October 25, 1972

Borel,

I justify my observation to the effect that:

*If $G$ is semi-simple, $\pi$ irreducible unitary, $\Lambda - g$ the highest weight of $\tilde{\mu}$ ($2g = \sum_{\alpha>0}\alpha$), $G/K$ a bounded symmetric domain, and $\Lambda$ sufficiently non-singular, then $H^q(\pi,\mu) = 0$ unless $q = \frac{1}{2}\dim(G/K)$, $\pi$ is a member $\pi_s\Lambda$, $s \in \Omega_{\mathbb{C}}$, of the Weyl group when $H^q(\pi,\mu) \cong \mathbb{C}$.*

I first reinterpret some results of Matsushima-Murakami (Osaka J. Math 1965) representation-theoretically. $s$ unitary, acts on $U$, $U^\infty$ infinitely differentiable vectors. $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ as usual. The complex structure on $G/K$ determines a decomposition $\mathfrak{m}' = \mathfrak{m}'_+ + \mathfrak{m}'_-$ if $\mathfrak{m}' = \mathrm{Hom}_{\mathbb{R}}(\mathfrak{m},\mathbb{C})$. $C^n(U^\infty,\mu)$ is the set of $K$-invariant elements in $U^\infty \otimes L(\mathbb{C}) \otimes \Lambda^n\mathfrak{m}'$. Thus

$$C^n(U^n,\mu) = \bigoplus_{p+q=n} C^{p,q}(U^\infty,\mu)$$

with

$$C^{p,q}(U^\infty,\mu) = U^\infty \otimes L(\mathbb{C}) \otimes \Lambda^p\mathfrak{m}'_+ \otimes \Lambda^q\mathfrak{m}'_-.$$

If $s = \pi$ is unitary, irreducible, $\pi(\omega) = \mu(\omega)$ then $C^n(U^\infty,\mu) = H^n(\pi,\mu)$ and there is a Hodge decomposition.

Assume now, it is the interesting case, that $s = \pi$, $\pi(\omega) = \mu(\omega)$. Fix a Cartan subalgebra $\mathfrak{A}$ of $\mathfrak{G}$ contained in $k$ and an order on the roots so that the annihilator of $\mathfrak{m}'_+$ in $\mathfrak{m} \otimes \mathbb{C}$ is spanned by the positive roots. According to Proposition 10.1 of Matsushima-Murakami the space $L(\mathbb{C}) \otimes \Lambda\mathfrak{m}'_+$ is a direct sum

$$\left(\bigoplus_{\omega\in\Omega'} X_\omega\right) \oplus X$$

of spaces invariant under $K$. $X$ is of no importance.

$$\Omega' = \{\omega \in \Omega \mid \omega^{-1} \text{ takes positive compact roots to positive roots}\}.$$

The lowest weight of the representation of $k$ on $X_\omega$ is $-\omega\wedge + g$. As in their paragraph 11 one deduces the very strong vanishing theorem:

$H^{p,q}(\pi,\mu) = 0$ *if* $p + q \neq \pm\dim G/K$ *provided* $(\Lambda,\alpha) > (q,\alpha)$ *for all positive roots* $\alpha$.

My notation (in particular my $\Lambda$) differs from theirs. To treat the case $p + q = N$, I reformulate things in the language of people like Okamoto and Schmid although this is probably ultimately unnecessary.

