means that it is the web associated to an algebraic curve by the construction in figure (ii).

Another remark is that it is sometimes useful to give an abelian relation (4.1) in integrated form as

$$\sum_i G_i(F_i(x,y)) = \text{constant} \tag{4.7}$$

where $G_i(\xi)$ are functions of ξ with $G'_i(\xi) = g_i(\xi)$.

Turning to the problem (4.5), for $n = 3$ we have

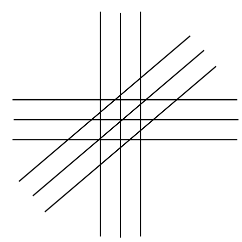
$$r(\mathcal{W}(3)) \leqq 1\ ,$$

and it was proved by Blaschke that if equality holds then the web is algebraic of type (ii) above. Moreover, the integrated form (4.7) of the abelian relation may be written as a functional equation for the logarithm

$$\varphi_1(x) - \varphi_1(y) + \varphi_1\left(\frac{y}{x}\right) = 0 \tag{4.8}$$

and up to a local diffeomorphism the web looks like

(iii)



For $n = 4$ we have

$$r(\mathcal{W}(4)) \leqq 3\ ,$$

and again if equality holds then the web is algebraic of type (ii) above. Using the result (4.6) this may be seen as follows: Writing a basis for the abelian relations as

$$\sum_j g_{ij}(F_j)dF_j = 0 \qquad i = 1, 2, 3 \tag{4.9}$$

we consider the matrix

$$\begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix} .$$

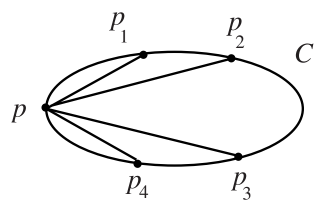
The rows give a basis for the abelian relations and the columns give a map

$$U \to \mathbb{P}^2 \tag{4.10}$$

that, using (4.9), maps $\mathcal{W}(3)$ to a *linear* web in an open set in the projective plane. We may now apply (4.6) to conclude the result.

For $n = 5$ we have $r(\mathcal{W}(5)) \leqq 6$ and a non-linearizable (and therefore non-algebraic) web of maximum rank was found by Bol; it may be pictured as

(iv)



Here, the p_i are fixed points in general position. Through a general variable point p the leaves of the foliation are the lines $L_i = \overline{pp_i}$ and the conic C passing through $pp_1 \cdots p_5$. Five of the six independent abelian relations are derived from Abel's theorem applied to figure (iv). The integrated form of the sixth may be expressed by

$$\varphi_2(x) - \varphi_2(y) + \varphi_2\left(\frac{y}{x}\right) - \varphi_2\left(\frac{1-y}{1-x}\right) + \varphi_2\left(\frac{x}{y}\frac{1-y}{1-x}\right) = 0 \ , \tag{4.11}$$

which we recognize as Abel's form of the functional equation for the dilogarithm. So again Abel enters our story, only this time from a completely different perspective.

For $n = 6, 7, 8$ there exist exceptional planar webs of maximum rank — i.e., webs of maximum rank not of type (ii) above. All are based on the dilogarithm. For example, for $n = 6$ there are two exceptional webs; one uses the six term relation for the dilogarithm, and the other the usual five term relation (4.11). For $n = 9$ Hénaut has shown that the trilogarithm appears as an abelian relation in a 9-web of maximum rank not composed of one of type (ii) with the Bol web. This leads to the obvious

Question: *Are all webs of maximum rank which are not algebraizable of this type?*

We do not attempt to formulate this question precisely — intuitively, we are asking whether or not for each k there is an integer $n(k)$ such that there is a "new" $n(k)$-web of maximum rank one of whose abelian relations is a (the?) functional equation with $n(k)$ terms for the k^{th} polylogarithm $\mathcal{L}i_k$? Here, "new" means the general extension of the phenomena above for the logarithm when $k = 1$, where $n(1) = 3$, for the Bol web when $k = 2$ and $n(2) = 5$, and for the Hénaut web when $k = 3$ and $n(3) = 9$.

