June 12, 1974

Contents

Letter

Dear Meiki,

What I want to do in these letters—there may be more than one—is to formulate exactly the conjecture of the appendix, which, as it stands, is not only a little vague but also, in various places, incorrect, and to prove it for certain Shimura "varieties" which can be realized as coarse moduli schemes for problems whose data involve only polarizations, endomorphisms, and points of finite order.

The proof then will involve the formulation of the moduli problem M, the verification that its course moduli scheme exists, and the study of $M(\bar{k}) = M(\operatorname{Spec}\bar{k})$. Here $\bar{k}$ is the algebraic closure of the finite field k and a morphism of $\operatorname{Spec} k$ into the base scheme for the moduli problem is given. There are presumably two aspects to the study of the set $M(\bar{k})$. It must first be decomposed into isogeny classes and then the structure of the individual isogeny classes must be analyzed.

You will find that the letter degenerates very rapidly. This is because on the whole it is less an attempt to explain an idea to you than to justify the idea to myself. This relieves you of any responsibility to read it.

We suppose we have been given the following:

(i) A semi-simple algebra L over $\mathbb{Q}$ with positive involution $\ell \to \ell^*$.
(ii) An order O_L in L.
(iii) An algebraic number field E of finite degree over $\mathbb{Q}$ and an order O_E in E.
(iv) A free finitely generated module $V = V(\mathbb{Z})$. We may extend ψ to $V(\mathbb{Q})$, on which L acts; we demand that

$$\psi(\ell u, v) = \psi(u, \ell^* v).$$

(v) An exact sequence

$$0 \to U \to V \otimes O_E \to W \to 0$$

where U and W are finitely generated, locally free O_E-modules. If τ_U and τ_W denote the traces of representations of O_L on U and W at best τ_U and τ_W may be extended to maps $L \to E$. We demand that

$$\tau_W(\ell) = \tau_U(\ell^*),$$

that $\tau_W(\ell)$ lie in O_E if ℓ lies in O_L, and that ψ is zero on U.

(vi) If Z is the centre of L and Z^0 consists of the elements of Z fixed by the involution, and if $\underline{Z}^0$ and $\underline{\mathbb{Q}}$ are the algebraic groups over $\mathbb{Q}$ associated to Z and $\mathbb{Q}$ in the usual way, a torus C with

$$\underline{\mathbb{Q}} \subseteq C \subseteq \underline{Z}^0.$$

There are further conditions to be imposed on these objects. As they are not yet all clear to me, I shall not impose them until they become necessary in the course of the discussion.

Let G be the group of all L-automorphisms g of V such that for some $\mu(g) \in C$

$$\psi(gu, gv) \equiv \psi(u, \mu(g)v).$$

Let $G(\mathbb{Z}_f)$ be the stabilizer of $V(\mathbb{Z}_f)$ in $G(\mathbb{A}_f)$. To each open compact subgroup $K \subseteq G(\mathbb{Z}_f)$ and each finite set of primes Q such that

$$K = K^p G(\mathbb{Z}_p)$$

with $K^p = K \cap G(\mathbb{Z}_f^p)$, for $p \notin Q$ I want to associate a moduli problem $M = M_{K,Q}$.

If $Q = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$, where $\mathfrak{p}_i$ is a prime of O_E the moduli problem will be formulated for locally noetherian schemes S over $\operatorname{Spec} O_E - \bigcup \mathfrak{p}_i$. If A is an abelian scheme over S let

$$T_p(A) = \varprojlim A_{p^n}.$$

If O_L acts on A and λ, λ' are two polarizations of A I call λ and λ' C-equivalent if there exist c, c' in $C(\mathbb{Q}) \cap O_L$ so that

$$\lambda \circ c = \lambda' \circ c'.$$

Two O_L-isomorphisms φ, $\varphi' : T_1(A) \xrightarrow{\sim} V(\mathbb{Z}_p)$ will be called K-equivalent if $\varphi' = k\varphi$ with

$$k \in G(\mathbb{Q}_p) \cap K.$$

The sheaf $H^1_{DR}(A)$ on S has been studied by, among others, Mazur-Messing. They show that it is isomorphic to the Lie algebra of the universal vector extension $E(\hat{A})$ of the dual abelian variety $\hat{A}$. However (cf. Note 1) the sheaf of invariant differential forms on $E(\hat{A})$ is isomorphic to the sheaf $H^1_{DR}(\hat{A})$. Thus $H^1_{DR}(\hat{A})$ and $H^1_{DR}(\hat{A})$ are paired. So are

$$\operatorname{Hom}(H^1_{DR}(A), O_S) \otimes_{O_E} R$$

and

$$\operatorname{Hom}(H^1_{DR}(\hat{A}), O_S) \otimes_{O_E} R.$$

But we take the negative of the natural pairing.

That data for the moduli problem M consist of an abelian variety, a C-equivalence class Λ of polarizations of A given locally on S, a homomorphism $i : O_L \to \operatorname{End} A$, with $i(1) = 1$ and for each p a K-equivalence class of isomorphisms $\varphi_p : T(A) \simeq V(\mathbb{Z}_p)$ (see Note 4) defined however only over S_p. The data are subject to the following conditions:

(a) If $*$ denotes the involution of $\operatorname{End} A \otimes \mathbb{Q}$ defined by a polarization on Λ then

$$i(\ell^*) = i(\ell)^*.$$

(b) If we choose on S_p an isomorphism of $\mu_{p^\infty} = \varprojlim \mu_{p^n}$ with $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p\mathbb{Z}$, so that the pairing $\langle \alpha, \hat{\alpha} \rangle$ of $T_p(A)$ with $T_p(\hat{A})$ may be regarded as taking values in $\mathbb{Z}_p$, there is, locally on S_p, a c in $C(\mathbb{Q}_p)$ so that

$$\langle \alpha, \lambda(\beta) \rangle = \psi(\varphi(\alpha), c\varphi(\beta)).$$

Here c depends on the polarization λ. If we replace λ by $\lambda \circ \alpha$ we must replace c by $c\alpha$.

(c) For all $\lambda \in L$, there is a c in $C(O_S)(C(\mathbb{Q}) \cap O_L)$, and an O_L-isomorphism

$$\eta : \operatorname{Hom}(H^1_{DR}(A), O_S) \simeq V \otimes O_E \otimes_{O_E} O_S$$

which preserves the filtrations and satisfies

$$\langle \alpha, \lambda(\beta) \rangle = \psi(\eta(\alpha), c\eta(\beta)).$$

Moreover for each p dividing the prime not in Q and each $\lambda \in \Lambda$ there is a $c \in C(\mathbb{Q}) \cap O_L$ so that $\lambda = \lambda' \circ c$ where the kernel of λ' has order prime to p. Here λ is also used to denote the associated map

$$\operatorname{Hom}(H^1_{DR}(A), O_S) \to \operatorname{Hom}(H^1_{DR}(\hat{A}), O_S).$$

Suppose an imbedding $O_E \to \mathbb{C}$ is given as well as a homomorphism $h_0 : \mathbb{C}^\times \to G(\mathbb{R})$. This homomorphism allows $\mathbb{C}^\times$ to act on $V(\mathbb{C})$ and we suppose

$$V(\mathbb{C}) = V_0^+(\mathbb{C}) \oplus V_0^-(\mathbb{C})$$

where

$$V_0^+ = \{v \in V(\mathbb{C}) \mid h_0(z)v = z^{-1}v \ \forall z \in \mathbb{C}^\times\}$$
$$V_0^- = \{v \in V(\mathbb{C}) \mid h_0(z)v = \bar{z}^{-1}v \ \forall z \in \mathbb{C}^\times\}.$$

We suppose moreover that the form

$$\psi(u, h_0(i)v)$$

on $V(\mathbb{R})$ is symmetric and positive definite and that the trace of $\ell \in O_L$ on $V^+(\mathbb{C})$ is $\tau_w(\ell)$.

We may define a moduli problem $M_{\mathbb{C}}$ over $\mathbb{C}$. The data consist now of an abelian variety over $\mathbb{C}$, a $\mathbb{C}$-equivalence class Λ of polarizations of A, a homomorphism $i : O_L \to \operatorname{End} A$ with $i(1) = 1$, and a K-equivalence class of isomorphisms $\varphi : T_f(A) \xrightarrow{\sim} V(\mathbb{Z}_f)$, if

$$T_f(A) = \varprojlim A_n.$$

The following conditions are to be satisfied:

(a′) The same as (a) above.

(b′) The same as (b) above, except that $T_f(A)$, $T_f(\hat{A})$, and $C(A_f)$ replace $T_{\mathbb{Q}}(A)$, $T_{\mathbb{Q}}(\hat{A})$, and $C_{\mathbb{Q}}$.

(c′) The trace of the action of $\ell \in O_L$ on the tangent space to A is $\tau_w(\ell)$.

(d′) Suppose $\langle \cdot, \cdot \rangle_\lambda$ is a bilinear form on $H_1(A, \mathbb{Q})$ defined by a polarization $\lambda \in \Lambda$. There is an O_L-isomorphism η of $H_1(A, \mathbb{Q})$ with $V(\mathbb{Q})$ and a $c \in C(\mathbb{Q})$ so that

$$\langle x, y \rangle_\lambda = \psi(\eta(x), c\eta(y)).$$

(e′) If h is the action of $\mathbb{C}^\times$ on $H_1(A, \mathbb{Q}) \otimes \mathbb{R}$ defined by the identification of this space with the tangent space to A at 0 so that $\eta h \eta^{-1} : \mathbb{C}^\times \to G(\mathbb{R})$ then $\eta h \eta^{-1}$ and h_0 are conjugate under $G(\mathbb{R})$.

As you well know the two problems just formulated are not sufficiently rigid. To stiffen them we introduce and additional datum. $c \in C(\mathbb{Q})$ will be called totally positive if every eigenvalue of the associated linear transformation of $V(\mathbb{R})$ is positive. If $c \in C(\mathbb{Q}) \cap O_L$ and is totally positive and if $\lambda \in \Lambda$ so is $\lambda \circ c$. The additional datum is an injection (of sheaves in the Zariski topology)

$$\epsilon : \Lambda \to C(\mathbb{Q})$$

such that

$$\epsilon(\lambda)c = \epsilon(\lambda')c'$$

if

$$\lambda \circ c = \lambda' \circ c'.$$

The stiffened moduli problems $\tilde{M}$ and $\tilde{M}_{\mathbb{C}}$ allow in action by $C(\mathbb{Q})$. The element c acts by replacing ϵ by $\epsilon' : \lambda \to c\epsilon(\lambda)$ but leaves the remaining data alone.

An isomorphism of $(A, \Lambda, i, \varphi_p, \epsilon)$ with $(A,' \Lambda', i', \varphi'_p, \epsilon')$ is an isomorphism $a : A \to A'$ and a bijection $\lambda \leftrightarrow \lambda'$ of Λ with Λ' so that $\epsilon(\lambda) = \epsilon'(\lambda')$, so that

$$\begin{array}{ccc} A & \xrightarrow{\lambda} & \hat{A} \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\hat{\alpha}} \\ A' & \xrightarrow[\lambda']{} & \hat{A}' \end{array}$$

is commutative, and so that

$$\varphi = \varphi' \circ \alpha$$

and

$$\varphi_p = \varphi'_p \circ \alpha.$$

An isomorphism of two collections of data for $\tilde{M}_{\mathbb{C}}$ is defined in a similar manner.

I observe that if K is chosen sufficiently small then no collection of data admits an automorphism. Any automorphism α of $(A, \Lambda, i, \varphi, \varphi_p, \epsilon)$ fixes each polarization $\lambda \in \Lambda$. Moreover

$$\beta \to \beta^* = \lambda^{-1}\hat{\beta}\lambda$$

is a positive involution of $\operatorname{End}^0(A) = \operatorname{End} A \otimes \mathbb{Q}$. Since $\alpha^* = \alpha^{-1}$ and the eigenvalues of α are algebraic integers, the eigenvalues are roots of unity and α is of finite order. If $\rho \in \mathbb{Q}$ the image of α in $G(\mathbb{Q}_p)$ lies in the image of K. However if we demand that K acts trivially on $V(\mathbb{Z}/p^n\mathbb{Z})$, $p^n \geqslant 3$, then no element but 1 of this image has finite order. We conclude that $\alpha = 1$.

I next observe that the set $M_{\mathbb{C}}$ is the quotient of $\tilde{M}_{\mathbb{C}}$ by $C(\mathbb{Q})$ and that $M(k)$ is the quotient of $\tilde{M}(k)$ by $C(\mathbb{Q})$ if k is a field.[1] The existence of injections

$$\tilde{M}_{\mathbb{C}}/C(\mathbb{Q}) \hookrightarrow M_{\mathbb{C}}$$

and

$$\tilde{M}(k)/C(\mathbb{Q}) \hookrightarrow M(k)$$

[1]algebraically closed.

is clear. It is a matter of ignoring ϵ. To check that these maps are also surjective, it is enough to verify that ϵ always exists. Fix $\lambda_0 \in \Lambda$. If

$$\lambda \circ c = \lambda_0 \circ c_0$$

and

$$\lambda' \circ c' = \lambda_0 \circ c_0'$$

then

$$\lambda_0 \circ (c_0 c') = \lambda_0 \circ (c_0' c).$$

We conclude that

$$c_0 c' = c_0' c$$

or

$$c_0 c^{-1} = c_0' c'^{-1}.$$

We define

$$\epsilon(\lambda) = c_0 c^{-1}.$$

As the next step I introduce a structure of complex analytic spaces on $M_{\mathbb{C}}$ and $\tilde{M}_{\mathbb{C}}$. Consider pairs consisting of

(a) A $\mathbb{Q}$-valued bilinear form Ψ on $V(\mathbb{Q})$ such that for some $c \in C(\mathbb{Q})$

$$\Psi(u, v) \equiv \psi(u, cv).$$

(b) A homomorphism h of $\mathbb{C}^\times$ into $G(\mathbb{R})$ such that

$$V(\mathbb{C}) = V^+(\mathbb{C}) \oplus V^-(\mathbb{C})$$

if

$$V^+(\mathbb{C}) = \{v \in V(\mathbb{C}) \mid h(z)v = z^{-1}v \text{ for all } z \in \mathbb{C}^\times\}$$
$$V^-(\mathbb{C}) = \{v \in V(\mathbb{C}) \mid h(z)v = \bar{z}^{-1}v \text{ for all } z \in \mathbb{C}^\times\}$$

Moreover h should be conjugate under $G(\mathbb{R})$ to h_0.

If $g_f \in G(\mathbb{A}_f)$ then $g_f V(\mathbb{Z})$ is the lattice

$$g_f V(\mathbb{Z}_f) \cap V(\mathbb{Q}).$$

If $x \in G(\mathbb{R})$ and

$$x^{-1} h(z) x = h_0(z)$$

for all z, then

$$\Psi(xu, \mu^{-1}(x) c^{-1} h(i) x v) = \psi(u, h_0(i) v)$$

is positive definite and symmetric on $V(\mathbb{R})$. However, by Serre's lemma, $C(\mathbb{Q})$ is dense in $C(\mathbb{R})$; so there is a $b \in C(\mathbb{Q})$ so that

$$\Psi(u, bh(i)v)$$

is positive definite and symmetric on $V(\mathbb{R})$. Multiplying b by a positive integer if necessary, we may also suppose that $\Psi(u, bv)$ takes integral values in $g_f V(\mathbb{Z})$.

