May 29, 1978

Dear Jim,

I have given a little thought to the suggestion of my previous letter. It seems to lead nowhere. I will set down the reasons below, but there is no point in reading them. They are just for my own record. Incidentally I heard from Hsiang that the PUP has decided to go ahead with your book. Apparently it will get in touch with you.

Consider the Shimura variety associated to the group of symplectic similitudes. I want to see what the conjecture implies about conjugation of special points. Take a torus T and the cocycle $c_\rho^{-1} = c_\rho^{-1}(\tau, \mu) = v\rho(v^{-2})$ as in p. 6.18 of the Märchen. Then we may, according to the conjecture take the f_τ on p. 6.19 to be the identity maps.

Then for any h and g_f, φ_τ is the composite

$$h \times g_f \to \operatorname{ad} v \circ h,\ v g_f v^{-1} \tau^{-1} = \operatorname{ad} v \circ h,\ v g_f b.$$

Let T' be the torus $v^{-1}Tv$. It is also defined over $\mathbb{Q}$. Let H factor through T, and let $h' = \operatorname{ad} v^{-1} \circ h^\tau$. Then

$$\psi_\tau(h', 1) = (h^\tau, vb) = (h^\tau, \tau^{-1})$$

Consequently,

$$\tau(h', 1) = \tau\varphi_\tau^{-1}(h^\tau, \tau^{-1}) = \mathfrak{F}(b\tau^{-1}b^{-1})\tau\varphi_\tau^{-1}(b^\tau, 1).$$

The second equality follows from part (b) of the conjecture. However, according to part (a)

$$\tau\varphi_\tau^{-1}(h^\tau, 1) = (h, 1)$$

Thus

$$\tau(h', 1) = (h, bv)$$

or

$$\tau^{-1}(h, 1) = (h', v^{-1}, b^{-1})$$

It remains to interpret this equation concretely. Within the full matrix algebra the centralizer of T is a commutative algebra E. Let E^* be the multiplicative group of E. Within E^* the cocycle $\{c_\rho^{-1}\}$ splits

$$c_\rho^{-1} = a\rho(a^{-1}).$$

Let

$$B = v^{-1}a.$$

Then B is a matrix over $\mathbb{Q}$.

Let A be the abelian variety defined by $\mathbb{R}^{2n}/\mathbb{Z}^{2n}$ multiplication by i being given by $h(1, -i) = C$, and let φ be the obvious isomorphism of $T(A)$ with $\mathbb{Z}_f^{2n}$. Thus (A, φ) is the special point $(h, 1)$. Let (A', φ') be defined in exactly the same way except that C is replaced by $C' = h^\tau(i, -1)$.

With no loss of generality we may suppose that B is integral. Consider the isogeny $A' \to A''$ defined by the commutativity of

$$\begin{tikzcd} T(A') \arrow[r, "\sim", "\varphi^1"'] \arrow[d] & \mathbb{Z}_f^{2n} \arrow[d, "B"] \\ T(A'') \arrow[r, "\sim", "\varphi^n"'] & \mathbb{Z}_f^{2n} \end{tikzcd}$$

Conversely

$$A'' = \mathbb{R}^{2n}/B^{-1}\mathbb{Z}^{2n},$$

multiplication being given by C'. The isomorphism φ'' is defined by $B : B^{-1}\mathbb{Z}^{2n} \to \mathbb{Z}^{2n}$. The polarization on A'' inherited from A' will have the property that

$$\langle \varphi''(x), \varphi''(y) \rangle = \langle Bx, By \rangle = \langle r^{-1}ax, r^{-1}ay \rangle = \langle a^*ax, y \rangle.$$

Since

$$\rho(a^*a) = \rho(a)^*\rho(a) = c_\rho^* c_\rho a^*a = \lambda_\rho a^*a$$

with λ_ρ a scalar, we may choose ϵ with

$$\lambda_\rho = \epsilon\rho(\epsilon^{-1})$$

and define a new polarization by composing the old with $(\epsilon a^*a)^{-1}$.

The pair A'', φ'' is the special point defined by $(h', 1)$. To see this we multiply by B to take

$$\mathbb{R}^{2n} \to \mathbb{R}^{2n} \qquad B^{-1}\mathbb{Z}^{2n} \to \mathbb{Z}^{2n} \qquad C^1 \to BC^1B^{-1} = h'(i, -i).$$

Finally to obtain the point corresponding to $(h', v^{-1}b^{-1})$, we define A''' and φ''' by

$$\begin{tikzcd} T(A'') \arrow[r, "\varphi''"] \arrow[d] & \mathbb{Z}_f^{2n} \arrow[d, "b_v"] \\ T(A''') \arrow[r, "\varphi'''"'] & \mathbb{Z}_f^{2n} \end{tikzcd}$$

In other words A''', φ''' can be obtained by isogeny from (A', φ'), according to the diagram

$$\begin{tikzcd} T(A') \arrow[r, "\varphi'"] \arrow[d] & \mathbb{Z}_f^{2n} \arrow[d, "ba"] \\ T(A''') \arrow[r, "\varphi'''"'] & \mathbb{Z}_f^{2n} \end{tikzcd}$$

Now the isogeny deformed by ba is simple enough, for b and a both lie in E^*. The difficulty is that A' is defined by C^1 and not by C, and there is no way I can see of comparing A' and A after reduction modulo a prime—unless of course $C^1 = C$, but this is the standard case.

I hope you wasted no time with my suggestions.

Yours,

Bob

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