I mention first a general type of cohomology problem which might interest you. It has been studied by M.S. Osborne. $G$ : real semi-simple (or reductive), $\mathfrak{g}$ its Lie algebra, $\mathfrak{h}$ a reductive subalgebra, the Lie algebra of $H$, $\mathfrak{p}$ a parabolic subalgebra of $\mathfrak{g} \otimes \mathbb{C}$, $\eta$: nilpotent radical of $\mathfrak{p}$. We suppose $\mathfrak{h} \otimes \mathbb{C}$ is the reductive part of $\mathfrak{p}$. $s$ irreducible, admissible representation of $G$ on Banach space $U$. $U^\infty$ defined in obvious manner. $\mathfrak{g} \otimes \mathbb{C}$ acts on $U^\infty$. Consider Lie algebra cohomology of $\mathfrak{n}$ on $U^\infty$. $C^q(\mathfrak{n},U^\infty)$ standard cochain complex. Semi-norm $\|Xu\|$, $X \in U(\mathfrak{g})$

gives topology on $U^\infty$ and hence on $C^q(\mathfrak{n}, U^\infty)$. Are coboundaries closed? Then $H^q(\mathfrak{n}, U^\infty)$ is a Banach space on which $\mathfrak{h}$ acts. Does it have a finite composition series? If so, what is the relationship between $s$ and the representations in the composition series?

In any case Okamoto-Ozeki (Osaka J. Math.) and Okamoto-Narasimhan (Annals) study these for our original $\mathfrak{g}$ when $\mathfrak{h} = \mathfrak{a}$, the Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{k}$ and $\mathfrak{n} = \mathfrak{m}_+$ with $s = \pi$ unitary. They study rather, if $\tau_\lambda$ is an irreducible representation of $K$ on $V_\lambda$ with highest weight $\lambda$,

$$\operatorname{Hom}_K(V_\lambda, H^q(\mathfrak{m}_+, U^\infty))$$

which is the cohomology of the complex

$$\operatorname{Hom}_K(V_\lambda, C^q(\mathfrak{m}_+, U^\infty))$$

or, if $\tilde{V}_\lambda = \operatorname{Hom}_{\mathbb{C}}(V_\lambda, \mathbb{C})$, of the $K$-invariant elements in

$$\tilde{V}_\lambda \otimes C^q(\mathfrak{m}_+, U^\infty).$$

In Okamoto-Ozeki, a Lapacian $\square$ is introduced. On this complex it equals a scalar

$$\frac{1}{2}\{(\lambda + g, \lambda + g) - (g, g) - \pi(\omega)\}$$

if $\omega =$ Casimir operator. Thus the cohomology groups are 0 or the boundary operator is 0. Comparing this with the results of Matsushima-Murakami we see that $H^{p,q}(U^\infty, \mu)$ is isomorphic to

$$(*) \qquad \bigoplus_{\substack{\omega \in \Omega' \\ \eta(\omega) = p}} \operatorname{Hom}_K(V_{\lambda(\omega), H^q(\mathfrak{m}_+, U^\infty)})$$

where $\lambda(\omega) = \omega \wedge -g$. $\eta(\omega)$ is defined in Matsushima-Murakami. Note these results (in §§5, 11, 12) imply in particular that if $\pi(\omega) = \mu(\omega)$ and as on p. 1 of this letter

$$L(\mathbb{C}) \otimes \Lambda\mathfrak{m}'_+ = \left(\bigoplus_{\omega \in \Omega'} X_\omega\right) \oplus X$$

then there are no $K$-invariant elements in $U^\infty \otimes X \otimes \Lambda\mathfrak{m}_-^-$.

Schmid however considers the general problem above with $\mathfrak{h} = \mathfrak{a}$ (as above), $\mathfrak{n}$ chosen to contain $\mathfrak{m}_+$, the subalgebra spanned by positive roots.

The groups $H^q(\mathfrak{m}_+, U^\infty)$ are finite-dimensional. Let $\mathfrak{n}_0 = \mathfrak{n} \cap (\mathfrak{k} \otimes \mathbb{C})$. According to my notes (not always reliable) a spectral sequence argument + finite-dimensional Borel-Weil shows that

$$H^n(\mathfrak{n}, U^n) \cong \bigoplus_{p+q=n} H^p(\mathfrak{n}_0, H^q(\mathfrak{m}_+, U^\infty))$$