We may also regard h as being defined by an element $g_\infty \in G(\mathbb{R})$, namely

$$g_\infty = x.$$

To Ψ and $g = (g_\infty, g_f)$ we assign the abelian variety

$$A = V^-(\mathbb{C}) \backslash V(\mathbb{C}) / g_f V(\mathbb{Z}) = V(\mathbb{R}) / g_f V(\mathbb{Z}).$$

The set of all $c \in C(\mathbb{Q})$ such that $\Psi(u, cv)$ is integral on $g_f V(\mathbb{Z})$ and $\Psi(u, ch(i)v)$ is positive, symmetric in $V(\mathbb{R})$, defines a $\mathbb{C}$-equivalence class Λ of polarizations on A. Certainly O_L acts on A. We have an obvious isomorphism (namely $\varphi = g_f^{-1}$)

$$\varphi : T_f(A) \to V(\mathbb{Z}_f)$$

and an obvious injection

$$\epsilon : \Lambda \hookrightarrow C(\mathbb{Q}).$$

It is easy to verify that conditions (a′) to (e′) are fulfilled (cf. Note 2).

We now ask when (Ψ, g) and (Ψ', g') yield the same element of $\tilde{M}_{\mathbb{C}}$. The isomorphism between A and A' must be given by a linear transformation γ of $V(\mathbb{Q})$ which takes $g_f V(\mathbb{Z})$ to $g_f V'(\mathbb{Z})$. Thus γ must be a linear transformation of $V(\mathbb{Q})$ and

$$\gamma g_f \equiv g'_f \pmod{K}.$$

α must commute with the action of O_L. Moreover

$$h'(z)\gamma = \gamma h(z)$$

so that

$$\gamma g_\infty h_0 g_\infty^{-1} \gamma^{-1} = g'_\infty h_0 g'^{-1}_\infty.$$

Also

$$\psi(u, cv) = \Psi(u, v) = \Psi'(\gamma u, \gamma v) = \psi(\gamma u, c'\gamma v).$$

Thus

$$\mu(\gamma) = cc'^{-1}$$

and

$$\gamma \in G(\mathbb{Q}).$$

The condition is therefore that g and g' lie in the same double coset.

$$G(\mathbb{Q}) \backslash G(\mathbb{A}) / K_\infty K$$

and that if

$$\gamma g \equiv g' \pmod{K_\infty K}$$

then

$$\Psi(u, v) = \Psi'(\gamma u, \gamma v)$$

K_∞ is the centralizer of h_0 in $G(\mathbb{R})$.

Observe that the action of c in $\tilde{M}_{\mathbb{C}}$ replaces Ψ by

$$u, v \to \Psi(u, cv)$$

but leaves g untouched. We shall verify shortly that every element of $\tilde{M}_{\mathbb{C}}$ corresponds to some pair (Ψ, g). To divide the action of $C(\mathbb{Q})$ is therefore simply to ignore Ψ.

If $(A, \Lambda, i, \varphi, \epsilon)$ defines a point of $\tilde{M}_{\mathbb{C}}$ then, by (d′), we may identify $H_1(A, \mathbb{Q})$ with $V(\mathbb{Q})$. If $\lambda \in \Lambda$, $\epsilon(\lambda) \in c_1$, and

$$\langle u, v \rangle_\lambda = \psi(u, c_2 v)$$

set

$$\Psi(u, v) = \psi(u, cv)$$

with

$$c = c_2 c_1^{-1}.$$

Introduce h as in (e′) and let

$$h(z) = g_\infty h_0(z) g_\infty^{-1} \qquad \forall z \in \mathbb{C}^\times.$$

The inverse of φ yields an automorphism g_f of $V(\mathbb{A}_f)$. Because of (b′) (cf. Note 2), $g_f \in G(\mathbb{A}_f)$. Moreover A is the quotient of $V(\mathbb{R})$ by $g_f V(\mathbb{Z})$.

We have established that, as sets,

$$\tilde{M}_{\mathbb{C}} \simeq C(\mathbb{Q}) \times G(\mathbb{Q})\backslash G(A)/K_\infty K$$

and

$$M_{\mathbb{C}} \simeq G(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty K.$$

We introduce in the usual way that complex structure on $G(\mathbb{R})/K_\infty$ for which $h_0(z)$ acts on the holomorphic tangent space at the coset K_∞ as z^{-1}. This complex structure on $G(\mathbb{R})/K_\infty$ then yields one in $\tilde{M}_{\mathbb{C}}$ and on $M_{\mathbb{C}}$. By Baily-Borel $\tilde{M}_{\mathbb{C}}$ and $M_{\mathbb{C}}$ then appear as the set of complex points on locally noetherian schemes $\widetilde{\mathcal{M}}_{\mathbb{C}}$ and $\mathcal{M}_{\mathbb{C}}$ over $\mathbb{C}$.

I next observe that, if K is sufficiently small, we also have available a point in $\tilde{M}(\widetilde{\mathcal{M}}_{\mathbb{C}})$. If $g_f \in G(\mathbb{A}_f)$ and Ψ are fixed, then

$$V^-(\mathbb{C})\backslash V(\mathbb{C})/g_f V(\mathbb{Z})$$

is a family of abelian varieties over

$$G(\mathbb{R}) g_f K/K_\infty K \simeq G(\mathbb{R})/K_\infty.$$

To give it a holomorphic structure we have only to give a holomorphic structure to the family of vector spaces $V^-(\mathbb{C})/V(\mathbb{C})$ over $G(\mathbb{R})/K_\infty$. The latter is, as a complex manifold, an open submanifold of $G(\mathbb{C})/P(\mathbb{C})$ if $P(\mathbb{C})$ is the parabolic subgroup of $G(\mathbb{C})$ with Lie algebra

$$\mathfrak{g}^{(0,0)} + \mathfrak{g}^{(0,-1)}.$$

Also $V_0^{-1}(\mathbb{C})$ is stable under $P(\mathbb{C})$. The space $V^-(\mathbb{C})$ over the coset of g is

$$g V_0^-(\mathbb{C}).$$

Thus the family is parametrized locally by

$$V^-(\mathbb{C})\backslash V(\mathbb{C}) \times U = V^+(\mathbb{C}) \times U$$

if U is a complex analytic slice of $G(\mathbb{C})/P(\mathbb{C})$.

Any c such that $\Psi(u, cv)$ is integral valued on $g_f V(\mathbb{Z})$ defines a polarization λ (cf. Note 2). We shall define $\epsilon(\lambda) = c$. If K is so small that $G(\mathbb{Q}) \cap g_f K g_f^{-1}$ acts freely on $G(\mathbb{R}) g_f K/K_\infty K$, as it will if the previous condition of smallness is met, then we divide out to obtain a family of abelian varieties on

$$G(\mathbb{Q}) \cap g_f K g_f^{-1} \backslash G(\mathbb{R}) g_f K/K_\infty K.$$

We may put these together to obtain an analytic family of abelian varieties on $\widetilde{\mathcal{M}}_{\mathbb{C}}$. So far as I can see one has to do a little work to verify that it is in fact an algebraic family. This work I omit for now. However it is done it will be clear that we have an element of $\tilde{M}(\widetilde{\mathcal{M}}_{\mathbb{C}})$. This work may not be necessary. It is probably sufficient for our purposes to know that $\widetilde{\mathcal{M}}_{\mathbb{C}}$ with the given family is universal for the continuous version of $M_{\mathbb{C}}$. For the continuous version we would demand that the abelian varieties and their duals formed an analytic family of abelian varieties, that

$$\lambda : A \to \hat{A}$$

and $\epsilon : \Lambda \to C(\mathbb{Q})$ were given locally, and on the fibres λ was associated to polarizations.

I will sort this out to some other time. What we have to find now are conditions on the objects defining our moduli problem which guarantee that $\tilde{M}_{\mathbb{C}}$ equals $\tilde{M}(\mathbb{C})$.

It follows from the computations in Note 3 that (c′) implies (c). What we have to do is to find conditions which guarantee that (d′) and (e′) are redundant. Such conditions are given in the first version of "Travaux de Shimura." I review them here with some slight variations.

Suppose we have a representation of $L \otimes \mathbb{R}$ on the vector space H over $\mathbb{R}$, an alternating bilinear form $\langle x, y\rangle$ on H, and an action $z \to h(z)$ of $\mathbb{C}^K$ on the $L \otimes \mathbb{R}$ moduli H so that

$$\langle h(z)x, h(z)y\rangle = (z\bar{z})^{-1}\langle x, y\rangle$$

so that

$$\langle x, h(i)y\rangle$$

is symmetric and positive definite, and so that

$$H \otimes \mathbb{C} = H^+(\mathbb{C}) \oplus H^-(\mathbb{C})$$

where $H^+(\mathbb{C})$ and $H^-(\mathbb{C})$ are defined in the usual way by h. Suppose moreover that the trace of $\ell \in L$ on $H^+(\mathbb{C})$ is $\tau_W(\ell)$ and that

$$\langle \ell x, y\rangle = \langle x, \ell^* y\rangle.$$

We want to verify that there is then an isomorphism η of H with $V(\mathbb{R})$ respecting the action of L and of $\mathbb{C}^\times$ so that

$$\langle x, y\rangle = \psi(\eta(x), \eta(y)).$$

Since the involution is positive each simple factor of $L \otimes \mathbb{R}$ is invariant under it. Thus decomposing H and $V \otimes \mathbb{R}$ into direct sums we suppose that $L \otimes \mathbb{R}$ is simple. Call it, for brevity, M. M is then a matrix algebra over $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{K}$ with the standard involution. Let e_{ij} be the usual idempotents in M. Then if $e_{ll}x = x$, $e_{ll}y = y$

$$\langle e_{jl}, e_{kl}y\rangle = \delta_{jk}\langle x, y\rangle.$$

Moreover

$$h(z)e_{jk} = e_{jk}h(z).$$

Replacing H or $V(\mathbb{R})$ by $e_{ll}H$ or $e_{ll}V(\mathbb{R})$ we reduce ourselves to the cases $M = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{K}$.

Define an M-valued bilinear form (x, y) on H by

$$\mathrm{Tr}_{M/\mathbb{R}}\{\ell(x, y)\} = \langle \ell x, y\rangle$$

for all $\ell \in M$. Define the M-valued bilinear form $\Psi(u, v)$ on $V(\mathbb{R})$ in a similar fashion. The relations

$$(\ell x, y) = \ell(x, y) = (x, \ell y) = (x, y)\ell^* \qquad (y, x) = -(x, y)^*$$

are satisfied. Decompose H and $V(\mathbb{R})$ into a direct sum of spaces, invariant under the actions of M and $\mathbb{C}$ and mutually orthogonal, which is as fine as possible. Since x and $h(i)x$ are not orthogonal unless $x = 0$, these spaces are at most two-dimensional over M.

If $M = \mathbb{R}$ then each one of these spaces has a basis x, $h(i)x$ with

$$(x, h(i)x) = 1.$$

The existence of η follows. If $M = \mathbb{C}$ then h is diagonalizable so each of these spaces has dimension 1. Again we may suppose $(x, h(i)x) = 1$. If j is a fixed square root of -1 in M the number of subspaces for which $h(i) = j^{-1}$ is, because of the conditions in the traces, the same for H and for $V(\mathbb{R})$. The existence of η follows again.

If $M = \mathbb{K}$ then one of these subspaces is isomorphic to $\mathbb{K}$ and the form (x, y) is given by

$$(x, y) = xty^*$$

with $t^* = -t$. Also $h(z)$ is given by right multiplication by $h(z) \in \mathbb{K}^\times$. A change of basis replaces t by btb^*; hence t can be taken to be any element s of $\mathbb{K}^\times$ with $s^* = -s$. We choose s so that $s^2 = -1$. Since

$$(h(i)x, h(i)y) = (x, y)$$

and

$$(x, h(i)x) > 0$$

if $x \neq 0$, we have

$$h(i)sh(i)^{-1} = h(i)sh(i)^* = s$$

and

$$\operatorname{tr} sh(i) > 0.$$

We conclude that

$$h(i) = s^{-1}.$$

The existence of η follows again.

We now introduce an assumption which will remain in force throughout this sequence of letters.

Assumption I. *If G_1 is the kernel of $\mu : G \to C$ then the image of*

$$H^1(\mathfrak{G}(\bar{Q}/Q), G_1(\bar{Q})) \to H^1(\mathfrak{G}(\bar{\mathbb{Q}}/\mathbb{Q}), G(\bar{\mathbb{Q}}))$$

satisfies the Hasse principle.

Deligne verifies that this assumption is fulfilled in many cases. What we need to verify now is that the assumption renders (d′) and (e′) superfluous.

If (d′) is satisfied than every eigenvalue of the c occurring there is positive so that $c = d^2$, with $d \in C(\mathbb{R})$. Then

$$\langle x, y\rangle_\lambda = \psi(d\eta(x), d\eta(y)).$$

It follows from the discussion above that

$$\eta^{-1}h\eta = \eta^{-1}d^{-1}hd\eta = g^{-1}h_0g$$

with $g \in G(\mathbb{R})$ and $\mu(g) = 1$. Thus (e′) as a consequence of (d′).

There is certainly an isomorphism $\eta : H_1(A, \mathbb{Q}) \otimes \bar{\mathbb{Q}} \to V(\bar{\mathbb{Q}})$ which commutes with L and satisfies

$$\langle x, y\rangle_\lambda = \psi(\eta(x), \eta(y)).$$

Then $\sigma(\eta)\eta^{-1}$ defines an element of $H^1(\mathfrak{G}(\bar{Q}(Q), G_1(\bar{Q})))$. However condition (b′) assures us that this cohomology class is trivial in $H^1(\mathfrak{G}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p), G(\mathbb{Q}_p))$ for every p. The discussion above together with (e′) implies that it is also trivial at infinity. It remains to apply Hasse's principle.

We have next to discover conditions which would guarantee that $\tilde{M}$ is represented by a scheme smooth over $O_E[1/p_1, \dots, 1/p_n]$ and that the quotient of $\tilde{M}$ by $C(Q)$ is smooth, quasi-projective, and yields a coarse moduli scheme for M. I shall give these conditions now, but I do not yet want to try to prove that they have the desired consequences. I shall, however, give brief arguments which at least indicate that we are on the right track.

Consider Theorem 3.4 of *Algebraization of formal moduli I.* To verify condition [0] of that theorem I have to learn descent theory. If we can show that the existence of φ and the φ_p,

$p \notin Q$ impose in reality only a finite number of conditions on the variety then [1] follows from the imbedding theorem for abelian varieties. A little attention must also be given to the local properties of ϵ.

The projective representability is dealt with by Schlessinger's paper. As he observes the conditions H_1 and H_2 of his Theorem 2.11 are almost automatically satisfied. I think that as far as these two conditions are concerned our deformation problem is not more complicated than his. The condition H_3 follows from the observation that the tangent space to our problem is contained in the tangent space of all deformations of the abelian variety. By Schlessinger's Lemma 3.8 the condition H_4 will be fulfilled if we take K so small that the problem has no automorphisms. Of course, if we do not want to take K small we will have to content ourselves with a coarse moduli scheme.

By the way, Schlessinger employs a ring Λ. I suppose that if the characteristic of k is 0 one may take $\Lambda = k$ but that if the characteristic is p one takes Λ to be the ring of Witt vectors over k. Then any complete local ring with residue field k is a ring over Λ.

A critical test of the correctness of my formulation of the moduli problem is the effective projective representability. Making use of the polarization and the smoothness of G one can I suppose carry all data from the finite levels to the limit. The only difficulty is that at the finite levels a given p may be nilpotent but may cease to be nilpotent in the limit. There is apparently a new datum to be introduced in the limit, namely φ_p. We have to impose supplementary assumptions which make the new datum superfluous.

There are two assumptions to be made. The first is natural enough in terms of the conjectures in the appendix but rather restrictive from the point of view of moduli. The second can be fulfilled by choosing Q large enough.

