JAN 30 2022. 

Dear Sarah and Valeriya,

The question that you ask about a "prime number theorem" for determinants of integral matrices is a very interesting one. I have thought about related problems but not this one. It can be tackled using linear groups via ergodic theory, or spectral theory of automorphic forms, or homogeneous dynamics as indicated below. The nature of the problem is clarified by formulating it in general.

Schinzel and Bateman/Horne put forth conjectures about prime values of polynomials in one variable. A recent extension of the Bateman/Horne Conjecture to several variables is formulated in Destagnol-Sofos [D-S]:

Let $f \in \mathbb{Z}[X_1, \ldots, X_n]$ be irreducible with leading part of degree $d$ and suitably non-singular. A compact box $B$ in $\mathbb{R}^n$ is positive if it satisfies

$$f(B) \subset (0, \infty) \quad \text{———} \quad (1)$$

For any box $B$ in $\mathbb{R}^n$ let

$$\pi_f(B) = \left| \{ x \in \mathbb{Z}^n \cap B \mid f(x) \text{ is prime} \} \right|. \qquad (2)$$

The conjecture is that for any positive $B$ the scaled boxes $TB$ satisfy

$$\pi_f(TB) \sim c_f \, \mathrm{Li}_f(TB) \quad \text{as } T \to \infty. \qquad (3)$$

Here $c_f$ is the product of local masses

$$c_f := \prod_p (1-p^{-1})\left(1 - p^{-n} \left| \{ x \in \mathbb{F}_p^n : f(x) = 0 \} \right| \right) \qquad (4)$$

(which converges under the assumptions on $f$), and as in Gauss' formulation of the prime number theorem

$$\mathrm{Li}_f(B) = \int_B \frac{dx}{\log f(x)} \, . \qquad (5)$$

It is natural to allow any box $B$ in the formulation of (3) with the only change needed being that the integral in (5) is restricted to $B \cap \{ x \mid f(x) > 0 \}$. To make this extension which we call (3'), one has to examine what happens when the points $x$ are close to the zero set of $f$.

Note that for satisfying (1)

$$Li_f(TB) \sim \frac{\lambda(B)}{d} \frac{T^n}{\log T} \quad \text{as } T \to \infty, \qquad (6)$$

where $\lambda(B)$ is the Lebesgue measure of $B$ normalized so that $\lambda([0,1]^n) = 1$.

$C_f \neq 0$ is a necessary condition for there being a Zariski dense set of primes $x$ (i.e. $x$'s for which $f(x)$ is a prime) and (3) implies that if $C_f \neq 0$ then this natural formulation of Schinzel's hypothesis for $f$'s in several variables is true; namely the set of prime's $x$ is Zariski dense in affine $n$-space iff $C_f \neq 0$.

Destagnol and Sofos develop the circle method following Birch [B] and establish Conjecture (3) when $n$ is suitably large in terms of $d$. Another setting for which one can establish (3) is in linear group setting for which the corresponding Linnik problem has been established.

For $f$ homogeneous (which we assume henceforth)

$$V_m \left(= V_m(f)\right) = \{ x : f(x) = m \}. \quad \text{—— (7)}$$

The Linnik problem asks about the distribution of the radial projections of $V_m(\mathbb{Z})$ onto $V_1(\mathbb{R})$. For $x \in V_m(\mathbb{Z})$ let $x' = x/m^{1/d}$ so that $x' \in V_1(\mathbb{R})$. Assuming that there are a suitably growing number of points on $V_m(\mathbb{Z})$ with $m$ increasing Linnik asks about the local equidistribution of these $x'$.

For a fixed nice compact set $K \subset V_1(\mathbb{R})$ this equidistribution in the cases where it has been established reads as follows:

$$\left| \{ x \in V_m(\mathbb{Z}) : x' \in K \} \right| \sim \omega(m) \mu_1(K), \text{ as } m \to \infty. \quad \text{—— (8)}$$