4.2. **Abel's DE's for points on a surface.** The geometry of an algebraic variety is reflected by the configuration of its algebraic subvarieties. Stemming from Abel one has learned to study subvarieties modulo the relation of rational equivalence. That is, in a smooth complex algebraic variety X two subvarieties Z, Z' are *rationally equivalent* if there is a family $\{Z_t\}_{t \in \mathbb{P}^1}$ of subvarieties with $Z_0 = Z$, $Z_\infty = Z'$. Passing to the group $Z^p(X)$ of codimension-p algebraic cycles modulo the relation generated by rational equivalence one obtains the *Chow groups* $CH^p(X) = Z^p(X)/Z^p_{\mathrm{rat}}(X)$.

For X an algebraic curve, Abel's theorem and its converse give a complete set of Hodge-theoretic invariants for the identity component $CH^1(X)_0$ (which is of course the *Jacobian variety* of X). In general, $CH^1(X)$ is the Picard variety whose identity component is an abelian variety — the story has much the same general flavor as in the case of algebraic curves.

However, already for configurations of points on an algebraic surface the story is much different — since Mumford's result in the 1960's we know that $CH^2(X)$ may be infinite dimensional. A few years ago, motivated by Spencer Bloch's formula for the *formal* tangent space $T_f CH^2(X)$, Mark Green and I wanted to understand what geometric content might lie behind Spencer's formula. This led us to propose a geometric definition for the tangent space $TZ^2(X)$ (cf. the example below) and to then define the *geometric* tangent space

$$T_g CH^2(X) = TZ^2(X)/TZ^2_{\mathrm{rat}}(X) \tag{4.12}$$

where $TZ^2_{\mathrm{rat}}(X)$ is the tangent space to the subgroup of 0-cycles rationally equivalent to zero. It is then a theorem that

$$T_g CH^2(X) \cong T_f CH^2(X)\ . \tag{4.13}$$

We shall denote either of these simply by $TCH^2(X)$ and refer to this vector space as *the* tangent space to $CH^2(X)$.

Implicit in (4.12) and (4.13) are the infinitesimal geometric conditions that a configuration of points move to 1st order in a rational equivalence class. Recall that the condition a 0-cycle Z on X be rationally equivalent to zero is that

$$Z = \sum_{\nu} (f_\nu)$$

where f_ν is a rational function on an irreducible curve Y_ν and (f_ν) is its divisor. A 1st order variation of the data (Y_ν, f_ν) gives a 1st order variation of $\sum_\nu (f_\nu)$, and we seek the geometric conditions on a configuration of points $p_1 + \cdots + p_d$ (assumed for simplicity to be distinct) and tangent vectors $\tau_i \in T_{p_i}X$ to be a 1st order variation of $\sum_\nu (f_\nu)$.

The answer to the corresponding question for configurations of points on an algebraic curve is given by Abel's DE's (3.5). We shall now explain the answer in the case of an algebraic surface. For this we first observe that Abel's construction of the trace extends to differential forms of any degree on a smooth algebraic variety X; the formula

$$\omega(p_1 + \cdots + p_d) = \omega(p_1) + \cdots + \omega(p_d)$$

defines a map

$$H^0\left(\Omega^q_X\right) \xrightarrow{\text{Tr}} H^0\left(\Omega^q_{X^{(d)}}\right) .$$

If $\dim X \geqq 2$ the symmetric products are singular along the diagonal; and regular differential forms are then defined to be rational forms that are regular on any desingularization.

We now let p_i, τ_i be as above and set

$$\tau = \sum_i (p_i, \tau_i) \in TX^{(d)} .$$

The first set of conditions that

$$\tau \in TZ^2_{\text{rat}}(X) \tag{4.14}$$

are, as in the curve case, that

$$\langle \operatorname{Tr} \varphi, \tau \rangle = \sum_i \langle \varphi(p_i), \tau_i \rangle = 0 \tag{4.15}$$

for all regular 1-forms $\varphi \in H^0(\Omega^1_X)$. Equation (4.15) simply says that τ should be in the kernel of the differential of the Albanese map.