Assumption II. *The group G is connected.*

Assumption III. *If $\mathfrak{p} \notin Q$ and $\bar{O}_E$ is the completion of O_E at $\mathfrak{p}$ then $\bar{O}_E$ is unramified over $\mathbb{Z}_p$ and $O_L \otimes \mathbb{Z}_p$ is a direct sum of matrix algebras over unramified extensions of $\mathbb{Z}_p$. Moreover K contains $G(\mathbb{Z}_p)$, the stabilizer of $V(\mathbb{Z}_p)$ in $G(\mathbb{Q}_p)$. Finally the maps $G(\mathbb{Z}_p) \to G(\mathbb{Z}/p^n\mathbb{Z})$, $n \geqslant 1$, are to be surjective, the determinant of ψ is to be a unit in $\mathbb{Z}_p$, and $O_L \otimes \mathbb{Z}_p$ is to be stable under the involution.*

In order to check that these assumptions do the trick I have to partially verify the formal smoothness. Suppose we are given

$$\begin{tikzcd} 0 & R_0 \arrow[l] & R \arrow[l] & J \arrow[l] & 0 \arrow[l] \\ & & O_E \arrow[ul] \arrow[u] & & \end{tikzcd}$$

where J is an ideal of square 0 as well as moduli data over R_0. We have to lift the data to R. I do not yet know how to handle a general $\mathbb{R}$ but for the immediate purposes we may suppose R is local artinian with residue characteristic ρ and apply the deformation theory of Messing's book.

We may take the divided powers to be 0. Since

$$D(G_0)_{S_0} \simeq \mathrm{Hom}(H^1_{DR}(A_0), R_0) \simeq V \otimes O_E \otimes_{O_E} R_0$$

we may identify Messing's $D(G_0)_S$ (p. 150 of his book) with

$$V \otimes O_E \otimes_{O_E} R.$$

It comes with a filtration, and action of O_L, and a bilinear form, and hence a mapping into its dual (see Note 7). Since the bilinear form is 0 on the filtering subspace, this mapping into the dual preserves filtrations. This allows us to lift A, the action of O_L, and the polarization. The other data lift automatically. In particular a factorization $\lambda = \lambda' \circ c$, $c \in C(\mathbb{Q}) \cap O_L$, on the special fibre lifts.

To come back to the effectiveness, we have to show that if R is the ring of formal power series over a finite unramified extension of $\bar{O}_E$, the completion of O_E at some $\mathfrak{p} \notin Q$, and if the moduli data for $\tilde{M}$ are given over R, except that φ_p is not prescribed or even assumed to exist, then φ_p does in fact exist, provided the assumptions are fulfilled, and is unique.

Suppose R is a ring of formal power series over O_F where F is an unramified extension of E. Then we have

$$\operatorname{Spec} F \to \operatorname{Spec} R.$$

To show the existence of φ_p, whose uniqueness is manifest, we just have to exhibit it as an O_L-isomorphism

$$\varphi_p : A(\bar{F})_{p^\infty} \xrightarrow{\sim} V(\mathbb{Z}_p)$$

for which

$$\langle \alpha, \lambda(\beta) \rangle = \psi(\varphi_p(\alpha), zc\varphi_p(\beta))$$

for some $z \in C(\mathbb{Q}) \cap O_L$ and $c \in C(\mathbb{Z}_p)$.

The traces of $\ell \in O_L$ on $A(\bar{F})_{p^\infty}$ and on

$$\operatorname{Hom}(H^1_{DR}(A/\bar{F}), \bar{F})$$

are the same. Since $O_L \otimes \mathbb{Z}_p$ is a direct sum of matrix algebras over unramified extensions the existence of an O_L-isomorphism between $A(\bar{F})_{p^\infty}$ and $V(\mathbb{Z})$ follows immediately. Using this isomorphism we pull the bilinear form on $A(\bar{F})_{p^\infty}$ over to a bilinear form ψ' on $V(\mathbb{Z}_p)$. If we show that there is a $z \in C(\mathbb{Q}) \cap O_L$, $z \neq 0$, so that

$$\psi'(x, z^{-1}y)$$

Is integral and has unit determinant, the existence of φ_p will follow from Note 5.

From (c) we know that, over O_F, there exists

$$\eta : \operatorname{Hom}(H^1_{DR}(A), O_F) \simeq V \otimes O_F$$

which satisfies

$$\langle \alpha, \lambda(\beta) \rangle = \psi(\eta(\alpha), z_1 c_1 \eta(\beta))$$

with $c_1 \in C(O_F)$, $z_1 \in C(\mathbb{Q}) \cap O_L$. There is a λ', the order of whose kernel is prime to p, so that $\lambda = \lambda' \circ c$, $c \in C(\mathbb{Q}) \cap O_L$. Replacing λ by λ' we may suppose that λ itself has a kernel whose order is prime to p.[2]

One still has to verify conditions (3) and (4) of Artin's Theorem 3.4. Condition (4), which I still do not really understand, can I believe be handled by his Theorem 3.9. This leaves (3) which you have shown me how to verify.

I believe you suggested that if $\mathcal{A}$, $\mathcal{A}'$ represent two abelian schemes over S provided with the necessary auxiliary data one checks that

$$\underline{\mathrm{Iso}}_S(\mathcal{A}, \mathcal{A}')$$

is represented by a scheme Y and then uses the fact that

$$\operatorname{Hom}(T, Y) \to \operatorname{Hom}(T, S)$$

[2]Then the above condition is satisfied with $z = 1$.

is an injection for noetherian T to show that Y is the subscheme of S. However, I shall not worry about (3) at the moment.

Finally in order to solve the coarse moduli problem for M one presumably has to prove that the quotient by $C(\mathbb{Q})$ exists. This should not be too difficult.

In the appendix I suggested that the isogeny classes should be indexed by pairs (γ, h^0). However, it seems to me now that the notion of equivalence I introduced there is not the correct one. The correct conditions defining the equivalence of (γ_1, h_1^0) and (γ_2, h_2^0) seem to be local, one condition each finite prime, but none at the infinite prime. First of all, for some positive integers m, n, γ_1^m and γ_2^n must be conjugate in $G(\mathbb{Q}_\ell)$ for each $\ell \neq p$. At p the condition is more complicated.

Suppose k_p is a finite Galois extension of $\mathbb{Q}_p$ sufficiently large with respect to some choice of (γ, h^0), and some Cartan subgroup T over $\mathbb{Q}$ which contains γ and through which h^0 factors. In the appendix I defined an element of $H^1(W_{k_p/\mathbb{Q}_p}, T(k_p))$ and hence an element of $H^1(W_{k_p/\mathbb{Q}_p}, H^0(k_p))$. I did not verify that this element is independent of T and that it does not change if h^0 is conjugated within $H^0(\mathbb{R})$ or, to be more precise, within the normalizer of $T(\mathbb{R})$ in $H^0(\mathbb{R})$. Let me indicate now how this is done.

Suppose b_w, $\bar{b}_w$ are two cocycles obtained in the above manner. Then $\operatorname{ad} b_w$ depends only on the image σ of w in $\mathfrak{G}(k_p/\mathbb{Q}_p)$; so we can use it to twist H^0 and obtain H_1^0.

$$c_w = \bar{b}_w b_w^{-1}$$

is a cocycle in $H_1^0(k_p)$. It has to be shown to be trivial. Let D be the maximal split torus in the centre of H^0 and therefore of H_1^0. Modulo D,

$$b_w = x^{\hat{\nu}} \alpha_\sigma$$

reduces to α_σ. Using this and various results from Galois cohomology, such as the vanishing of H^1 of a simply-connected group, the Tate-Nakayama theory, and Tate duality, one shows that $\{c_w\}$ which is clearly the pullback of α, $\{c_\sigma\}$, $\sigma \in \mathfrak{G}(k_p/\mathbb{Q}_p)$, is trivial modulo D. However D is split so that, by Theorem 90, it carries no H^1. We conclude that $\{c_\sigma\}$ is trivial.

The condition for equivalence of (γ_1, h_1^0), (γ_2, h_2^0) which I impose at p demands first of all that γ_1^n and γ_2^m be conjugate in $G(\overline{\mathbb{Q}_p^{un}})$ (m, n as before)

$$\gamma_2^m = g\gamma_1^n g^{-1}.$$

In the paragraph to follow I associate to (γ, h) and $F \in H^0(\mathbb{Q}_p^{un})$ and hence $H(\mathbb{Q}_p^{un})$ for, again by the following discussion, H^0 and H become isomorphic over $\mathbb{Q}_p^{un}$. Suppose F_1 and F_2 are the elements associated to γ_1 and γ_2. F_1 is determined modulo $F_1 \to cF_1\sigma(c^{-1})$, $c \in H_1^0(\overline{\mathbb{Q}_p^{un}})$. Here σ is the Frobenius on $\overline{\mathbb{Q}_p^{un}}$. Since

$$gF_1\sigma(g^{-1}) = (gF_1g^{-1})(g\sigma(g^{-1}))$$

and both gF_1g^{-1} and $g\sigma(g^{-1})$ lie in $H_2^0(\overline{\mathbb{Q}_p^{un}})$ it makes sense to demand that their be a $c \in H_2^0(\overline{\mathbb{Q}_p^{un}})$ so that

$$F_2 = cgF_1\sigma(g^{-1})\sigma(c^{-1}).$$

This is in fact demanded.

Of course I am still not willing to stake all my money on the notion of equivalence introduced above. However, I do not have anything better to offer and it does work for the groups we are considering in this letter.

There is something else that I did not anticipate in the appendix. To one equivalence class of (γ, h^0) can correspond several isogeny classes, all with the same structure. These are parameterized by certain elements of

$$H^1(\mathfrak{G}(\bar{\mathbb{Q}}/\mathbb{Q}), H).$$

The elements in question must first of all be trivial every place except p, including infinity. Moreover the map $H^0 \to G_{\mathrm{der}}\backslash G$ leads to $H \to G_{\mathrm{der}}\backslash G$. The elements must have an image in

$$H^1(\mathfrak{G}(\bar{\mathbb{Q}}/\mathbb{Q}), G_{\mathrm{der}}\backslash G)$$

which is trivial.

I should also at this point correct the definition of X given in the appendix. That definition suffers more than usual from my hastiness. Taken modulo D, b_w defines an element of $H^1(\mathfrak{G}(k_p/\mathbb{Q}_p), T/D(k_p))$ and hence an element of $H^1(\mathfrak{G}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p), T/D(\mathbb{Q}_p))$. By a lemma which is I believe to be found in "Corps locaux," there is a finite unramified extension k'_p of $\mathbb{Q}_p$ so that this element can be realized in $H^1(\mathfrak{G}(k'_p/\mathbb{Q}_p), T/D(k'_p))$. This means that by an appropriate choice of the cocycle b_w within its cohomology class and with a sufficiently large k_p we may suppose that the restriction of $\{b_w\}$ to W^0 (defined as in the appendix) takes values in $D(k_p)$. In particular the image of b_w in T/D lies in $T/D(\mathbb{Q}_p^{un})$.

W^0 contains as a subgroup of finite index the units U of O_{k_p}. If $x \in U$ then b_x is of the form

$$x^{\hat{\nu}}.$$

Thus for all $w \in W^0$ and all rational characters λ of D

$$|\lambda(b_w)| = 1.$$

Suppose $\Phi \in W$ maps to the Frobenius and σ is the image of Φ in $\mathfrak{G}(k_p^{un}/k_p)$. Let $\overline{k_p^{un}}$ be the completion of k_p^{un} and consider the map

$$x \to \sigma(x)x^{-1} \qquad x \in \overline{k_p^{un}}^{\times}.$$

Its image consists of units; I claim that every unit lies in the image.

Since $p \to 1$, its image is closed. Moreover if $\mathfrak{p}$ is the maximal ideal of O_{k_p} and y is a unit we can always find x so that

$$y \equiv \sigma(x)x^{-1} \pmod{\mathfrak{p}}.$$

If $\mathfrak{p} = (\Pi)$, we have

$$\frac{\sigma(1+\alpha\Pi^k)}{1+\Pi^k} \equiv 1 + (\sigma(\alpha)\theta - \alpha)\Pi^k \pmod{\Pi^{2k}}$$

if

$$\frac{\sigma(\Pi^k)}{\Pi^k} = \theta.$$

Since

$$\sigma(\alpha) \equiv \alpha^q \pmod{\Pi}$$

and the equation

$$\alpha^q\theta - \alpha \equiv \beta \pmod{\Pi}$$

can be solved in k_p^{un}, we can establish our assertion by iteration.

Now because the map

$$T(\mathbb{Q}_p^{un}) \to T/D(\mathbb{Q}_p^{un})$$

is surjective we may represent b_Φ as $b_1 b_2$ with b_2 in $T(\mathbb{Q}_p^{un})$ and $b_1 \in D(k_p^{un})$. I claim that

$$b_1 = b_3 b_4$$

with

$$b_4 \in D(\mathbb{Q}_p^{un})$$

and with

$$|\lambda(b_3)|_p = 1$$

for any rational character λ of D. To show this I have to show that there is a $\hat{\lambda} \in \hat{L}(D)$ so that

$$|\lambda(b_1)|_p = p^{-\langle \lambda, \hat{\lambda} \rangle}$$

for all λ. It is in fact enough to establish this relation for rational characters of D which are restrictions of rational characters of T defined over $\mathbb{Q}_p$. This allows us to replace b_1 by b_Φ.

Suppose λ is a rational character of T over $\mathbb{Q}_p$. If $w = (x, \sigma)$ then

$$|\lambda(b_w)|_p = |x|_p^{\langle \lambda, \hat{\nu} \rangle} \prod_\tau |\alpha_{\sigma,\tau}|_p^{\langle \lambda, \hat{\mu} \rangle}$$

because the left side depends only on the class of $\{b_w\}$. The right side equals

$$|N_{k_p/\mathbb{Q}_p^\times}|_p^{\langle \lambda, \hat{\mu} \rangle} \left| \prod_\tau \alpha_{\sigma,\tau} \right|_p^{\langle \lambda, \hat{\mu} \rangle}.$$

However if v is the image of w in $\mathbb{Q}_p^\times$ under the usual map then

$$v = \{N_{k_p/\mathbb{Q}_p^\times}\} \left\{ \prod_\tau \tau^{-1}(\alpha_{\sigma,\tau}) \right\}$$

Thus

$$|\lambda(b_w)|_p = |v|_p^{\langle \lambda, \hat{\mu} \rangle}$$

as required. I observe at this point that I have been for some time very cavalier with respect to enlargements of the field k_p. For a justification of this, see Note 6.

I may suppose that $|\lambda(b_1)|_p < 1$ for every rational character of D. I set $F = b_2$. Then $F \in T(\mathbb{Q}_p^{un})$. Moreover any two choices of Φ and any choice of b_2 affect F only by multiplying it by $c\sigma(c^{-1})$ with c in $T(\overline{\mathbb{Q}_p^{un}})$.

K_p determines a parahoric subgroup $K_p(\overline{\mathbb{Q}_p^{un}})$ of $G(\overline{\mathbb{Q}_p^{un}})$ and F_0 acts on

$$X'' = G(\overline{\mathbb{Q}_p^{un}})/K_p(\overline{\mathbb{Q}_p^{un}}).$$

Any point of this quotient determines a special vertex in the Bruhat-Tits building of each simple factor of $G(\overline{\mathbb{Q}_p^{un}})$. Let $r = [E_\mathfrak{p} : \mathbb{Q}_p]$. Both F_0 and F_0^r act on this quotient. If $x' \in X''$ consider x' and $y' = F_0^r x'$. We consider only those x' such that for each simple factor the vertices x'_c and y'_c are either the same or lie on an edge. (By the way, the abelian factors play no role here.) Thus x' and y' determine a parahoric subgroup of $G(\overline{\mathbb{Q}_p^{un}})$. X consists of those x' for which this parahoric subgroup is conjugate to $I(\overline{\mathbb{Q}_p^{un}})$, where I is defined in basically the same way as $\bar{I}$ of the appendix, except that G is used instead of $\bar{G}$. I am, however, nagged by the thought that somewhere in my notes I have imposed an extra condition associated with the abelian part. Since I do not see what it could have been and do not have my notes with me as I write, I shall forget about it.