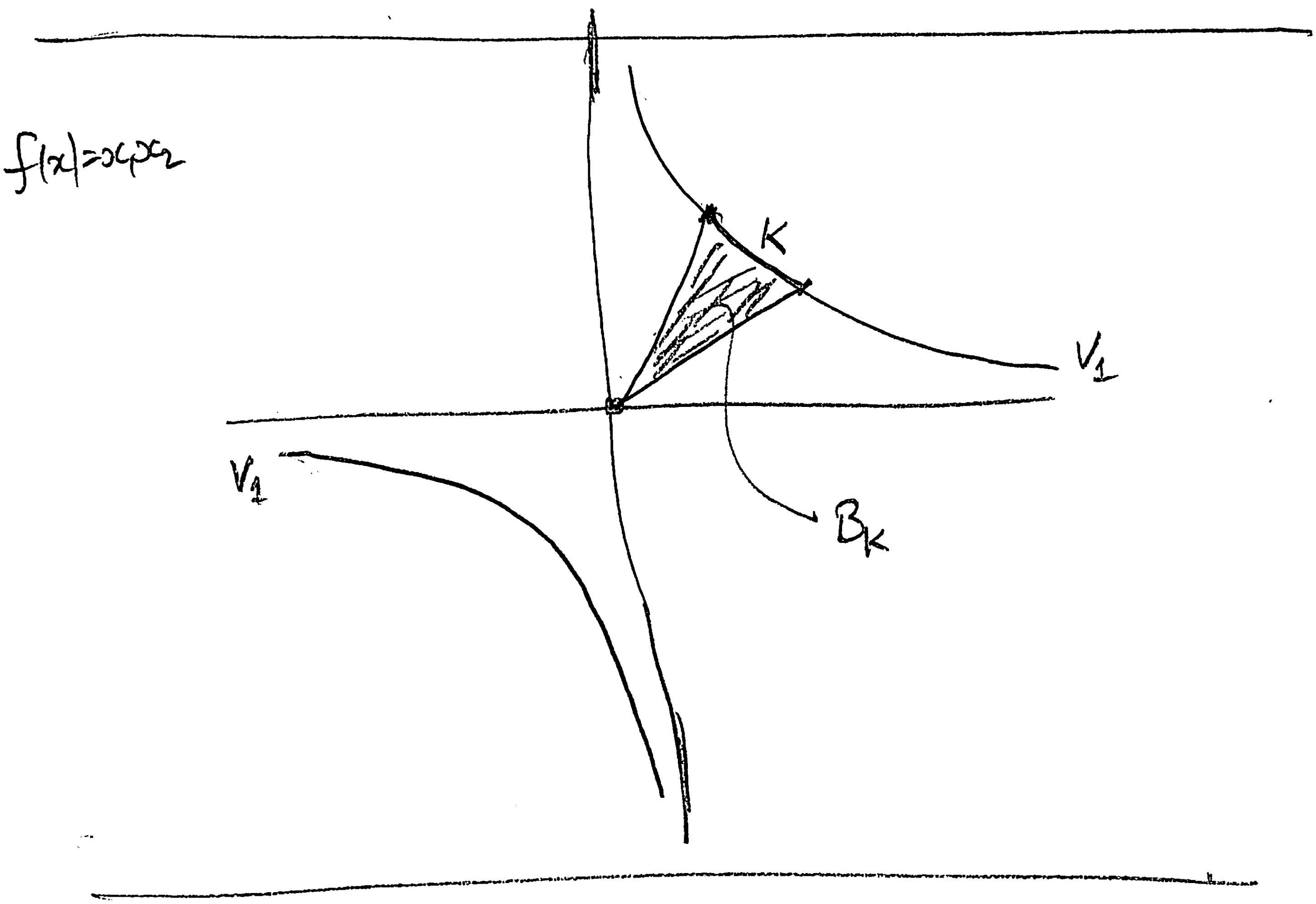
Here the $\omega(m)$ depends only on $m$ and not on $K$ and is given as a product of local masses as would be formally predicted by the Hardy–Littlewood circle method. The measure $\mu_1$ on $V_1(\mathbb{R})$ is a fixed positive number $\alpha = \alpha(f)$

times the Lebesgue measure of the cone at K. That is if $B_K$ is the cone

$$B_K = \{ tk \in \mathbb{R}^n ; 0 \le t \le 1, \ k \in K \} \qquad (9)$$

Then

$$\mu_1(K) = \alpha(f) \cdot \lambda(B_K). \qquad (10)$$




When the circle method can be applied to study $V_m(\mathbb{Z})$ as in Birch [B], (8) can be established. While being general they require that $n$ be much larger than $d$. For $f$'s which are invariants for a linear action $G$, Linnik developed an ergodic method to prove (8). The spectral method [Sa] establishes (8) quite generally and with a uniform power saving for the remainder terms (see [C-H-U], [G-O] and also [Te]). The most general cases for which (8) use homogeneous dynamics and in particular Ratner's theory of unipotent flows, and ~~to~~ due to Eskin and Oh [ ]. To date this method has not been made effective in terms of the remainder term.