The new ingredient comes from the 2-forms on X. Already from the works of Mumford and Bloch one knew that the 2-forms are relevant; the following is a geometric explanation. First remark that if one considers any n-dimensional complex manifold X; e.g. an open set in $\mathbb{C}^n$, and if one then considers collections of forms $\varphi_d \in H^0(\Omega^q_{X^{(d)}})$ that have the *hereditary property*

$$\varphi_{d+1} \mid_{X^{(d)}} = \varphi_d$$

where the inclusion $X^{(d)} \hookrightarrow X^{(d+1)}$ is given by

$$p_1 + \cdots + p_d \to p + p_1 + \cdots + p_d$$

for some fixed point $p \in X$, then it is theorem that

(4.16) *The hereditary forms are generated as an exterior algebra by the traces of the q-forms on X where $0 \leqq q \leqq n = \dim X$. All of these forms are needed to generate.*

The geometric point is this: Taking X to be a germ of a neighborhood of a point in $\mathbb{C}^n$, were $X^{(d)}$ smooth then of course the 1-forms would generate the forms of all degrees. Exactly along the diagonals — which reflect the infinitesimal structure of X —- to generate we need forms of all degrees up to $\dim X$.

The other new ingredient is that

(4.17) *The field of definition of the $p_i \in X$ enters into the condition (4.14).*

To explain this we assume for simplicity of exposition that the algebraic surface X is defined over $\mathbb{Q}$ (or over a number field); e.g., we may think of $X \subset \mathbb{P}^N$ as being projected to $X^0 \subset \mathbb{P}^3$ where X^0 has an affine equation

$$f(x,y,z) = 0$$

where $f \in \mathbb{Q}[x,y,z]$. We may assume that x, y give local uniformizing parameters around $p_i = (x_i, y_i, z_i)$ and write

$$\tau_i = \lambda_i \frac{\partial}{\partial x} + \mu_i \frac{\partial}{\partial y} \ .$$

The regular 2-forms on X are given by the pullbacks to X of

$$\omega = \left. \frac{g(x,y,z) dx \wedge dy}{f_z(x,y,z)} \right|_{X^0} \tag{4.18}$$

where $\deg g \leqq \deg f - 4$ and g vanishes on the double curve of X^0. Since X is defined over $\mathbb{Q}$ we may take a basis for $H^0(\Omega^2_X)$ to be given by 2-forms (4.18) where $g \in \mathbb{Q}[x,y,z]$. Recalling that the *Kähler differentials*

$$\Omega^1_{\mathbb{C}/\mathbb{Q}}$$

are the complex vector space generated by expressions $\delta a, a \in \mathbb{C}$, modulo the relations

$$\begin{cases} \delta(a+b) & = \delta a + \delta b \\ \delta(ab) & = a\delta b + b \delta a \\ \delta a & = 0 \text{ if } a \in \mathbb{Q} \ , \end{cases}$$

we now *define*

$$\langle \omega(p_i), \tau_i \rangle = \frac{g(x_i, y_i, z_i)}{f_z(x_i, y_i, z_i)} \left(\mu_i \delta y_i - \lambda_i \delta x_i \right) \in \Omega^1_{\mathbb{C}/\mathbb{Q}} \tag{4.19}$$

and

$$\langle \operatorname{Tr} \omega, \tau \rangle = \sum_i \langle \omega(p_i), \tau_i \rangle \ . \tag{4.20}$$

Abel's DE's for the 2-forms are then defined to be the $\Omega^1_{\mathbb{C}/\mathbb{Q}}$-valued equations

$$\operatorname{Tr} \omega = 0 \tag{4.21}$$

where ω is as above. It is then a theorem that *the equations (4.15) and (4.21) define infinitesimal rational motion as explained above.*

(4.22) **Corollary:** *If $\omega(p_i) \neq 0$ and if the x_i, y_i are independent transcendentals, then (4.20) has no non-zero solutions.*

In other words, no matter how large d is the 0-cycle $z = p_1 + \cdots + p_d$ is rigid in its rational equivalence class. (This includes allowing rational motions of $(z + z') - z'$ for any $z' \in X^{(d')}$.) This result gives a proof of Mumford's theorem and provides rather precise meaning to the use of "generic" in Mumford's argument and the subsequent developments by Roitman, Voisin and others.