What has to be checked is that the suggestions of the appendix, with the corrections above, are valid for the groups studied in this letter. We fix a prime $\mathfrak{p}$ of E not in Q, let $k_{\mathfrak{p}}$ be the residue field of O_E at $\mathfrak{p}$, and study $M(\bar{k}_{\mathfrak{p}})$.

I recall first how $G(\mathbb{A}_f^p)$ acts on $M(\bar{k}_{\mathfrak{p}})$. Suppose $\mathcal{A} = (A, \{\varphi_\ell\}, \Lambda)$ represents an element of $M(\bar{k}_{\mathfrak{p}})$. Let $g = \prod g_\ell$ lie in $G(\mathbb{A}_f^p)$. Suppose we can find an isogeny the order of whose kernel is prime to p

$$A' \xrightarrow{\psi} A$$

as well as, for each $\ell \neq p$,

$$\varphi'_\ell : T_\ell(A') \xrightarrow{\sim} V(\mathbb{Z}_\ell)$$

so that

$$\begin{tikzcd}
T_\ell(A') \arrow[r, "\psi"] \arrow[d, "\varphi'_\ell"] & T_\ell(A) \arrow[d, "\varphi_\ell"] \\
V(\mathbb{Z}_\ell) \arrow[d, hook] & V(\mathbb{Z}_\ell) \arrow[d, hook] \\
V(\mathbb{Q}_\ell) \arrow[r, "g_\ell"] & V(\mathbb{Q}_\ell)
\end{tikzcd}$$

is commutative. This implies of course that $gV(\mathbb{Z}_\ell) \subseteq V(\mathbb{Z}_\ell)$. It also implies that O_L acts on A'. Moreover the φ'_ℓ are determined by the φ_ℓ. We define the class Λ' to be that of

$$\lambda' = \hat{\psi}\lambda\psi \qquad \lambda \in \Lambda.$$

Then $\mathcal{A}' = (A', \{\varphi'_\ell\}, \Lambda')$ yields a new element of $M(\bar{k}_{\mathfrak{p}})$. If g_ℓ takes $V(\mathbb{Z}_\ell)$ to itself for all ℓ we can find A' and $\{\varphi'_\ell\}$. We define $\mathcal{A}_g$ to be $\mathcal{A}'$. This action can be extended to all of $G(\mathbb{A}_f^p)$ by setting $\mathcal{A}g = \mathcal{A}$ if $g = n$ is a positive integer prime to p.

Suppose M is a Dieudonné module of $\mathcal{A}$ and

$$N = \varinjlim_n \frac{M}{p^n}.$$

Any submodule M' of N, as a module over $W(\bar{k}_{\mathfrak{p}})$, which is invariant under F and V, defines an abelian variety A' isogenous to A. If, as we may suppose after multiplying by a rational integer, $M \subseteq M'$ the map

$$\psi : A' \to A$$

corresponds to

$$M \subseteq M'.$$

In order that O_L acts on A', M' must be invariant under O_L. We may define φ'_ℓ, $\ell \neq p$ as $\varphi_\ell \circ p$ because the order of the kernel of ψ is a power of p. Define Λ' as the class of

$$\lambda' : \hat{\psi}\lambda\psi \qquad \lambda \in \Lambda.$$

In order to check that (c) is satisfied, I observe that it must be interpreted to hold for any $\lambda : A \to \hat{A}$ such that $n\lambda$ arises from a polarization in Λ for some $n > 0$. If I had been more careful in my discussion of Condition 2 of Artin's Theorem 3.4, this would have become clear earlier. Anyhow we may suppose that the kernel of λ is prime to p and that the bilinear form

$$\langle \alpha, \lambda(\beta) \rangle$$

on

$$\mathrm{Hom}(H^1_{DR}(A), \bar{k}_{\mathfrak{p}})$$

is non-degenerate. The Dieudonné module $\hat{M}$ of $\hat{A}$ is

$$\mathrm{Hom}_{W(\bar{k}_{\mathfrak{p}})}(M, W(k_{\mathfrak{p}})).$$

Define $\hat{N}$ in the same way as N so that $\hat{M}' \subseteq \hat{M} \subseteq \hat{N}$. λ defines $\hat{M} \to M$ and hence a bilinear form on $\hat{M}$ as well as one on $\hat{N}$, which we use to identify N with $\hat{N}$. (Note: $\hat{N}$ takes precedence, so that O_L acts to the left.) Thus λ corresponds to the inclusion $\hat{M} \hookrightarrow M$ and λ' to

$$\hat{M}' \hookrightarrow \hat{M} \hookrightarrow M \hookrightarrow M'.$$

We need to know that the bilinear form $\hat{M}$ is alternating and that its reduction modulo p is compatible with the isomorphism

$$\hat{M}/p\hat{M} \simeq \mathrm{Hom}(H^1_{DR}(A), \bar{k}_{\mathfrak{p}}).$$

I do not know what the best way to verify this. As a stopgap, one could combine §15 of Mazur-Messing with the skew-symmetry established in Note 7.

Since we have taken the order of the kernel of λ to be prime to p,

$$M = \hat{M}.$$

Condition (c) will be satisfied only if there is a $c \in C(Q) \cap O_L$ so that

$$(*) \qquad \hat{M}' = cM'.$$

I observe also that $\hat{M}'$ is a Dieudonné module if and only if

$$(**) \qquad p\hat{M}' \subseteq F\hat{M}' \subseteq \hat{M}'$$

and that since

$$\langle Fx, Fy \rangle = p\sigma(\langle x, y \rangle)$$

the form $\langle x, cy \rangle$ yields a non-degenerate bilinear form on $\hat{M}'/p\hat{M}'$ with respect to which $F\hat{M}$ (mod $p\hat{M}'$) is a maximal isotropic subspace. If (c) is satisfied the fibration of O_L-modules[3]

$$F\hat{M}'/p\hat{M}' \hookrightarrow \hat{M}'/p\hat{M}'$$

is isomorphic to

$$U \otimes \bar{k}_{\mathfrak{p}} \hookrightarrow V \otimes \bar{k}_{\mathfrak{p}}.$$

If all these conditions are fulfilled then $(A', \{\varphi'_\ell\}, \Lambda')$ will satisfy (c).

In order to have consistency I have to associate to $\hat{M}'/p^r$ the collection $\{A', \{\frac{1}{p^r}\varphi'_\ell\}, \Lambda'\}$.

Given A let X be the set of $\hat{M}' \subseteq \hat{N}$ satisfying the above conditions. To every element of $G(\mathbb{A}_f^p) \times X$ we can now associate the point $\mathcal{A}'$ in $M(\bar{k}_{\mathfrak{p}})$. Observe that the group $\bar{G}(\mathbb{Q}_p)$ of all O_L-automorphisms of $\hat{N}$ which preserve the bilinear form up to an element of $C(\overline{\mathbb{Q}_p^{un}})$ and commute with F acts on X. Let $H(\mathbb{Q})$ be the group of invertible elements in the tensor product of the O_L-endomorphisms of A with $\mathbb{Q}$ which preserves the bilinear form up to elements of $C(Q)$. We have imbeddings

$$H(\mathbb{Q}) \hookrightarrow G(\mathbb{Q}_\ell)$$
$$H(\mathbb{Q}) \hookrightarrow \bar{G}(\mathbb{Q}_p)$$

[3]A small blunder: the filtering subspace is $V\hat{M}'/p\hat{M}'$! This however has very little effect on the arguments.

and

$$H(\mathbb{Q}) \hookrightarrow G(\mathbb{A}_f^p) \times \bar{G}(\mathbb{Q}_p).$$

To keep the discussion formally simple, I passed the limit over all K containing $G(\mathbb{Z}_p)$. Observe that (g, x) and (g_1, x_1) yield isomorphic $\mathcal{A}'$ if and only if there is an $h \in H(\mathbb{Q})$ so that

$$g_1 = hg \qquad x_1 = hx.$$

The point $\mathcal{A}'$ associated to (g, x) is defined by

$$A' \xrightarrow{\psi'} A'' \xrightarrow{\psi''} A$$

where, if $x = \bar{M}'$, A'' has dual Dieudonné module

$$p^r \hat{M}' \hookrightarrow \hat{M}. \qquad \text{(We may suppose } r = 0.)$$

Also A' is defined by

$$\begin{tikzcd} T_\ell \arrow[r, "\psi'"] \arrow[d, hook, "\varphi'_\ell"] & T_\ell(A'') \arrow[d, hook, "\varphi''_\ell"] \\ V(\mathbb{Q}_\ell) \arrow[r, "ng_\ell"'] & V(\mathbb{Q}_\ell) \end{tikzcd}$$

with n prime to p. If $\mathcal{A}'$ is isomorphic to $\mathcal{A}'_1$, we have

$$\eta : A' \to A'_1$$

so that

$$\varphi'_\ell = \varphi'_{\ell,1} \circ \eta.$$

If $\psi = \psi'' \circ \psi'$, $\psi_1 = \psi''_1 \circ \psi'_1$, then

$$n^{-1} g_\ell^{-1} \circ \varphi_\ell \circ \psi = n_1^{-1} g_{\ell,1}^{-1} \circ \psi_1 \circ \eta$$

or

$$\varphi_\ell^{-1} g_{\ell,1} g_\ell^{-1} \varphi_\ell = \psi_1 \circ \eta \circ \psi^{-1}$$

and $\frac{n}{n_1} \psi_1 \circ \eta \circ \psi^{-1}$ lies in $\operatorname{End} A \otimes \mathbb{Q}$ and is invertible. If we call it h, then

$$g_{\ell,1} = hg_\ell.$$

In particular $h \in G(\mathbb{Q}_\ell)$ and therefore in $H(\mathbb{Q})$. A' also has dual Dieudonné module $\hat{M}'$. The map $\psi : A' \to A$ corresponds to

$$\hat{M}' \hookrightarrow \hat{M}.$$

Since η is an isomorphism and

$$\begin{tikzcd} \hat{M}' \arrow[r, hook, "\psi"] \arrow[d, "\eta"] & \hat{M} \\ \hat{M}'_1 \arrow[ur, hook, "\psi_1"'] & \end{tikzcd}$$

is commutative, while n and n_1 are prime to p, h takes $\hat{M}'$ to $\hat{M}'_1$. The above computations may of course be reversed. Thus our isogeny class is parametrized by

$$H(\mathbb{Q}) \backslash G(\mathbb{A}_f^p) \times X.$$

We have now to check that H, $\bar{G}$ and X are as predicted and that our description of the isogeny classes is correct.

The first step is to find a γ and an h_0 associated to $\mathcal{A}$. For this I use a lifting. Let T be a Cartan subgroup of H over $\mathbb{Q}$ and let R be the centralizer of T in $\mathrm{End}_{O_L} A \otimes \mathbb{Q}$. Examining the situation in $T_\ell(A) \otimes_{\mathbb{Z}_\ell} \bar{\mathbb{Q}}_\ell$ for some $\ell \neq p$ and using Tate's theorem we see that R is commutative. I want to show that there is a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$, and a point $(\tilde{A}'', \{\tilde{\varphi}''_\ell\}, \tilde{\Lambda}'')$ in $M(\mathcal{O})$ with reduction $(A'', \{\varphi''_\ell\}, \Lambda'')$ so that there is an isogeny

$$\xi : A'' \to A$$

whose kernel has order of power of p and with

$$\varphi''_\ell = \varphi_\ell \circ \xi$$

and

$$\hat{\xi} \circ \lambda \circ \xi \in \Lambda''$$

if $\lambda \in \Lambda$. Moreover I want an order S in R to act as O_L-endomorphisms of $\tilde{A}''$ in such a way that the involution in R defined by $\tilde{\lambda}'' \in \tilde{\Lambda}''$ is that defined by $\lambda \in \Lambda$ and that the action of $S \cap Z$ is that defined by the imbedding $S \to O_L$. Moreover ξ should commute with R.

Since we always fix an imbedding $\bar{\mathbb{Q}}_p \hookrightarrow \mathbb{C}$, we have in particular $O \hookrightarrow \mathbb{C}$ so that $(\tilde{A}'', \{\tilde{\varphi}''_\ell\}, \tilde{\Lambda}'')$ also yields a point of $M(\mathbb{C})$. Thinking, for the moment of $\tilde{A}''$ as been defined over $\mathbb{C}$, we have therefore an isomorphism

$$H_1(\tilde{A}'') \to V(\mathbb{Q}).$$

Since the Frobenius on A'' is given by an element of R, this isomorphism associates to the Frobenius an element γ in $V(\mathbb{Q})$ (or rather a conjugacy class). Since the finite field over which the data is defined is never specified, we do not really know γ, just γ^m for m sufficiently large in a multiplicative sense. Moreover $\tilde{A}''$ as a variety over $\mathbb{C}$ defines an h^0. Since $\tilde{A}''$ admits complex multiplication by some integer times γ, h^0 maps into $H^0(\mathbb{R})$. Before continuing with the discussion of (γ, h^0), I verify the existence of $\tilde{A}''$.

I may take my Dieudonné modules not over $W(\bar{k}_\mathfrak{p})$ but over $W(k'_\mathfrak{p})$ where $k'_\mathfrak{p}$ is a large but unspecified finite extension of $k_\mathfrak{p}$. (This is a technical point; its purpose is to keep $\mathcal{O}$ inside $\bar{\mathbb{Q}}_p$.) Set

$$B = O_L \otimes_{S \cap Z} S.$$

What I want to show is that there is a Dieudonné module.

$$\hat{M}' \subseteq \hat{M}$$

which is invariant under B so that the filtration

$$F\hat{M}'/p\hat{M}' \subseteq \hat{M}'/p\hat{M}' = \hat{M}' \otimes_{O'} O/\omega O$$

lifts to a B-invariant filtration of $\hat{M}' \otimes_{O'} O$ buy a subspace which is anisotropic with respect to the alternating form defined by an element of Λ. Here ω generates the maximal ideal of O and $O' = W(k'_\mathfrak{p})$ is also the largest unramified subring of O.