Returning to (3), if $B = B_K$ is a positive cone as in (9) then for $T \geq 1$,

$$\pi_f(TB_K) = \left|\{x \in TB_K \mid f(x) \text{ is prime}\}\right|$$

$$= \left|\{x \in TB_K \mid x' \in K,\ f(x) \text{ is prime}\}\right|$$

$$= \sum_{p \leq T^d} \left|\{x \in V_p(\mathbb{Z}) : x' \in K\}\right| .$$

Hence if (8) holds we get that

$$\pi_f(TB_K) \sim \mu_1(K) \sum_{p \le T^d} \omega(p) \quad . \qquad (11)$$

and from (10) this

$$= \alpha\, \lambda(B_K) \sum_{p \le T^d} \omega(p) \qquad (12)$$

Thus (3) for such conic $B_K$'s is reduced to executing the sum on prime in (12). In many cases (such as the determinant case of interest) $\omega(m)$ is a multiplicative function of $m$, and the sum in (12) can be understood using L-functions. Even when $\omega$ is not multiplicative it is a product of local densities and the sum over primes can be analyzed. An example is $f(x_1,x_2,x_3) = x_1^2 + x_2^2 + x_3^2$, where Siegel's mass formula gives $\omega(m)$ in terms of Dirichlet L-functions at $s=1$. Using this the sum over $p$ can be executed easily as in Friedlander-Iwaniec [F-I]. Combining this with (8), which in this case was proved by Duke [D] yields (2) for this $f$ and positive cones $B_K$.

By a simple approximation argument, once one has (3) for cones one derives it for positive boxes and more general positive compact sets. If one is interested in extending (3) to (3') that is general boxes, one needs to control the count for points near $f=0$. Using elementary upper bound sieves (for example Selberg's $\Lambda^2$ bound) one can show that for any $B$ (and any $f$)

$$\overline{\lim_{T\to\infty}} \frac{\log T}{T^n} \sum_{\substack{x \in TB \\ f(x) \text{ a prime}}} 1 \ll \lambda(B), \tag{13}$$

the implied constant depending on $f$ only.

With this one can pass from (3) to (3'). This applies even in cases where (8) is only known ineffectively, for example when applying Ratner.

Finally turning to your case of

$$f(X) = \det X, \quad X \in \mathbb{Z}^{k^2} \text{ a } k \times k.$$

integral matrix. So $n = k^2$ and $d = k$. In this case (8) was first proved by Linnik and Skubenko [L-S] and mentioned before the spectral method gives it with a power saving. (there are also uniform bounds for the singular level set $m = 0$ [Ka]). The numbers $w(m)$ are multiplicative in $m$ and a polynomial in $p$ for $m$ a prime. For large $p$, $w(p) \sim p^{k-1}$. Hence by the usual prime number Theorem we have that

$$\pi_f(TB_K) \sim \mu_1(K) \frac{T^{k^2}}{k^2 \log T} \text{ as } T \to \infty. \qquad (14)$$

Moreover $\alpha(\det) = \dfrac{1}{k\,\zeta(2)\cdots\zeta(k)}$, ——— (15)

coming from $\mathrm{Vol}(SL_n(\mathbb{Z})\backslash SL_n(\mathbb{R}))$ as computed by Minkowski.

The constant $c_f$ can also be computed easily and is

$$c_f = \prod_p \left(1-\frac{1}{p^k}\right)\left(1-\frac{1}{p^{k-1}}\right)\cdots\left(1-\frac{1}{p^2}\right)$$

$$= \frac{1}{\zeta(2)\cdots\zeta(k)}. \qquad \text{——— (16)}$$

Putting these together we conclude that (3) and (3') hold for $f = \det$.

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With best regards

Peter