

October 25, 1972

Borel,

I justify my observation to the effect that:

*If  $G$  is semi-simple,  $\pi$  irreducible unitary,  $\Lambda - g$  the highest weight of  $\tilde{\mu}$  ( $2g = \sum_{\alpha > 0} \alpha$ ),  $G/K$  a bounded symmetric domain, and  $\Lambda$  sufficiently non-singular, then  $H^q(\pi, \mu) = 0$  unless  $q = \frac{1}{2} \dim(G/K)$ ,  $\pi$  is a member  $\pi_s \Lambda$ ,  $s \in \Omega_{\mathbb{C}}$ , of the Weyl group when  $H^q(\pi, \mu) \cong \mathbb{C}$ .*

I first reinterpret some results of Matsushima-Murakami (Osaka J. Math 1965) representation-theoretically.  $s$  unitary, acts on  $U$ ,  $U^\infty$  infinitely differentiable vectors.  $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$  as usual. The complex structure on  $G/K$  determines a decomposition  $\mathfrak{m}' = \mathfrak{m}'_+ + \mathfrak{m}'_-$  if  $\mathfrak{m}' = \text{Hom}_{\mathbb{R}}(\mathfrak{m}, \mathbb{C})$ .  $C^n(U^\infty, \mu)$  is the set of  $K$ -invariant elements in  $U^\infty \otimes L(\mathbb{C}) \otimes \Lambda^n \mathfrak{m}'$ . Thus

$$C^n(U^\infty, \mu) = \bigoplus_{p+q=n} C^{p,q}(U^\infty, \mu)$$

with

$$C^{p,q}(U^\infty, \mu) = U^\infty \otimes L(\mathbb{C}) \otimes \Lambda^p \mathfrak{m}'_+ \otimes \Lambda^q \mathfrak{m}'_-.$$

If  $s = \pi$  is unitary, irreducible,  $\pi(\omega) = \mu(\omega)$  then  $C^n(U^\infty, \mu) = H^n(\pi, \mu)$  and there is a Hodge decomposition.

Assume now, it is the interesting case, that  $s = \pi$ ,  $\pi(\omega) = \mu(\omega)$ . Fix a Cartan subalgebra  $\mathfrak{A}$  of  $\mathfrak{G}$  contained in  $\mathfrak{k}$  and an order on the roots so that the annihilator of  $\mathfrak{m}'_+$  in  $\mathfrak{m} \otimes \mathbb{C}$  is spanned by the positive roots. According to Proposition 10.1 of Matsushima-Murakami the space  $L(\mathbb{C}) \otimes \Lambda \mathfrak{m}'_+$  is a direct sum

$$\left( \bigoplus_{\omega \in \Omega'} X_\omega \right) \oplus X$$

of spaces invariant under  $K$ .  $X$  is of no importance.

$$\Omega' = \{\omega \in \Omega \mid \omega^{-1} \text{ takes positive compact roots to positive roots}\}.$$

The lowest weight of the representation of  $\mathfrak{k}$  on  $X_\omega$  is  $-\omega \wedge +g$ . As in their paragraph 11 one deduces the very strong vanishing theorem:

$$H^{p,q}(\pi, \mu) = 0 \text{ if } p + q \neq \pm \dim G/K \text{ provided } (\Lambda, \alpha) > (q, \alpha) \text{ for all positive roots } \alpha.$$

My notation (in particular my  $\Lambda$ ) differs from theirs. To treat the case  $p + q = N$ , I reformulate things in the language of people like Okamoto and Schmid although this is probably ultimately unnecessary.

I mention first a general type of cohomology problem which might interest you. It has been studied by M.S. Osborne.  $G$ : real semi-simple (or reductive),  $\mathfrak{g}$  its Lie algebra,  $\mathfrak{h}$  a reductive subalgebra, the Lie algebra of  $H$ ,  $\mathfrak{p}$  a parabolic subalgebra of  $\mathfrak{g} \otimes \mathbb{C}$ ,  $\eta$ : nilpotent radical of  $\mathfrak{p}$ . We suppose  $\mathfrak{h} \otimes \mathbb{C}$  is the reductive part of  $\mathfrak{p}$ .  $s$  irreducible, admissible representation of  $G$  on

---

William Casselman rewrote this letter of Langlands by hand. This typed version is based on Casselman's handwritten version, not the original one by Langlands.