At the other extreme we have the

(4.23) **Corollary:** *If $x_i, y_i \in \bar{\mathbb{Q}}$, then (4.20) is zero for any choice of the τ_i.*

This is an infinitesimal version of a well known conjecture of Beilinson-Bloch — it gives a geometric existence result, albeit only to 1^{st} order. Understanding the "integration" of Abel's DE's (4.15) and (4.21) is a deep and fundamental question.[2]

One may quite reasonably ask how the essentially arithmetic object $\Omega^1_{\mathbb{C}/\mathbb{Q}}$ gets into the purely geometric question of tangents to arcs in the space of 0-cycles on an algebraic surface. The following example illustrates how this comes about.

Example: The issue already appears locally, so we consider the space of arcs $z(t)$ in $Z^2(\mathbb{C}^2)$. We may define an arc to be a finite linear combination with integer coefficients of analytic maps of the t-disc into the symmetric products $(\mathbb{C}^2)^{(d)}$. One may then define an equivalence relation $\sim$ on the space of arcs and the tangent space is the complex

[2] Integrating a DE means finding a solution by an iterative process. Since there are no derivations of $\mathbb{Q}$ the methods of calculus break down — one must break the problem into "increments" by some other means, perhaps either by an iteration process that at each stage decreases the "arithmetic complexity" of the 0-cycle, or by analyzing the DE's (4.15) and (9.20) in the completions of $\mathbb{Q}$ under all valuations.

vector space defined by

$$TZ^2(\mathbb{C}^2) = \{\text{arcs in } Z^2(\mathbb{C}^2)\}/\sim \ .$$

The equivalence relation $\sim$ is characterized by the properties:

(i) $z_i(t) \sim \tilde{z}_i(t)$ for $i = 1, 2$
$\Rightarrow z_1(t) \pm z_2(t) \sim \tilde{z}_1(t) \pm \tilde{z}_2(t)$;

(ii) $z(\alpha t) \sim \alpha z(t), \qquad \alpha \in \mathbb{Z}$

(iii) $\alpha z(t) \sim \alpha\tilde{z} \Rightarrow z(t) \sim \tilde{z}(t)$ for $\alpha \in \mathbb{Z}^*$; and

(iv) if $z(t), \tilde{z}(t)$ are arcs in $\operatorname{Hilb}_0(X)$ with the same tangent vector in $T\operatorname{Hilb}_0(X)$, then

$$z(t) \sim \tilde{z}(t) \ .$$

Now let

$$z_{\alpha\beta}(t) = \operatorname{Var}\left(x^2 - \alpha y^2, xy - \beta t\right) \ , \qquad \alpha \neq 0$$

and F be the free group generated by the 0-cycles

$$w_{\alpha\beta}(t) = z_{\alpha\beta}(t) - z_{1\beta}(t).$$

Then we have the result:

(4.24) *The map*

$$F/\sim \ \to \Omega^1_{\mathbb{C}/\mathbb{Q}}$$

given by

$$w_{\alpha\beta}(t) \to \beta\frac{\delta\alpha}{\alpha}$$

is a well-defined isomorpism.

As a non-obvious geometric corollary, we see that if α is a root of unity then

$$z_{\alpha\beta}(t) \sim z_{1\beta}(t) \ .$$

This result illustrates the very interesting and subtle interplay between geometry and arithmetic in higher codimension.

In summary, Abel's DE's (4.21) for the rational motion of configurations of points on a surface have an arithmetic/geoemtric character — the integration of these equations presents a major challenge (cf. footnote [(2)]).

5. Reprise

In the beginning we considered the integral (1.1)

$$\int y(x)dx$$

of an algebraic function; there we observed that at the time of Abel such integrals were seen as "highly transcendental" functions of the upper limit of integration, and Abel's great insight was to find a general recipe for generating simple relations among them. In recent years there has been renewed interest in the integrals (1.1), *exactly because they are generally transcendental.* Whether or not relations of the Abel type generate all such relations then leads into one of the deepest questions in arithmetic algebraic geometry. We shall now briefly discuss this.