I may suppose that

$$O_L \otimes \mathbb{Z}_p = \bigoplus M_i$$

is a direct sum of matrix algebras over unramified extensions of $\mathbb{Z}_p$. Let $\{e'_{jj}\}$ be the standard idempotents in M_i. It is really only a question of choosing the $e^i_{jj}\hat{M}'$ for

$$\hat{M}' = \bigoplus_{i,j} e^i_{jj} \hat{M}'.$$

So for simplicity I suppose $O_L \otimes \mathbb{Z}_p$ it is an integral domain which is an unramified extension of $\mathbb{Z}_p$. A composition series for the $W(k'_\mathfrak{p})$ module $\hat{M}/\hat{M}'$ consists of one-dimensional spaces over $k'_\mathfrak{p}$. I shall also arrange that the length of this composition series is even. Observe that this will actually yield the same condition for the composition series of

$$e^i_{jj}\hat{M}/e^i_{jj}\hat{M}'.$$

S has not yet been chosen. Tentatively I take it to be the ring of integers in R. Let

$$O' \otimes_{\mathbb{Z}} S = \bigoplus S_j.$$

(Note: in the general case of the left side would be $e_i(O' \otimes_{\mathbb{Z}} S)$.) Since F and V almost commute with $O' \otimes_{\mathbb{Z}} S$, $F(\alpha \otimes \beta) = (\sigma(\alpha) \otimes \beta)F$, I can certainly find $\hat{M}'$ which is invariant under $O' \otimes_{\mathbb{Z}} S$. Observe this is again a tentative choice of $\hat{M}'$. The problem of lifting the filtration becomes that of lifting a filtration on

$$S_i \otimes_{O'} O/\omega O$$

to one on

$$S_i \otimes_{O'} O.$$

If $[S_i : O'] = r_i$ and $\bar{\omega}_i$ is a generator of the maximal ideal of S_i the first filtration must be of form $S_i/\omega_i^{k_i} S_i$, $0 \leqslant k_i \leqslant r_i$. In other words there is only one possible filtering subspace of a given dimension. Any ideal of $S_i \otimes_{O'} O$ with torsion-free quotient will therefore lift of the filtration. If k_i, k', and k are the quotient fields of S_i, O', and O and if $\varphi_1, \ldots, \varphi_{r_i}$ are the different imbeddings of k_i into k' over k then

$$\alpha \otimes \beta \to \bigoplus \varphi_i(\alpha)\beta$$

imbeds $S_i \otimes_{O'} O$ into

$$k \oplus \cdots \oplus k \qquad (r_i \text{ factors}).$$

The intersection of $S_i \otimes_{O'} O$ with the subspace obtained by setting $r_i - k_i$ factors equal to 0 has the correct rank.

We have to make sure that the listed submodule is anisotropic. If $S_i \neq S_i^*$ we take the factors for S_i^* to be the complement of those obtained by applying the involution to the ones chosen for S_i. If $S_i = S_i^*$ then $k_i = r_1/2$ and we just choose the factors so that one is not obtained from the other by applying the involution.

However, our choices were only tentative. Since

$$O' \otimes_{\mathbb{Z}} S \simeq \hat{M}'$$

it has the structure of a Dieudonné module. Let us examine this structure. Since

$$O' \otimes_{\mathbb{Z}} S \simeq O' \otimes_{\mathbb{Z}_p} (\mathbb{Z}_p \otimes_{\mathbb{Z}} S)$$

we may without any loss of generality suppose $\mathbb{Z}_p \otimes_{\mathbb{Z}} S$ is an integral domain for otherwise we have a direct sum. Let S_1 be the ring of integers in the smallest extension of k' containing S and regard $\mathbb{Z}_p \otimes_{\mathbb{Z}} S$ as a subring of S_1. Let u be the the degree of the maximal unramified extension of $\mathbb{Q}_p$ in the quotient field of $\mathbb{Z}_p \otimes_{\mathbb{Z}} S$. If σ is the Frobenius over $\mathbb{Q}_p$ then

$$\alpha \otimes \beta \to (\alpha\beta, \sigma(\alpha)\beta, \ldots, \sigma^{u-1}(\alpha)\beta)$$

defines an isomorphism of $O' \otimes_{\mathbb{Z}_p} (\mathbb{Z}_p \otimes_{\mathbb{Z}} S)$ with

$$S_1 \oplus \cdots \oplus S_1 \qquad (u \text{ times}).$$

F acts as

$$(\beta_1, \ldots, \beta_r) \to (\gamma_2\beta_2, \gamma_3\beta_3, \ldots, \gamma_u\beta_u, \gamma_1\sigma^u(\beta_1))$$

and V as

$$(\beta_1, \ldots, \beta_u) \to \left(\frac{p}{\gamma_1}\sigma^{-u}(\beta_\omega), \frac{p}{\gamma_2}\beta_1, \ldots, \frac{p}{\gamma_u}\beta_{u-1}\right)$$

If π generates the maximal ideal of S_1 we may take γ_i to be π^{ℓ_i} with

$$0 \leqslant \ell_i \leqslant [S_1 : O']$$

times a unit. Also σ^u is extended to S^1 in such a way that it is trivial on S. If the ℓ_i are not all equal to 0 or not all equal to $[S_1 : O']$ there is an i so that $\ell_i > 0$ and $\ell_{i-1} < [S_1 : O']$. (Take the subscripts to be of period r.) Then we can modify $\hat{M}'$ by taking only those $(\beta_1, \ldots, \beta_u)$ with $\beta_{i-1} \equiv 0 \pmod{\pi}$. This changes the length of a composition series by 1.

We still run into difficulty if all ℓ_i equal 0 or all ℓ_i equal $[S_1 : O']$. In this case, however, there is no difficulty in lifting the filtration, for it is trivial. Thus we may modify $\hat{M}'$ without attempting to preserve the invariance under the ring of integers of R. We can let S become another order. Moreover F and V act now in such a trivial fashion that an O' submodule is a Dieudonné submodule.[4] Thus it is easy to so modify $\hat{M}'$ that the length of a composition series changes by one.

Suppose A' is defined by the condition that there is an isogeny

$$\zeta : A' \to A$$

the order of whose kernel is a power of p so that the corresponding map on dual Dieudonné modules is

$$\hat{M}' \hookrightarrow \hat{M}.$$

Define φ'_ℓ as $\varphi_\ell \circ \zeta$ and define λ' by

$$\begin{tikzcd} A' \arrow[r, "\zeta"] \arrow[d, "\lambda'"] & A \arrow[d, "\lambda"] \\ \hat{A}' & \hat{A} \arrow[l, "\hat{\zeta}"] \end{tikzcd}$$

The lift of the filtering subspace of $\hat{M}'/p\hat{M}'$ to an anisotropic submodule of $\hat{M}' \otimes_{O'} O$ should allow us to lift A' and λ' to $\tilde{A}'$ and $\tilde{\lambda}'$ over O. This I take for granted. We can also lift φ'_ℓ, $\ell \neq p$, to $\tilde{\varphi}'_\ell$.

We start from λ, the order of whose kernel is prime to p. If the order of $\tilde{\lambda}'$ were prime to p it would follow from Note 5 that $\tilde{\varphi}'_p$ could be defined in a unique way and then, because of $O' \hookrightarrow \mathbb{C}$,

$$(\tilde{A}', \{\tilde{\varphi}'_\ell\}, \{\lambda'\})$$

would define a point of $M(\mathbb{C})$. In general what we have to check is that, over $\mathbb{C}$, we can find an isogeny $\psi : \tilde{A}' \to \tilde{A}''$, commuting with the action of O_L and having a kernel whose order is a power of p, so that $\tilde{\lambda}'$ factors as

[4] This is not correct. The desired multiplication is none the less easily effected.

$$\begin{array}{ccc} \tilde{A}' & \xrightarrow{\psi} & \tilde{A}'' \\ \Big\downarrow{\scriptstyle \tilde{\lambda}'} & & \Big\downarrow{\scriptstyle \tilde{\lambda}''} \\ \hat{\tilde{A}}' & \longleftarrow & \hat{\tilde{A}}'' \end{array}$$

where the kernel of $\tilde{\lambda}''$ has order prime to p. Then we could replace $\tilde{A}'$ by $\tilde{A}''$, $\tilde{\varphi}'_\ell$ by $\tilde{\varphi}''_\ell$ $(\ell \neq p)$ where

$$\tilde{\varphi}''_\ell \circ \psi = \tilde{\varphi}'_\ell$$

and $\tilde{\lambda}'$ by $\tilde{\lambda}''$ to obtain a point of $M(\mathbb{C})$. We might also have to change the order S.

What we have to do is show that there is a lattice U in $T_p(\tilde{A}') \otimes \mathbb{Q}_p$ so that

$$T_p(\tilde{A}') \subseteq U$$

And so that the bilinear form associated to $\tilde{\lambda}'$ is integral on U with unit determinant. Observe that $\tilde{A}'$ is not being treated as a variety over $\mathbb{C}$.

We find a standard form for the bilinear form on $T_p(\tilde{A}')$. We use the notation of Note 5, except that $T_p(\tilde{A}')$ replaces $V(\mathbb{Z}_p)$. If $M_1^* = M_j$, $i \neq j$, then we find x_ℓ^k, y_ℓ^k as there except that

$$\langle x_\ell^k, y_{\ell'}^{k'} \rangle = \alpha_\ell \delta_{kk'} \delta_{\ell\ell'}$$

where α_ℓ is a power of p. We obtain a partial basis for U by taking

$$\{\alpha_\ell^{-1} x_{\ell'}^k, y_{\ell'}^{k'}\}.$$

Suppose $M_i^* = M_i$. Let M and V have the same meaning as in Note 5. If the involution is trivial on the centre of M then we can find a basis $\{x_1, \ldots, x_r, y_1, \ldots, y_r\}$ of V_{11} so that

$$\langle x_i, x_j \rangle = \langle y_i, y_j \rangle = 0$$

and so that

$$\langle x_i, x_j \rangle = \alpha_i \delta_{ij}$$

where α_i is a power of p. We take as basis for the analogue for U of V_{11},

$$\{\alpha_1^{-1} x_1, \ldots, \alpha_r^{-1} x_r, y_1, \ldots, y_r\}.$$

Suppose the involution is not trivial on the centre of M. Then on V_{11} there are coordinates so that ψ''_2 has the form

$$\sum \alpha_i x_i y_i^*$$

where α_i is a power of p. Enlarging $T_p(\tilde{A}')$ if necessary we may assume that each α_i is either 1 or p. The number of p is equal, modulo 2, to the length of the composition series of

$$e_{11}^i \hat{M} / e_{11}^i \hat{M}'$$

and is therefore even. Thus choosing a new basis we can put ψ''_2 in the form

$$\sum_{i=1}^{a} x_i y_i^* + p \sum_{j=1}^{b} (u_j v_{b+j} - u_{b+j} v_j^*)$$

This makes it easy to define U.

You may have observed that there is a gap in the above discussion; I just did. I have to guarantee that the trace of the action of O_L on the tangent space of $\tilde{A}''$ is correct. This is the same as the trace of the action on the tangent space of $\tilde{A}'$. To obtain the correct trace, we have to exercise more care in our choice of $\hat{M}'$. What we have to ensure is that the elements of Z have the same trace on

$$e^i_{jj}(\hat{M}'/F\hat{M}')$$

as on

$$e^i_{jj}(\hat{M}/F\hat{M}).$$

Again when checking this we may suppose $O_L \otimes \mathbb{Z}_p$ is already a commutative integral domain. It is unramified over $\mathbb{Z}_p$. We regard it it as a subring O'' of O'. Let $[O'' : \mathbb{Z}_p] = v$. There are integers $k_0, \ldots, k_{v-1}$ so that the action on the tangent space will be correct if the trace of $a \in O'$ on $\hat{M}'/F\hat{M}'$ is

$$k_0 a + k_0 \sigma^{-1}(a) + \cdots + k_{v-1}\sigma^{-(v-1)}(a) \pmod{p}.$$

Let

$$\mathbb{Z}_p \otimes_{\mathbb{Z}} S = \bigoplus S^j.$$

Consider the corresponding decompositions into

$$S^j_1 \oplus \cdots \oplus S^j_1.$$

The action of F on one of these pieces is given by integers ℓ^j_i. Only

$$\sum_i \ell^j_i$$

is *a priori* determined. This direct sum decomposition of $\hat{M}'$ does not yield a direct sum decomposition of $\hat{M}$. However, it does yield a decomposition series which is just as good for computing traces as well as indices so we may suppose once again that $\mathbb{Z}_p \otimes_{\mathbb{Z}} S$ is already an integral domain. We drop the subscript j. Since

$$\sum_{i=1}^{u} [S_1 : O'] - \ell_i = \sum_{j=0}^{v-1} k_j$$

and since

$$k_j \leqslant \frac{u}{v}[S_1 : O']$$

we may suppose

$$\sum_{i \equiv j+1 \pmod{v}} [S_1 : O'] - \ell_i = k_j.$$

This gives the required trace. But we have to worry once again about the conditions on the indices.

$O'' \otimes_{\mathbb{Z}_p} O'$ acts on both $\hat{M}'$ and $\hat{M}$. It contains the projections on $L'_0, \ldots, L'^p_{v-1}$ where L'_j is the set of

$$(\beta_1, \ldots, \beta_u) \in S_1 \oplus \cdots \oplus S_1$$

such that $\beta_i = 0$ unless $i \equiv j+1 \pmod{v}$. Then

$$FL'_j = L'_{j-1}.$$

Let L_j be the corresponding space for $\hat{M}$. Then $FL_j \subseteq L_{j-1}$. But if $[L_j : L'_j]$ denotes the length of a composition series for a quotient

$$[L_j : L'_j] = [FL_j : FL'_j]$$

and

$$[L'_j : FL'_{j+1}] = k_j = [L_j : FL_{j+1}].$$

Since

$$[L_j : FL'_{j+1}] = [L_j : L'_j] + [L'_j : FL'_{j+1}] = [L_j : FL_{j+1}] + [FL_{j+1} : FL'_{j+1}]$$

we conclude that

$$[L_j : L'_j] = [L_{j+1} : L'_{j+1}].$$

Thus if v is even

$$[\hat{M}' : \hat{M}] = \sum_j [L_j : L'_j]$$

is automatically even.

Since it is only for v even that we actually exploited the condition on the length of the composition series, we may forget about odd v.

We verify first that H, defined in terms of A and Λ, and H^0, defined by γ, bear the predicted relation to each other and that X, defined in terms of A and Λ, has the structure defined by γ and h^0. I observe first that it is clear, from Tate's theorem, that H and H^0 are isomorphic over $\mathbb{Q}_\ell$ for $\ell \neq p$. It is also clear that any two choices of γ are, if we define them in terms of the Froebenius automorphisms over the same extension of $k_{\mathfrak{p}}$, conjugate in $G(\mathbb{Q}_\ell)$.

Suppose T is the Cartan subgroup of H used to define γ and h^0. The isomorphism of H with H^0 over $\mathbb{Q}_\ell$ takes T to a Cartan subgroup T^0 of H^0 which is again defined over $\mathbb{Q}$, namely to the Cartan subgroup of H^0 defined by taking the centralizer of R, which acts on $\tilde{A}''$ and hence on $V(\mathbb{Q})$, in G. The map $T \to T^0$ is, however, defined over $\mathbb{Q}$. It corresponds to taking the identity map on R. Thus H is obtained from H^0 by an inner twisting with a cocycle from the image of T^0 in H^0_{ad}. Since $H_{\mathrm{der}}(\mathbb{R})$ is compact because the Rosati involution is positive, the twisting is the predicted one at infinity.

I again take the Dieudonné module $\hat{M}$ to be defined over $W(\bar{k}_{\mathfrak{p}})$. Then $\hat{N}$ is defined over the quotient field k of $W(\bar{k}_{\mathfrak{p}})$. There is an $O_L \otimes_{\mathbb{Z}} R$ isomorphism of $\hat{N}$ with

$$V \otimes k$$

which preserves the bilinear form. We want to analyze the structure of $\hat{N}$. As before there is no loss of generality in assuming that

$$O_L \otimes_{\mathbb{Z}} \mathbb{Q}_p$$

is already a field. We can also assume that $R \otimes \mathbb{Q}_p$ is a field k_1. Then we may identify $V(\mathbb{Q})$ with k_1.

I observe that, although it was not made explicit, our argument showed that S, which acts on $\tilde{A}''$, can be so chosen that its completion contains the projections on the factors of $R \otimes \mathbb{Q}_p$. I assume it has been so chosen. Let $\hat{M}'' \subseteq \hat{N}$ be the dual Dieudonné module of the reduction A'' of $\tilde{A}''$.