Banach space  $U$ .  $U^\infty$  defined in obvious manner.  $\mathfrak{g} \otimes \mathbb{C}$  acts on  $U^\infty$ . Consider Lie algebra cohomology of  $\mathfrak{n}$  on  $U^\infty$ .  $C^q(\mathfrak{n}, U^\infty)$  standard cochain complex. Semi-norm  $\|Xu\|$ ,  $X \in U(\mathfrak{g})$  gives topology on  $U^\infty$  and hence on  $C^q(\mathfrak{n}, U^\infty)$ . Are coboundaries closed? Then  $H^q(\mathfrak{n}, U^\infty)$  is a Banach space on which  $\mathfrak{h}$  acts. Does it have a finite composition series? If so, what is the relationship between  $s$  and the representations in the composition series?

In any case Okamoto-Ozeki (Osaka J. Math.) and Okamoto-Narasimhan (Annals) study these for our original  $\mathfrak{g}$  when  $\mathfrak{h} = \mathfrak{a}$ , the Cartan subalgebra of  $\mathfrak{g}$  contained in  $\mathfrak{k}$  and  $\mathfrak{n} = \mathfrak{m}_+$  with  $s = \pi$  unitary. They study rather, if  $\tau_\lambda$  is an irreducible representation of  $K$  on  $V_\lambda$  with highest weight  $\lambda$ ,

$$\mathrm{Hom}_K(V_\lambda, H^q(\mathfrak{m}_+, U^\infty))$$

which is the cohomology of the complex

$$\mathrm{Hom}_K(V_\lambda, C^q(\mathfrak{m}_+, U^\infty))$$

or, if  $\tilde{V}_\lambda = \mathrm{Hom}_{\mathbb{C}}(V_\lambda, \mathbb{C})$ , of the  $K$ -invariant elements in

$$\tilde{V}_\lambda \otimes C^q(\mathfrak{m}_+, U^\infty).$$

In Okamoto-Ozeki, a Lapacian  $\square$  is introduced. On this complex it equals a scalar

$$\frac{1}{2}\{(\lambda + g, \lambda + g) - (g, g) - \pi(\omega)\}$$

if  $\omega =$  Casimir operator. Thus the cohomology groups are 0 or the boundary operator is 0. Comparing this with the results of Matsushima-Murakami we see that  $H^{p,q}(U^\infty, \mu)$  is isomorphic to

$$(*) \quad \bigoplus_{\substack{\omega \in \Omega' \\ \eta(\omega) = p}} \mathrm{Hom}_K(V_{\lambda(\omega)}, H^q(\mathfrak{m}_+, U^\infty))$$

where  $\lambda(\omega) = \omega \wedge -g$ .  $\eta(\omega)$  is defined in Matsushima-Murakami. Note these results (in §§5, 11, 12) imply in particular that if  $\pi(\omega) = \mu(\omega)$  and as on p. 1 of this letter

$$L(\mathbb{C}) \otimes \Lambda \mathfrak{m}'_+ = \left( \bigoplus_{\omega \in \Omega'} X_\omega \right) \oplus X$$

then there are no  $K$ -invariant elements in  $U^\infty \otimes X \otimes \Lambda \mathfrak{m}^-$ .

Schmid however considers the general problem above with  $\mathfrak{h} = \mathfrak{a}$  (as above),  $\mathfrak{n}$  chosen to contain  $\mathfrak{m}_+$ , the subalgebra spanned by positive roots.