For this we assume that the algebraic equation

$$f(x, y(x)) = 0$$

satisfied by $y(x)$ is defined over $\mathbb{Q}$ (or over a number field); i.e., $f(x, y) \in \mathbb{Q}[x, y]$. For $\xi_0, \xi \in \mathbb{Q}$ we set

$$u(\xi) = \int_{\xi_0}^{\xi} y(x)dx \ . \tag{5.1}$$

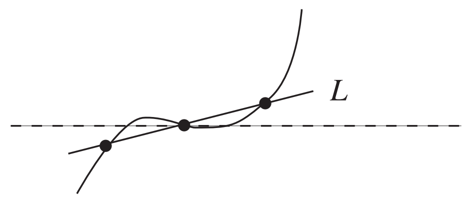
The transcendence properties of the numbers $u(\xi)$ have been studied over many years by many mathematicians, including Enrico Bombieri who in 1981 proved a result first enunciated by Siegel in 1929 which may informally be stated as follows:

(5.2) *Assume that $u(\xi)$ is not an algebraic function of ξ, and for convenience take $\xi_0 = 0$, assumed to be a regular value of $y(x)$. Then for each integer l there is a constant $C(l)$ such that if*

$$|\xi| < C(l)$$

then $u(\xi)$ does not satisfy an algebraic equation over $\mathbb{Q}$ of degree l.

This result also applies to the more general integrals (1.4), in particular to the integral of a differential of the first kind. Taking F to be a cubic curve and ξ_0 the x-coordinate of a flex, small perturbations of the flex tangent



lead by Abel's theorem to linear relations

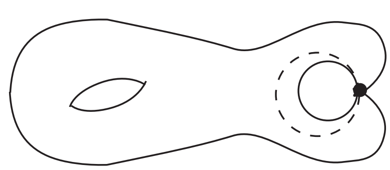
$$u(\xi_1) + u(\xi_2) + u(\xi_3) = 0 \tag{5.3}$$

for arbitrarily small $|\xi_i|$. Thus, thinking of the $u(\xi_i)$ as nearly transcendental numbers in the sense of (5.2), for geometric reasons stemming from Abel's theorem they satisfy algebraic relations defined over $\mathbb{Q}$ (actually, linear relations).

Integrals of the form

$$\int_{(\xi_0,\eta_0)}^{(\xi,\eta)} r(x, y(x))dx \tag{5.4}$$

where (ξ_0, η_0), $(\xi, \eta) \in F(\bar{\mathbb{Q}})$, together with their higher dimensional analytic analogues, are termed *periods* by Kontsevich and Zagier. Periods include the case when $(\xi, \eta) = (\xi_0, \eta_0)$; i.e., the integral around a closed loop $\gamma \in H_1(F, \mathbb{Z})$. In fact, by identifying (ξ, η) with (ξ_0, η_0) the integral (5.4) becomes an integral over a closed loop on a singular curve



The periods generate a countable field Π with

$$\bar{\mathbb{Q}} \subset \Pi \subset \mathbb{C}\,.$$

Kontsevich and Zagier point out that there is no known explicit example of a transcendental number that is not a period.

A general philosophy is

(5.5) *The relations of* Π *over* $\bar{\mathbb{Q}}$ *should be defined by geometric conditions.*

One example of this was just given. For another example, if we choose differentials $\omega_1, \dots, \omega_g$ of the 1st kind defined over $\bar{\mathbb{Q}}$ and which give a basis for $H^0(\Omega^1_F)$, and if $\gamma_1, \dots, \gamma_{2g} \in H_1(F, \mathbb{Z})$ is a canonical basis for the integral 1st homology, then the periods

$$\pi_{\alpha j} = \int_{\gamma_j} \omega_\alpha$$

satisfy the 1st Riemann bilinear relations

$$\sum \pi_{\alpha i} Q_{ij} \pi_{\beta j} = 0 \tag{5.6}$$

where $Q = \|Q_{ij}\|$ is the inverse of the intersection matrix. Geometrically, this relation arises from the class of the diagonal $\Delta \subset F \times F$. More generally, any generalized correspondence $T \subset \underbrace{F \times \cdots \times F}_{n}$ gives a polynomial relation of degree n over $\mathbb{Q}$. A beautiful and deep conjecture of Grothendeick is that *all* relations of the $\pi_{\alpha j}$ over $\mathbb{Q}$ arise in this way. In fact, Grothendeick conjectures the analogous statement for smooth varieties of any dimension defined over $\bar{\mathbb{Q}}$ and for all of the algebraic de Rham cohomology defined over $\bar{\mathbb{Q}}$.