As before if k_2 is the extension of k determined by k_1

$$k \otimes k_1 \simeq k_2 \oplus \cdots \oplus k_2.$$

The isomorphism being of the form

$$\alpha \otimes \beta \to (\alpha\beta, \sigma(\alpha)\beta, \ldots, \sigma^{u-1}(\alpha)\beta)$$

F acts as

$$(\beta_1, \ldots, \beta_r) \to (\gamma_2\beta_2, \ldots, \gamma_r\beta_r, \gamma_1\sigma^u(\beta_1)).$$

Because we can change coordinates on $k_2 \oplus \cdots \oplus k_2$ without changing the action of $k \otimes k_1$, the only invariant of F is

$$\left|\prod \gamma_i\right|.$$

We have seen before with this absolute value is. If π is a uniformizing parameter for k_2 it is

$$|\pi|^u$$

where u is the dimension over $\bar{k}_{\mathfrak{p}}$ of the range in $\hat{M}''/F\hat{M}''$ of the projection corresponding to the factor k_1. This is the number of imbeddings of k_1 in the closure of $\bar{\mathbb{Q}}_p$ yielding $R \to \mathbb{C}$ coming from the action of R on

$$e^i_{jj} Tang \tilde{A}''/C.$$

[5] These yield the characters of T on which $\hat{\mu}$ takes the value 1. By the way, with the present conventions I have to replace the $\hat{\mu}$ of the appendix by its negative.

The action of σ alone is

$$(\beta_1, \ldots, \beta_r) \to (\beta_2, \ldots, \beta_r, \sigma^u(\beta_1)).$$

Thus the twisting defining the action of F is given by

$$(\gamma_2, \ldots, \gamma_r, \gamma_1).$$

Since

$$\langle Fx, Fy\rangle = p\sigma(\langle x, y, \rangle)$$

the element b of $R \otimes k$ defined by putting together the $(\gamma_2, \ldots, \gamma_r, \gamma_1)$ actually lies in $T(k)$. We have just seen that b is basically determined by

$$|\lambda(b)|$$

where λ is a rational character of T defined over $\mathbb{Q}_p$. The expression $(*)$ shows that this absolute value is

$$p^{-\langle \lambda, \hat{\mu}\rangle}.$$

This shows that the present F is the F defined in terms of γ and h^0. Consequently any two (γ, h^0) associated to $(A, \{\varphi_\ell\}, \Lambda)$ are going to be equivalent in the sense described.

The $\bar{G}(\mathbb{Q}_p)$ defined in terms of $(A, \{\varphi_\ell\}, \Lambda)$ is the set of all

$$g \in G(k)$$

such that

$$gF = Fg.$$

Then

$$gF^r = F^r g$$

for all r. In order to compare with the prediction we take F in the form given earlier, that is, the predicted form. Then for a suitable r

$$F^r = p^{\hat{\nu}} \cdot \sigma^r.$$

Then

$$p^{-s\hat{\nu}} g p^{s\hat{\nu}} = \sigma^{rs}(g).$$

It follows readily that g commutes with $p^{s\hat{\nu}}$ and hence with $p^{\hat{\nu}}$. Then it follows that $\bar{G}$ is obtained from $\bar{G}^0$ by twisting in the predicted manner. Now $H(\mathbb{Q})$ is nothing but the

[5] What is $Tang$?

centralizer of large powers of γ in $\bar{G}(\mathbb{Q}_p)$. Therefore H over $\mathbb{Q}_p$ is obtained from H^0 over $\mathbb{Q}_p$ by twisting in the specified manner.

As I said we shall parameterize several isogeny classes by the same family of (γ, h^0). Let me now describe the isogeny classes which are lumped together with that of $(A, \{\varphi_\ell\}, \Lambda)$. Let $B = \mathrm{End}_{O_L} A_1 \otimes \mathbb{Q}$. Suppose d is a symmetric element in B which is positive in $B(\mathbb{R})$. Multiplying d by an integer if necessary we may suppose $\lambda \circ d$ is again a polarization. Over $\bar{\mathbb{Q}}$

$$d = b^* b$$

and $a_\sigma = b\sigma(b^{-1})$ defines an element of $H^1(\mathfrak{G}(\bar{\mathbb{Q}}/\mathbb{Q}), H_1)$ if H_1 is the kernel of $H \to C$. Suppose the image of $\{a_\sigma\}$ in $H^1(\mathfrak{G}(\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell), H)$ is trivial for $\ell \neq p$. Then

$$b\sigma(b^{-1}) = h_\ell \sigma(h_\ell^{-1}) \qquad \sigma \in \mathfrak{G}(\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$$

with $h_\ell \in H(\bar{\mathbb{Q}}_\ell)$ and

$$d = b^* h_\ell^{*-1} h_\ell^* h_\ell h_\ell^{-1} b = g_\ell^{*-1} g_\ell^{-1} c_\ell$$

with $c_\ell \in C(\mathbb{Q}_\ell)$ and $g_\ell \in B(\mathbb{Q}_\ell)^\times$. Thus if $\langle x, y \rangle$ is the bilinear form determined by λ then

$$\langle g_\ell x, d g_\ell y \rangle = \langle x, c_\ell y \rangle.$$

We define A' and $\varphi : A \to A'$ so that it is possible to construct the commutative diagram

$$\begin{tikzcd} T_\ell(A') \arrow[r, "\varphi"] \arrow[d, "\varphi'_\ell"'] & T_\ell(A) \arrow[d, "\varphi_\ell"] \\ V(\mathbb{Q}_\ell) \arrow[r, "g_\ell"'] & V(\mathbb{Q}_\ell) \end{tikzcd}$$

so that φ'_ℓ takes $T_\ell(A')$ to $V(\mathbb{Z}_\ell)$. For this we may have to multiply g_ℓ by an integer. We define λ' by pulling back $\lambda \circ d$. Then

$$\begin{aligned} \langle x, \lambda'(y) \rangle &= \langle \varphi(x), \lambda d \varphi(y) \rangle = \psi(\varphi_\ell \varphi(x), \varphi_\ell d \varphi(y)) \\ &= \psi(g_\ell \varphi'_\ell(x), d g_\ell \varphi'_\ell(y)) \\ &= \psi(\varphi'_\ell(x), c_\ell \varphi'_\ell(y)). \end{aligned}$$

We have here allowed d to operate on both $T_\ell(A)$ and on $V(\mathbb{Q}_\ell)$ and have used to the fact that it is symmetric.

To force $(A, \{\varphi_\ell\}, \Lambda)$ to lie in $M(\bar{k}_\mathfrak{p})$ we have to exercise some care in the choice of the p-component of the kernel of φ, which is still at our disposal. Let $\hat{M}' \hookrightarrow \hat{M}$ be the dual Dieudonné module of $\hat{A}'$. If the above triple is to lie in $M(\bar{k}_\mathfrak{p})$ there must be a $c \in C(\mathbb{Q})$ so that

$$d\hat{M}' = cM'.$$

Without any loss of generality we may suppose $c = 1$. There is another property of d I want to deduce from

$$d\hat{M}' = M'$$

and the fact that the traces of the elements of O_L on $\hat{M}'/F\hat{M}'$ are correct. For this I can proceed as before and suppose that $O_L \otimes \mathbb{Z}_p$ is a commutative integral domain O''. For simplicity let $O' = W(\bar{k}_\mathfrak{p})$. The decomposition

$$O'' \otimes_{\mathbb{Z}_p} O' \simeq O' \oplus \cdots \oplus O' \qquad v = [O'' : \mathbb{Z}_p] \text{ factors}$$

leads to corresponding decompositions of $\hat{M}'$ and $\hat{M}$.

$$\hat{M}' = \bigoplus \hat{L}'_j \qquad \hat{M} = \bigoplus \hat{L}_j.$$

We may define $[\hat{L}_j : \hat{L}'_j]$ as the difference $[\hat{L}_j : X] - [\hat{L}'_j : X]$ where X is a sufficiently small to lie in both $\hat{L}_j$ and $\hat{L}'_j$. An argument used before shows that

$$[\hat{L}_j : \hat{L}'_j]$$

is independent of j. Moreover if

$$M' = \bigoplus L'_j \qquad M = \bigoplus L_j$$

so that $L = \hat{L}_j$ then

$$[L'_j : L_j] = [\hat{L}_j : \hat{L}'_j].$$

Thus

$$[L'_j : \hat{L}'_j] = 2[\hat{L}_j : \hat{L}'_j].$$

Thus

$$|\det d| = p^{-2v[\hat{L}_1 : \hat{L}'_1]}.$$

In particular if v is even this exponent is a multiple of 4.

Conversely suppose there is a $c \in C(\mathbb{Q}) \cap O_L$ so that the order of the determinant of $e^i d$ is a multiple of $2v$ for every simple idempotent of $O_L \otimes \mathbb{Z}_p$ which is invariant under the involution. Then I claim there is a Dieudonné module $\hat{M}'$ so that

$$d\hat{M}' = cM'$$

and so that the trace of O_L on $\hat{M}'/F\hat{M}'$ is correct. Again we may suppose $O_L \otimes \mathbb{Z}_p = O''$ is already a commutative integral domain and that $c = 1$, except in one case. Suppose the involution interchanges two factors in the direct sum decomposition of the centre of $O_L \otimes \mathbb{Z}_p$. Suppose $\hat{M}_1$, $\hat{M}_2$ are the corresponding factors of $\hat{M}$. We can take $\hat{M}'_1$ to be $\hat{M}_1$ and $\hat{M}'_2$ to be the dual of $d\hat{M}'_1$. I remind you that we are using the original λ to identify $\hat{M}$ and M. Anyhow this trivial case disposed of, suppose that $O_L \otimes \mathbb{Z}_p = O''$. The involution can be trivial or not on O''. Suppose it is trivial.

Introduce the modules $\hat{L}_j$ as above. They are mutually orthogonal and

$$d\hat{L}_j \subseteq \mathbb{Q}\hat{L}_j.$$

We first try to find M' without trying to control the trace of the elements in O_L. The standard form of $\langle x, dy\rangle$ in $\hat{L}_j$ will have a matrix

$$\begin{pmatrix} & & & p^{e_1} & & \\ & & & & \ddots & \\ & & & & & p^{e_r} \\ -p^{e_1} & & & & & \\ & \ddots & & & & \\ & & -p^{e_r} & & & \end{pmatrix}$$

with $e_1 \leqslant e_2 \leqslant \cdots \leqslant e_r$. Suppose the maximum of the e_r as j varies is positive. Let it be e. Consider

$$X = \{x \in \hat{M} \mid \langle x, dy\rangle \equiv 0 \pmod{p^e} \text{ for all } y \in \hat{M}\}.$$

X is invariant under F and V because

$$\langle Fx, y\rangle = \sigma(x, Vy).$$

Also

$$X = \bigoplus X_j$$

with $X_j = X \cap \mathbb{Q}\hat{L}_j$. The form

$$\frac{1}{p^e}\langle x, dy\rangle$$

is non-degenerate on $X/X \cap p\hat{M}$. Since $\langle Fx, Fy\rangle = p\sigma\langle x, y\rangle$ and $\langle Vx, Vy\rangle = p\sigma^{-1}\langle x, y\rangle$ both FX and VX give isotropic subspaces of $X/X \cap p\hat{M}$. If $FX \not\subseteq pX + p\hat{M}$ or $VX \not\subseteq pX + p\hat{M}$ we take Y to be one of

$$FX + pX \text{ or } VX + pX.$$

Otherwise we just take Y so that $X \supseteq Y \supseteq pX$ and Y/pX is maximal isotropic in $X/pX + p\hat{M}$ and invariant under $O' \otimes O''$. We replace $\hat{M}$ by

$$\hat{M} + \frac{Y}{p}.$$

This has the effect of decreasing the number of e_k which equal e. Continuing we may eventually reach the stage that no e_r is positive.

Suppose some e_1 is negative. Let e be the smallest of the e_1.

Consider

$$X = \{x \in \hat{M} \mid \langle x, dy\rangle \equiv 0 \pmod{p^{e+1}}\}.$$

X is again invariant under F and V and p annihilates $\hat{M}/X$. If $F\hat{M} \not\subseteq X$ or $V\hat{M} \not\subseteq X$ we replace $\hat{M}$ by $F\hat{M} + X$ or $N\hat{M} + X$. Otherwise we replace it by Y so that Y/X is a maximal isotropic subspace of $\hat{M}/X$ with respect to

$$p^{-e}\langle x, dy\rangle \pmod{p}$$

and so that Y is invariant under $O'' \otimes O'$. Again we can repeat this process arriving finally at $\hat{M}'$ such that $d\hat{M}'$ is the dual of $\hat{M}'$.

$$\hat{M} = \bigoplus \hat{L}'_j.$$

Moreover

$$[\hat{L}'_j : \hat{L}_j] = \sum e_i$$

Since $\sum e_i$ is independent of j, a variant of the argument used before shows that

$$[\hat{L}'_{j-1} : F\hat{L}'_j] = [\hat{L}_{j-1} : F\hat{L}_j].$$

Thus there is no problem with the traces.

Suppose now the involution is not trivial. Then v is even. The involution is extended to $O'' \otimes_{\mathbb{Z}_p} O'$ by making it trivial on O'. If we regard the elements of

$$O' \oplus \cdots \oplus O'$$

as infinite sequences by setting

$$\beta_{i+v} = \sigma^{-v}(\beta_i)$$

then its effect is to replace (β_i) by β'_i with

$$\beta'_i = \sigma^{v/2}(\beta_{i+v/2}).$$

Then $\hat{L}_i$ is orthogonal to $\hat{L}_j$ unless $j \equiv i + r/2 \pmod{v}$. We may choose bases $\{x_k\}$ and $\{y_\ell\}$ for $\hat{L}_j$ and $\hat{L}_{i+v/2}$, $0 \leqslant i < v/2$ so that

$$\langle x_k, dy_\ell \rangle = \delta_{k\ell} p^{e_\ell}$$

with $e_1 \leqslant \cdots < e_r$. Our condition on the determinant of d says that

$$\sum e_\ell$$

which is independent of i, is even. Again suppose that some e_r is positive. Suppose the maximum e_r is e.

If $0 < 2f < e$ consider

$$X = \{x \in \hat{M} \mid \langle x, dy \rangle \equiv 0 \pmod{p^e} \text{ for all } y \in \hat{M}\}.$$

Again X is invariant under F and V and we may replace $\hat{M}$ by $\hat{M} + \frac{X}{p^f} = Y$. Then

$$[Y_j : \hat{L}_j] = [Y_{j+v/2} : \hat{L}_{j+v/2}].$$

This allows us to carry out a reduction until all e_ℓ are at most 1.

In the same way if the smallest e_1 is negative let it be e. If $e < 2f < 0$ and

$$X = \{x \in \hat{M} \mid \langle x, dy \rangle \equiv 0 \pmod{p^{e+1}} \text{ for all } y \in \hat{M}\}$$

we can replace $\hat{M}$ by

$$X + p^f \hat{M}.$$

The condition $(*)$ will again be satisfied.

We suppose then all e_ℓ are either -1, 0 or 1. A reduction similar to the above allows us to suppose that they are in fact all 0 or 1. Let

$$X = \{x \in \hat{M} \mid \langle x, dy \rangle \equiv 0 \pmod{p} \text{ for all } y \in \hat{M}\}.$$

Suppose $e_r = 1$ for some i. Then e_{r-1} is also 1. Choose a minimal non-zero subspace of $X_i / X_i \cap p\hat{M}$ which is invariant under F and V. This subspace must be annihilated by F and V and is therefore one-dimensional. Let the subspace be generated by the image of $x \in X_i$. The orthogonal complement of x in

$$X_{i+v/2} / X_{i+v/2} \cap p\hat{M}$$

with respect to $1/p\langle x, dy \rangle \pmod{p}$ is not 0. It is moreover invariant under F and V. Therefore it contains a one-dimensional subspace invariant under F and V. Let this space be spanned by γ. Then we can replace $\hat{M}$ by

$$\hat{M} + \frac{O'x}{p} + \frac{O'y}{p}.$$

This allows us to reduce the number of e_i which are 1. Continuing, we finally arrive at $\hat{M}'$. What we have arranged in addition is that

$$[\hat{L}'_i : \hat{L}_i] = [\hat{L}'_{i+v/2} : \hat{L}_{i+v/2}].$$

It is this condition which guarantees that the traces of the elements of O_L are correct. We can now proceed as before.