The groups  $H^q(\mathfrak{m}_+, U^\infty)$  are finite-dimensional. Let  $\mathfrak{n}_0 = \mathfrak{n} \cap (\mathfrak{k} \otimes \mathbb{C})$ . According to my notes (not always reliable) a spectral sequence argument + finite-dimensional Borel-Weil shows that

$$H^n(\mathfrak{n}, U^\infty) \cong \bigoplus_{p+q=n} H^p(\mathfrak{n}_0, H^q(\mathfrak{m}_+, U^\infty))$$

This must be in any case a generally accepted fact. Let  $T$  be the Cartan subgroup with Lie algebra  $\mathfrak{a}$ , and let  $\lambda$  be a highest weight of  $\mathfrak{a}$  with respect to  $k$  (not  $\mathfrak{g}$ ). Let  $W_\lambda$  be the corresponding 1-dimensional representation of  $T$ . By Borel-Weil again

$$\mathrm{Hom}_T(W_\lambda, H^n(\mathfrak{n}, U^\infty)) \cong \bigoplus_{\lambda_i = \lambda} \mathbb{C}$$

if as a  $K$ -module

$$H^q(\mathfrak{m}_+, U^\infty) \cong \bigoplus_i V_{\lambda_i}$$

But Schmid (Annals '71) shows that if  $\lambda$  is sufficiently regular the left-side is 0 unless  $\pi = \pi_{\lambda+g}$  in the discrete series. If  $\Lambda$  is sufficiently regular so are the  $\lambda(\omega)$  occurring in (\*) and  $\lambda(\omega) + g = \omega\Lambda$ . Retracing our steps backward from Schmid to Matsushima-Murakami via Okamoto-Ozeki, we get all the required vanishing.

To completely justify my assertion I have only to show that for sufficiently non-singular  $\Lambda$ , if  $\pi = \pi_{\omega\Lambda}$  the  $H^n(\pi, \mu) \cong \mathbb{C}$ .

It may be over-elaborate but I use the weak form of Blattner's conjecture has given in Schmid (Rice Studies) Theorem 2 to compute

$$(**) \quad \text{Hom}_K(\tilde{L}(\mathbb{C}) \otimes \Lambda^p \mathfrak{m}_+ \otimes \Lambda^q \mathfrak{m}_-, U^\infty)$$

where

$$\tilde{L}(\mathbb{C}) = \text{Hom}(L(\mathbb{C}), \mathbb{C})$$

Suppose  $s \in \Omega_{\mathbb{R}}$ , the real Weyl group, and

$$s\mu + s\rho_K - (\omega\Lambda + \rho_p) = \sum \delta$$

where  $\delta$  are positive with respect to an order putting  $\omega\Lambda$  in the positive chamber. With respect to this order  $\rho_K$  and  $\rho_p$  have the same meaning as in Schmid, so  $\rho = \rho_K + \rho_p = \omega g$ . If, with respect to this order,  $\mu$  is the highest weight of a representation occurring in (\*\*) (the first term) then  $s\mu$  is a weight and

$$s\mu = \omega\Lambda - \omega g = \sum \gamma + \sum_{i=1}^p \alpha_i + \sum_{j=1}^q \beta_j$$

where  $\gamma$  are positive in this new order,  $\alpha_i$  are positive and distinct in the old order,  $\beta_i$  are distinct and negative in the old order. Since  $\omega g = \rho$  and  $s\mu = \omega\Lambda + \rho_p - s\rho_K + \sum \delta$  we have

$$\sum \delta + \sum \gamma + \{\rho_K + s\rho_p\} + \{2\rho_p - \sum \alpha_i - \sum \rho_i\} = 0.$$

The two sums in brackets are sums of roots positive in the new order. Thus the sums over  $\gamma$  and  $\delta$  are 0 and  $s\rho_K = s$  so that  $s = 1$ . Moreover

$$2\rho_p = \sum \alpha_i + \sum \beta_i$$

This determines  $\mu$  uniquely. It is equal to

$$\omega\Lambda - \omega g + 2\rho_p = \omega\Lambda + \rho_p - \rho_K$$

Thus there is at most one  $\lambda$  such that  $\tau_\lambda$  occurs in the representation  $\tilde{L}(\mathbb{C}) \otimes \Lambda^p \mathfrak{m}_+ \otimes \Lambda^q \mathfrak{m}_-$  for which

$$\text{Hom}_K(V_\lambda, U^\infty) \neq 0$$

and for it the Hom is of dimension 1.

I hope that with this my work is justified.

Compiled on July 12, 2022 11:41am -04:00