I do not know a precise formulation of (5.5) which includes Grothendeick's conjecture — which is in some sense global — and relations of the Abel type — which are in some sense local although they arise from the global constraint $h^0(\Omega^1_{\mathbb{P}^1}) = 0$. In any case, taking into account the arithmetic aspect of Abel's DE's discussed in the preceeding section and the arithmetic questions concerning periods discussed above, I believe that one may with some confidence expect that *the arithmetic aspects of Abel's theorem and its legacies will be a central and deep topic for mathematicians in the third century after the time of Abel.*

6. Guide to the literature

Abel's famous "Paris memoiré" entitled *Memoiré sure une propriété générale d'une class très entendue des fonctions transcendantes* was presented to l'Academie des sciences à Paris in 1826 and published in t VII in 1841. It appears in Oeuvres complétes de Niels Henrik Abel, pages 145–211.

The paper referred to in footnote (1) is *sur l'integration de la formule differentielle $\rho dx/\sqrt{R}$, R and ρ étant des fonctions entières*, Oevres complétes, pages 104–144. The recent paper *Abel equations*, St. Petersburg Math. J., vol. 13 (2002), pages 1–45 by V. A. Malyshev gives an extension of Abel's result and a further guide to the literature.

The usual version of Abel's theorem and its (global) converse appears in standard books on Riemann surfaces, e.g., the famous *Die idee der Riemannischer flächen* by Herman Weyl.

The local converses, including the theorem of Soplus Lie and its extensions by Darboux and others, are discussed in the paper by the author *Variations on a theorem of Abel*, Inventiones Math., vol. 35 (1976), pages 321–390. The recent paper by G. Henkin, *Abelian differentials on singular varieties and variations on a theorem of Lie-Griffiths*, Invent. Math., vol. 135 (1991), pages 297–328 presents new results and references that have appeared after the paper mentioned above.

The theory of webs was first presented in the book *Geometrie der Geube. Topologische Frogen der Differentialgeometrie*, Springer, Berlin (1938) by W. Blaschke and G. Bol. Two recent works *Analytic web geometry*, Toulouse (1996), 6–47, World Sci. Publishing by A. Hénaut, and *Differential geometry of webs* in Handbook of differential geometry, vol. I, pages 1–152, North-Holland (2000) by M. Akivis and V. Goldberg give surveys of recent works and a further guide to the literature.

Abel's form of the functional equation for the dilogarithm is given in his paper *Note sur la fonction* $\psi x = x + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \cdots + \frac{x^n}{n^2} + \cdots$, Oeuvres complétes, pages 189–193.

Bloch's formula for $T_f CH^2(X)$ may be found in his *Lectures on algebraic cycles*, Duke Univ. Math. Ser. IV (1980).

A discussion of the extension of Abel's theorem to configurations of points on an algebraic surface may be found in *Abel's differential equations*, Houston J. of Math. (volume in honor of S. S. Chern), vol. 28 (2002), pages 329–351, by Mark Green and the author.

A general survey of the arithmetic properties of algebraic integrals is given in the paper *Periods*, Mathamatics unlimited – 2001 and beyond, Springer, Berlin (2001), pages 771–808 by M. Kontsevich and D. Zagier. The Bombieri-Siegel result appears in the paper by Bombieri *On G-functions*, Recent progress in analytic number theory, Vol. 2, Durham (1979), pages 1–67. The book *G-functions and geometry*, Aspects of mathematics, E/3, Friedr. Vieweg and Sohn, Brandenberg (1989) by I. André contains a "geometrization" of these issues, including a discussion of Grothendeick's conjecture.