This must be in any case a generally accepted fact. Let $T$ be the Cartan subgroup with Lie algebra $\mathfrak{a}$, and let $\lambda$ be a highest weight of $\mathfrak{a}$ with respect to $k$ (not $\mathfrak{g}$). Let $W_\lambda$ be the corresponding 1-dimensional representation of $T$. By Borel-Weil again

$$\operatorname{Hom}_T(W_\lambda, H^n(\mathfrak{n}, U^\infty)) \cong \bigoplus_{\lambda_i = \lambda} \mathbb{C}$$

if as a $K$-module

$$H^q(\mathfrak{m}_+, U^\infty) \cong \bigoplus_i V_{\lambda_i}$$

But Schmid (Annals '71) shows that if $\lambda$ is sufficiently regular the left-side is 0 unless $\pi = \pi_{\lambda+g}$ in the discrete series. If $\Lambda$ is sufficiently regular so are the $\lambda(\omega)$ occurring in $(*)$ and $\lambda(\omega) + g = \omega\Lambda$. Retracing our steps backward from Schmid to Matsushima-Murakami via Okamoto-Ozeki, we get all the required vanishing.

To completely justify my assertion I have only to show that for sufficiently non-singular $\Lambda$, if $\pi = \pi_{\omega\Lambda}$ the $H^n(\pi, \mu) \cong \mathbb{C}$.

It may be over-elaborate but I use the weak form of Blattner's conjecture has given in Schmid (Rice Studies) Theorem 2 to compute

$$(**) \qquad \mathrm{Hom}_K(\tilde{L}(\mathbb{C}) \otimes \Lambda^p \mathfrak{m}_+ \otimes \Lambda^q \mathfrak{m}_-, U^\infty)$$

where

$$\tilde{L}(\mathbb{C}) = \mathrm{Hom}(L(\mathbb{C}), \mathbb{C})$$

Suppose $s \in \Omega_\mathbb{R}$, the real Weyl group, and

$$s\mu + s\rho_K - (\omega\Lambda + \rho_p) = \sum \delta$$

where $\delta$ are positive with respect to an order putting $\omega\Lambda$ in the positive chamber. With respect to this order $\rho_K$ and $\rho_p$ have the same meaning as in Schmid, so $\rho = \rho_K + \rho_p = \omega g$. If, with respect to this order, $\mu$ is the highest weight of a representation occurring in $(**)$ (the first term) then $s\mu$ is a weight and

$$s\mu = \omega\Lambda - \omega g = \sum \gamma + \sum_{i=1}^{p} \alpha_i + \sum_{j=1}^{q} \beta_j$$

where $\gamma$ are positive in this new order, $\alpha_i$ are positive and distinct in the old order, $\beta_i$ are distinct and negative in the old order. Since $\omega g = \rho$ and $s\mu = \omega\Lambda + \rho_p - s\rho_K + \sum \delta$ we have

$$\sum \delta + \sum \gamma + \{\rho_K + s\rho_p\} + \{2\rho_p - \sum \alpha_i - \sum \rho_i\} = 0.$$

The two sums in brackets are sums of roots positive in the new order. Thus the sums over $\gamma$ and $\delta$ are 0 and $s\rho_K = s$ so that $s = 1$. Moreover

$$2\rho_p = \sum \alpha_i + \sum \beta_i$$

This determines $\mu$ uniquely. It is equal to

$$\omega\Lambda - \omega g + 2\rho_p = \omega\Lambda + \rho_p - \rho_K$$

Thus there is at most one $\lambda$ such that $\tau_\lambda$ occurs in the representation $\tilde{L}(\mathbb{C}) \otimes \Lambda^p \mathfrak{m}_+ \otimes \Lambda^q \mathfrak{m}_-$ for which

$$\mathrm{Hom}_K(V_\lambda, U^\infty) \neq 0$$

and for it the Hom is of dimension 1.

I hope that with this my work is justified.

Compiled on August 26, 2019 6:18pm -04:00