The condition on the determinants of b can be reinterpreted. It means simply that the image of $\{a_\sigma\}$ in

$$H^1(\mathfrak{G}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p), G_{\mathrm{der}} \backslash G_1)$$

is 0. Since Hasse's principle is valid for the group $G_{\mathrm{der}}\backslash G_1$ (cf. Travaux de Shimura), this, together with the demand that $\{a_\sigma\}$ be trivial away from p, implies that $\{a_\sigma\}$ has trivial image in $H^1(\mathfrak{G}(\bar{\mathbb{Q}}/\mathbb{Q}), G_{\mathrm{der}}\backslash G_1)$. Conversely suppose we have a cocycle $\{a_\sigma\}$ which is trivial away from p and has trivial image in $H^1(\mathfrak{G}(\bar{\mathbb{Q}}/\mathbb{Q}), G_{\mathrm{der}}\backslash G)$. Then in particular it has trivial image in $G_1\backslash G = H_1\backslash H$ so that it may be supposed to take values in H_1. Since, by Theorem 90, $B^\times$ has no cohomology

$$a_\sigma = b\sigma(b^{-1}) \qquad b \in B^\times(\bar{\mathbb{Q}}).$$

Set

$$d = b^* b.$$

Then d is a symmetric positive element in $B(\mathbb{Q})$. This shows that we have lumped the elements into isogeny classes in a way consistent with our predictions.

It remains to verify that every (γ, h^0) yields an isogeny class and that if (γ_1, h_1^0), (γ, h_2^0) are equivalent they yield the same isogeny classes F in a Cartan subgroup T of H^0 containing γ. Let R be the centralizer of T in the ring of O_L-endomorphisms of $V(\mathbb{Q})$. h^0 and $V(\mathbb{Z})$ define together an abelian variety $\tilde{A}$ over $\mathbb{C}$ on which O_L and an order S in R act. $\tilde{A}$ comes provided with $\tilde{\Lambda}$ and $\tilde{\varphi}_\ell$. $\tilde{A}$ and $\tilde{\Lambda}$ can be defined over some finite extension of E. Moreover, although this is something I have still to check, if we make the extension large enough they should reduce well at a prime dividing $\mathfrak{p}$. It is clear that (γ, h^0) correspond to the isogeny class obtained by reduction. One just has to check that some power of γ reduces to the Frobenius.

However, if γ_1 is the element of R given by the Frobenius then, replacing γ and γ_1 by integral powers if necessary, we see from Manin's theorem and the discussion above that for any homomorphism λ of R into $\bar{\mathbb{Q}}$

$$|\lambda(\gamma\gamma_1^{-1})|_v = 1$$

for all valuations v. Thus $\gamma\gamma_1^{-1})$ is a root of unity in R. Taking powers again

$$\gamma = \gamma_1.$$

Suppose (γ_1, h_1^0) and (γ_2, h_2^0) are equivalent. Replacing γ_1 and γ_2 by powers we may suppose $\gamma_1 = \gamma_2 = \gamma$. Let A_1, A_2 be corresponding varieties over $\bar{k}_{\mathfrak{p}}$. Tate's theorem guarantees the existence of an isogeny

$$A_1 \xrightarrow{\varphi} A_2.$$

For each $\ell \neq p$ let $h_\ell : V(\mathbb{Q}_\ell) \to V(\mathbb{Q}_\ell)$ be defined by the commutativity of

$$\begin{array}{ccc} T_\ell(A_1) & \xrightarrow{\varphi} & T_\ell(A_2) \\ \downarrow & & \downarrow \\ V(\mathbb{Q}_\ell) & \longrightarrow & V(\mathbb{Q}_\ell) \end{array}$$

h_ℓ commutes with O_L and with γ. Suppose $\lambda_1 \in \Lambda_1$, $\lambda_2 \in \Lambda_2$. Define λ_2' by

$$\begin{array}{ccc} A_1 & \xrightarrow{\varphi} & A_2 \\ \downarrow{\scriptstyle \lambda_1} & & \downarrow{\scriptstyle \lambda_2'} \\ \hat{A}_1 & \xleftarrow[\hat{\varphi}]{} & \hat{A}_2 \end{array}$$

Some multiple of λ_2' is a true isogeny. Set

$$\lambda_2' = \lambda_2 \circ d$$

where d is a positive symmetric element in $\mathrm{End}_{O_L} A_2 \otimes \mathbb{Q} = B$. Then

$$\langle \varphi(x), d\varphi(y)\rangle_2 = \langle h_\ell x, c_\ell d h_\ell y\rangle_1 \qquad c_\ell \in C(\mathbb{Q}_\ell)$$

This shows that (γ_1, h_1^0), (γ_2, h_2^0) define the same family of isogeny classes.

There is still the structure of X to be considered. As before, if k is the quotient field of $W(\bar{k}_\mathfrak{p})$ we identify $\hat{N}$ with $V \otimes k$. We may suppose $\hat{M}$ corresponds to $V \otimes W(\bar{k}_\mathfrak{p})$. We defined X'' as

$$G(k)/k_p(k).$$

This is just the set of $W(\bar{k}_\mathfrak{p})$ lattices $\hat{M}'$ in $\hat{N}$ invariant under O_L and with dual M' of the form

$$(*) \qquad M' = c\hat{M}' \qquad c \in C(k).$$

The parahoric subgroup I is just the stabilizer of the pair $F\hat{M} \hookrightarrow \hat{M}$.

If $\hat{M}_1'$, $\hat{M}_2'$ are any two lattices satisfying $(*)$ then they determine special vertices x_i^1, x_i^2 in the Bruhat-Tits building of each simple factor of G. The condition that x_i^1, x_i^2 are either the same or are joined by an edge and that the corresponding parahoric is of type I says that for some $z \in Z(k^\times)$ either

$$\text{(i)} \qquad p\hat{M}_1' \subseteq z\hat{M}_2' \subseteq \hat{M}_1'$$

or

$$\text{(ii)} \qquad p\hat{M}_2' \subseteq z\hat{M}_1' \subseteq \hat{M}_2'$$

and the trace of $o \in O_L$ on $z\hat{M}_2' \backslash \hat{M}_1'$, respectively $z\hat{M}_1' \backslash \hat{M}_2'$, is that on $F\hat{M} \backslash \hat{M}$. To see this observe that to give all x_i^1 is to give $\hat{M}_1'$ up to a factor in $Z^\times(k)$. Thus if the corresponding parahoric is of type I we can find z_1, z_2 and $g \in G(k)$ so that either $z_1 g\hat{M}_1'$, $z_2 g\hat{M}_2'$ or $z_2 g\hat{M}_2'$, $z_1 g\hat{M}_1'$ is the pair $F\hat{M}$, $\hat{M}$.

Each point of X'' determines a point in the Bruhat-Tits building of $G_{\mathrm{der}} \backslash G$, i.e. a point of $G_{\mathrm{der}} \backslash G(k)$ modulo elements all of whose eigenvalues have absolute value 1. Since $\hat{M}$ is fixed we may speak of the determinant of a lattice $\hat{M}'$. More precisely, if e is a standard idempotent of $O_L \otimes k$ we may speak of the determinant of $e\hat{M}'$ or at least of its order. It is these orders which give the point in the Bruhat-Tits building of $G_{\mathrm{der}} \backslash G$ corresponding to $\hat{M}'$. If

$$\dim(e\hat{M}/eF\hat{M}) = a(e)$$

then the difference between the orders of $e\hat{M}'$ and $eF\hat{M}'$ is $a(e)$.

Suppose $\hat{M}_1'$, $\hat{M}_2'$ are taken to be $\hat{M}'$, $F\hat{M}'$. In case (i) the difference between the orders of $e\hat{M}_1'$ and $e\hat{M}_2'$ is

$$a(e) + b(e)\,\mathrm{ord}(ez).$$

Here $b(e)$ is the rank of $e\hat{M}$ over $W(\bar{k}_\mathfrak{p})$ and ez is treated as a scalar. In case (ii) the difference is

$$-a(e) - b(e)\,\mathrm{ord}(ez).$$

Thus if case (i) occurs, z may be taken to be 1. Case (ii) can only occur if

$$-b(e)\,\mathrm{ord}(ez) = 2a(e)$$

for all e. If e^* is obtained from e by the involution then $b(e^*) = b(e)$ and $a(e^*) = b(e) - a(e)$. Thus this equation implies that $2a(e) = b(e)$ for all e and that $\mathrm{ord}(ez) = 1$ for all e; so that z may be taken to be p. Then (i) holds with $z = 1$. In other words the points of the predicted X are just those $\hat{M}'$ which satisfy $(*)$ and

$$(**) \qquad p\hat{M}' \subseteq F\hat{M}' \subseteq \hat{M}'.$$

and for which the trace of the elements of O_L on $F\hat{M}'\backslash\hat{M}'$ is correct. Thus the predicted X gives the actual X.

Yours,

Bob (September 2, 1974)

P.S.

(i) The notes will follow.
(ii) Because of the blunder on p. 16 all sorts of F's must be replaced by V. Moreover the parahoric I must be replaced by its opposite.

Notes

Note 1. Let A be an abelian scheme over S. An element of $H^1_{\mathrm{DR}}(A) = H^1_{\mathrm{DR}}(A/S)$ is obtained from an open covering $\{U_1\}$ of A by S-schemes U_i together with sections ω_i of $\Omega^1_{A/S}$ on U_i and sections g_{ij} of $\mathcal{O}_A$ on $U_i \cap U_j$ so that

$$\omega_i - \omega_j = dg_{ij}$$

on $U_i \cap U_j$.

This universal vector extension

$$0 \to \omega_A \to E(A) \to A \to 0$$

is a principal homogeneous space over A with group $\omega_{\hat{A}}$ in the Zariski topology. To verify this one must, so far as I can see, observe that although $E(A)$ is introduced as a principal homogeneous space for the fppf topology, it becomes trivial over every open affine subscheme of A, because $\omega_{\hat{A}}$ is a vector group, and is therefore a principal homogeneous space for the Zariski topology.

Choose an open covering U_i of A and isomorphisms

$$\varphi_i : E(A)|_{U_i} \simeq \omega_{\hat{A}} \times U_i.$$

Let

$$\varphi_i \varphi_j^{-1} : (x; u) \to (x + \psi_{ij}(u), u)$$

on $U_i \cap U_j$. Let $\nu_i \times \omega_i$ be the restriction of μ to $\omega_{\hat{A}} \times U_i$. ν_i is an invariant differential form on $\omega_{\hat{A}}$ and hence defines an element of the dual space of $\omega_{\hat{A}}$. Set

$$g_{ij}(u) = \nu_j(\psi_{ij}(u)).$$

Since μ is well-defined

$$\omega_i = d(\nu_j(\psi_{ij}(u))) + \omega_j = dg_{ij} + \omega_j.$$

If φ_i is modified by composing it with

$$(x, u) \to (x + \eta_i(u), u)$$

then ω_i is replaced by

$$g_{ij} + f_i - f_j.$$

This gives us a well-defined map $\omega_{E(A)} \to H^1_{\mathrm{DR}}(A)$.

It is compatible with the filtrations

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \omega_A & \longrightarrow & \omega_{E(A)} & \longrightarrow & \omega^*_{\hat{A}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^0(\Omega^1_{A/S}) & \longrightarrow & H^1_{\mathrm{DR}}(A/S) & \longrightarrow & H^1(\Omega^0_{A/S}) & \longrightarrow & 0
\end{array}$$

The middle arrow is an isomorphism because the first and last are.

Note 2. There is a possibility of error when passing from the analytic to the algebraic definition of various objects associated to an abelian variety. In this note I check some things against Mumford's book in an attempt to keep the number of errors down.

If the abelian variety A is represented as the quotient of $V(\mathbb{R})$ by a lattice U then $h(z)$ is defined so that $h(z^{-1})$ is the action of z on $V(\mathbb{R})$ regarded as the tangent space to A.

Our $\Psi(u, cv)$ is Mumford's $E(u, v)$. As in Mumford (p. 237) let π_ℓ denote the natural map from U to $T_\ell(A)$. Thus u is mapped to the sequence $u_n = u/\ell^n \pmod U$. We choose the isomorphism

$$x \to e^{-2\pi i_x/\ell^n}$$

of $\mathbb{Z}/\ell^n\mathbb{Z}$ with μ_{ℓ^n} and use these to identify $\mathbb{Z}_{\ell^\infty}$ with μ_{ℓ^∞}. According to p. 237 of Mumford

$$\langle \pi_\ell(u), \lambda\pi_\ell(v)\rangle = E(u, v)$$

if E is the form associated analytically to the polarization λ.

The description of the dual abelian variety on p. 86 of Mumford can be reformulated. Suppose A is the quotient of $V(\mathbb{R})$ by U. Set

$$\hat{V}(\mathbb{R}) = \mathrm{Hom}_{\mathbb{R}}(V(\mathbb{R}), \mathbb{R})$$
$$\hat{V}(\mathbb{C}) = \mathrm{Hom}_{\mathbb{C}}(V(\mathbb{C}), \mathbb{C}) = V(\mathbb{R}) \otimes \mathbb{C}.$$

Let $\hat{h}$ be defined by

$$\langle h(z)v, \hat{h}(z)\hat{v}\rangle = (z\bar{z})^{-1}\langle v, \hat{v}\rangle$$

and let

$$\hat{u} = \mathrm{Hom}(V(\mathbb{Z}), \mathbb{Z}).$$

Then Mumford's $\bar{T}$ is $\hat{V}^+(\mathbb{C})$ where $\hat{v} \in \hat{V}(\mathbb{C})$ defines $\ell \in \bar{T}$ by

$$\ell(v) = 2i\langle v, \hat{v}\rangle \qquad v \in V(\mathbb{R})$$

and his U' is the projection on $\hat{V}^+(\mathbb{C})$ along $\hat{V}^-(\mathbb{C})$ of $\hat{U}$ because for $\hat{v} \in \hat{V}(\mathbb{R})$ and $v \in V(\mathbb{R})$

$$\langle v, \hat{v}\rangle = \langle v, \hat{v}^+\rangle + \langle v, \hat{v}^-\rangle = 2\,\mathrm{Re}\langle v, \hat{v}^+\rangle$$

and his $\hat{T}/U'$ is isomorphic to

$$\hat{V}^-(\mathbb{C})\backslash\hat{V}(\mathbb{C})/\hat{U} \simeq \hat{V}(\mathbb{R})/\hat{U}.$$

The map $V(\mathbb{R}) \to \hat{V}(\mathbb{R})$ defined by the map $A \to \hat{A}$ associated to λ is $v \to \hat{v}$ with

$$\langle u, \hat{v}\rangle = E(u, v).$$

Consider the family $\mathcal{A}$ of abelian varieties over $G(\mathbb{R})g_fK/K_\infty K$ constructed in the text. The family of dual abelian varieties is given by $\hat{V}(\mathbb{C})$, $\hat{V}^-(\mathbb{C})$; defined with respect to $\hat{h}$, and $\hat{g}_f\hat{V}(\mathbb{Z})$ if $\hat{g}_f$ is defined by

$$\langle g_f v, \hat{g}_f\hat{v}\rangle = \langle v, \hat{v}\rangle.$$

These give us an analytic family $\hat{\mathcal{A}}$ on $\tilde{\mathcal{M}}_{\mathbb{C}}$. The map $v \to \hat{v}$ given on $G(\mathbb{R})g_fK/K_\infty K$ by $v \to \hat{v}$ with

$$\Psi(u, cv) = \langle u, \hat{v}\rangle$$

when $\Psi(u, cv)$ is integral-valued on $g_fV(\mathbb{Z})$ yields a map $\mathcal{A} \to \hat{\mathcal{A}}$ over $\hat{\mathcal{M}}_{\mathbb{C}}$.

Note 3. We have to work out the conventions in Note 1 over $\mathbb{C}$. We have the exact sequence

$$0 \to \hat{V}^-(\mathbb{C}) \to \hat{V}(\mathbb{C})/U \to \hat{A} \to 0.$$

What I want to check is that $\hat{V}(\mathbb{C})/U$ is a universal vector extension $E(\hat{A})$ of $\hat{A}$. If

$$0 \to \mathbb{C} \to W \to \hat{A} \to 0$$

is a vector extension then $\hat{V}(\mathbb{C}) \to \hat{A}$ lifts to a map $\varphi : \hat{V}(\mathbb{C}) \to W$, unique up to a map $\hat{V}(\mathbb{C}) \to \mathbb{C}$. There is therefore exactly one way of choosing φ so that φ is 0 on U. This gives then

$$\begin{tikzcd}
0 \arrow[r] & \hat{V}^-(\mathbb{C}) \arrow[r] \arrow[d] & \hat{V}(\mathbb{C})/\hat{U} \arrow[r] \arrow[d, "\varphi"] & \hat{A} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbb{C} \arrow[r] & W \arrow[r] & \hat{A} \arrow[r] & 0
\end{tikzcd} .$$

It follows that $\hat{V}(\mathbb{C})/U$ gives the universal vector extension. Thus the procedure in Note 1 identifies $H^1_{\mathrm{DR}}(\hat{A})$ with $V(\mathbb{C})$, the dual of $\hat{V}(\mathbb{C})$.

On the other hand if $\hat{v} \in \hat{V}(\mathbb{C})$ we associate it to a line bundle $L(\hat{v})$ on A by dividing

$$\mathbb{C} \times V(\mathbb{R})$$

by the action

$$(\cdot, v) \to (e^{2\pi i \langle u, \hat{v}\rangle}, v + u)$$

of U. The trivial connection on $\mathbb{C} \times V(\mathbb{R})$ induces a connection on $L(\hat{v})$. Also $L(\hat{v})$ together with its connection depends only on $\hat{v}$ modulo $\hat{U}$. If $\hat{v} = \hat{v}^+ + \hat{v}^-$ then the holomorphic line bundle depends only on $\hat{v}^+$ and is the same as that defined by

$$\hat{v}^+ + \overline{\hat{v}^+}$$

which is given by the holomorphy factor

$$u \to e^{2\pi i 2 \operatorname{Re}\langle u, \hat{v}^+\rangle}.$$

Thus the map $\hat{V}(\mathbb{C})/\hat{U} \to \hat{A}$ is compatible with passage from $L(\hat{v})$ to the point of $\hat{A}$ defined by the associated line bundle.

In any case we can now identify $\hat{V}(\mathbb{C})/\hat{U}$ with the group $E^\natural$ introduced on p. 48 of Mazur-Messing. Because of the universal property there is only one way of identifying $E^\natural \simeq \hat{V}(\mathbb{C})/\hat{U}$ with $E(\hat{A})$ and that is by the identity map

$$\hat{V}(\mathbb{C})/\hat{U} = \hat{V}(\mathbb{C})/\hat{U}.$$

The constructions of Mazur-Messing allow us therefore to identify $H_{\mathrm{DR}}(A)$ with $\hat{V}(\mathbb{C})$ and the pairing introduced between $H^1_D(A)$ and $H^1_{\mathrm{DR}}(\hat{A})$ is the negative of the natural pairing between $\hat{V}(\mathbb{C})$ and $V(\mathbb{C})$ and that between

$$\operatorname{Hom}(H^1_{\mathrm{DR}}(A), \mathbb{C})$$

and

$$\operatorname{Hom}(H^1_{\mathrm{DR}}(\hat{A}), \mathbb{C})$$

is the negative of the pairing between $V(\mathbb{C})$ and $\hat{V}(\mathbb{C})$. The map

$$\operatorname{Hom}(H^1_{\mathrm{DR}}(A), \mathbb{C}) \to \operatorname{Hom}(H^1_{\mathrm{DR}}(\hat{A}), \mathbb{C})$$

induced by a polarization is that on the cotangent spaces at 0 of the induced map

$$E(A) \to E(\hat{A}).$$

Now $A \to \hat{A}$ can be lifted to $V(\mathbb{C}) \to \hat{V}(\mathbb{C})$ as $v \to \hat{v}$ with

$$\langle u, \hat{v} \rangle = E(v, u).$$

Since this takes $U \to \hat{U}$ it gives the unique lifting of $V(\mathbb{C})/U \to \hat{V}(\mathbb{C})/\hat{U}$. Thus the bilinear form on $V(\mathbb{C})$ associated to λ is

$$E(u, v).$$

Note 4. You will have noticed that I have been very careless in my definition of φ_p. I suppose that to be exact one must proceed along the following lines. Over S_p one can consider the sheaf in the étale topology

$$\mathrm{Iso}(A_{p^n}, V(\mathbb{Z}/p^n\mathbb{Z})) \qquad n = 0, 1, 2, \ldots.$$

Regarding it as a presheaf one may divide by the action of K.

$$k : \varphi \to k \circ \varphi$$

and then construct the associated sheaf X_n. We have

$$X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots$$

φ_p should I suppose be taken to be a consistent family $\varphi_p^0, \varphi_p^1, \ldots$ of sections of these sheaves over S_p.

Note 5. We need to show that assumptions II and III have the following consequence. Suppose ψ' is an alternating form on $V(\mathbb{Z}_p)$ with

$$\psi'(\ell x, y) = \psi'(x, \ell^* y) \qquad \ell \in O_L \otimes \mathbb{Z}_p.$$

Suppose moreover that the determinant of ψ' is a unit. Then there is an O_L-automorphism η of $V(\mathbb{Z}_p)$ so that

$$\psi'(x, y) = \psi(\eta(x), \eta(y)).$$

Since ψ satisfies the same conditions as ψ' all we have to do is verify that the conditions allow us to transform ψ' into a canonical form.

As in the text let

$$O_L \otimes \mathbb{Z}_p \simeq \bigoplus_{i=1}^{n} M_i$$

and let $1 = \bigoplus e^i$ be the corresponding partition of unity. Set $V_i = e^i V(\mathbb{Z}_p)$. For a given i either $M_i^* = M_i$ or $M_i^* = M_j$, $i \neq j$. In the second instance we can choose bases $\{x_\ell^k\}$, $\{y_\ell^k\}$ for V_i and V_j so that

$$\begin{aligned} \psi'(x_{\ell'}^k, x_{\ell'}^{k'}) &= \delta_{kk'}\delta_{\ell\ell'} \\ e_{k'\ell'}^i x_\ell^k &= \delta_{\ell'\ell} x_{k'}^k \\ e_{k'\ell'}^{i*} y_\ell^k &= \delta_{k'\ell} y_{\ell'}^k. \end{aligned}$$

Moreover the spaces V_i and V_j are isotropic. This puts at least part of ψ' in standard form.

Suppose $M_i^* = M_i$. For simplicity of notation I drop the subscript and write simply M and V and e_{jk}. Let O be the centre of M. It is clear that M is the direct sum of the submodules $e_{jj}^* M e_{ii}$. Moreover

$$e_{kk}^* M e_{ii} = e_{kj}^* e_{jj}^* M e_{ii} e_{i\ell}.$$

Thus these spaces are isomorphic to each other and to O. Let

$$\alpha : e_{11}^* M e_{11} \to 0$$

be an isomorphism. Set $V_{jj} = e_{jj} V$. If $\operatorname{rank}_O M = n^2$ and W is the n-dimensional coordinate space over O then

$$(x_1, \dots, x_n) \otimes x \to \sum x_j e_{j1} x$$

yields an isomorphism of $W \otimes_O V_{11}$ with V. Define the skew hermitian form $\psi''(x, y)$ on V over O by

$$\operatorname{Tr}_{O/\mathbb{Z}_p} a\psi''(x, y) = \psi'(ax, y).$$

The determinant of ψ'' is also a unit. We just have to show that ψ'' can be put in a standard form.

Choose $a \in e_{11}^* M e_{11}$ so that $\alpha(a) = 1$. Define ψ_2'' on V_{11} by

$$\psi_2''(x, y) = \psi''(x, ay).$$

Define a form ψ_1'' on W by

$$\psi_1''((x_1, \dots, x_n), (y_1, \dots, y_n)) = \sum x_i t_{ij} y_j^*$$

with

$$t_{jk} = \alpha(e_{j1}^* e_{k1}).$$

Then

$$\psi'' = \psi_1'' \otimes \psi_2''.$$

One of ψ_1'' and ψ_2'' is hermitian. The other is skew-hermitian. They both have determinants which are units. Since

$$\psi_1''(\ell u, v) = \psi_1''(u, \ell^* v)$$

ψ_1'' is determined up to a unit factor by the involution alone. Since we can absorb scalar factors into ψ_2'', we may suppose ψ_1'' is given.

It becomes therefore merely a question of finding a standard form of ψ_2''. If G is connected so is the group G_1 obtained by replacing C by $C_1 = \underline{Z}^0$. If however the involution is trivial on O then G_1 cannot be connected unless ψ_2'' is alternating (see p. 28–29 of the suppressed version of *Travaux de Shimura*).

Thus if the involution is trivial ψ_2'' can be put in the usual standard form with matrix

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

Suppose the involution is not trivial on O. If $\psi_2''(x, x)$ were congruent to 0 modulo p for every x then

$$0 \equiv \psi_2''(\alpha x + y, \alpha x + y) \equiv (\alpha x, y) + (\alpha x, y)^*.$$

Since O is unramified

$$(x, y) \equiv 0 \pmod{p}.$$

Since this is impossible we conclude that ψ_2'' can be put in the form

$$\sum x_i x_i^*$$

with respect to appropriate coordinates.

Note 6. In the text, I have proceeded as though the cocycle $\{b_w\}$ was invariant under lifting. Although this is not quite so, it is sufficiently invariant for our purposes. Suppose F is a local field and $L_1 \subseteq L_2$ are two finite Galois extensions of F. Set

$$\mathfrak{G}_1 = \mathfrak{G}(L_1/F) \quad \mathfrak{G}_2 = \mathfrak{G}(L_2/F).$$

Let T be a torus over F which splits over L_1 and let $\hat{L} = \hat{L}(T)$.

The map

$$\hat{\lambda} \to \hat{\lambda}$$

sends $H^{-1}(\mathfrak{G}_1, \hat{L}) \to H^{-1}(\mathfrak{G}_2, \hat{L})$. Inflation sends $H^1(\mathfrak{G}_1, T)$ to $H^1(\mathfrak{G}_2, T)$. The square

$$\begin{array}{ccc} H^1(\mathfrak{G}_1, \hat{L}) & \xrightarrow{\sim} & H^1(\mathfrak{G}_1, T) \\ \downarrow & & \downarrow \\ H^1(\mathfrak{G}_2, \hat{L}) & \xrightarrow{\sim} & H^1(\mathfrak{G}_2, T) \end{array}$$

is commutative. The easiest way to check this is to choose $\hat{L}'$ free over $\mathfrak{G}_1$ so that $\hat{L}' \to \hat{L}$ is surjective. Associated to

$$0 \to \hat{L}'' \to \hat{L}' \to \hat{L} \to 0$$

is

$$0 \to T'' \to T' \to T \to 0$$

and

$$H^{-1}(\mathfrak{G}_1, \hat{L}') \simeq H^1(\mathfrak{G}_1, T') = 0 = H^1(\mathfrak{G}_2, T') \simeq H^{-1}(\mathfrak{G}_2, \hat{L}').$$

Thus we have

$$\begin{array}{ccc} H^1(\mathfrak{G}_1, T) & \hookrightarrow & H^2(\mathfrak{G}_1, T'') \\ \downarrow & & \downarrow \\ H^1(\mathfrak{G}_2, T) & \hookrightarrow & H^2(\mathfrak{G}_2, T''). \end{array}$$

This is commutative. Suppose $\hat{\lambda}$ is the image of $\hat{\lambda}'$ and

$$\sum_{\mathfrak{G}_1} \sigma\hat{\lambda}' = \hat{\nu}$$

so that

$$\sum_{\mathfrak{G}_2} \sigma\hat{\lambda}' = [L_2 : L_1]\hat{\nu}.$$

Let $\{a^1_{\sigma,\tau}\}$ be the fundamental class for $\mathfrak{G}_1$ and $\{a^2_{\sigma,\tau}\}$ that for $\mathfrak{G}_2$. Inflating $\{a^1_{\sigma,\tau}\}$ we obtain $[L_2 : L_1]$ times the class of $\{a^2_{\sigma,\tau}\}$. Under the map

$$H^1(\mathfrak{G}_1, T) \to H^2(\mathfrak{G}_1, T'')$$

the class

$$\sum_{\mathfrak{G}_1} \sigma\tau\hat{\lambda} \otimes a^1_{\sigma,\tau}$$

maps to

$$\hat{\nu} \otimes a^1_{\rho,\sigma}$$

while the class of
$$\sum_{\mathfrak{G}_2} \sigma\tau\hat{\lambda} \otimes a^2_{\sigma,\tau}$$
in $H^1(\mathfrak{G}_2, T)$ maps to
$$[L_2 : L_1]\hat{\nu} \otimes a^2_{\rho,\sigma}.$$
The commutativity is established.

In the situation of the text the previous observation can only be applied to T/D. However we can say that the cohomology class of $\{b_w\}$ is respected by the lifting up to an element s of D such that
$$|\lambda(s)|_p = 1$$
for all rational characters of D. To check this one has only to check that if λ is a rational character of T over $\mathbb{Q}_p$ then $|\lambda(b_w)|_p$ is independent of the cocycle representing the class and behaves properly on lifting. This results from a calculation made in the text.

Note 7. I just noticed a lacuna in this discussion. In order to make the desired identification we must show that λ defines an alternating form on $D(G_0)_S$ and then we have to carry out a verification similar to that of Note 5.

$D(G_0)_S$ and $V \otimes O_E \otimes_{O_E} R$ are isomorphic modulo J as O_L-modules provided with a bilinear form. Suppose $\mathfrak{p}$ is the inverse image of O_E of the maximal ideal of R. Then $O_E \to R$ extends to $O_{E_\mathfrak{p}} \to R$. Thus $O_L \otimes R$ is a direct sum of matrix algebras over unramified extensions of R. Thus the given isomorphism can be lifted to an isomorphism between the O_L-modules $D(G_0)_S$ and $V \otimes O_E \otimes_{O_E} R$. This gives us two bilinear forms on $V \otimes O_E \otimes_{O_E} R$. We can assume they both have unit determinant. Suppose we know they are both alternating. We have then to show that they are equivalent under an O_L-automorphism congruent to the identity modulo J. This is done as in Note 5.

The rest of Note 7 will follow. I am having trouble with the verifications.

Ich kann nämlich noch nicht zeigen, daß die Paarung zwischen $\operatorname{Lie} E(A)$ und $\operatorname{Lie} E(\hat{A})$ kristallisch ist. Ich werde Deligne danach fragen.

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