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Borel,

The following approach might work to yield a direct sum decomposition. Of course, you may not like it. G real reductive group. Γ-arithmetic subgroup. Slowly increasing is taken to mean that the condition of the first definition of p. 7 of H.C.—Automorphic Forms on s-s Lie groups is satisfied. V_G space of $C^\vee$-functions on $\Gamma\backslash G$ such that X_ψ is slowly increasing for all X in universal enveloping algebra *at a rate,* i.e. *with an r independent of X (an important condition)* P (provided with Λ, $M \supseteq A$, and N) cuspidal subgroup

$$\varphi_p(g) = \int_{\Gamma\cap N\backslash N} \varphi(ng)\, dn$$

Set

$$\varphi_p(m:k) = \varphi_p(mk)$$

Lemma 1. *If $\varphi \in V_G$ then $\varphi_p(\cdot : k) \in V_H$ uniformly in k.*

Let

$$\varphi(n:m:k) = \varphi(nmk)$$

Let $M = \Lambda M_0$. Let X be in n. Using HC's notation

$$\varphi\big(nam_0k; \operatorname{Ad} k \operatorname{Ad} m_0(\lambda)\big) = \varphi(n, \operatorname{Ad} nX, am_0, k)$$

If X is a root vector X_α this is

$$\alpha(a)\varphi(n, X : am_0 : k)$$

Using this fact one should not have too much trouble deducing:

Lemma 2. *For m_0 in a compact set Ω, $\varphi(n:am_0:k) - \varphi_p(n:am_0:k)$ is rapidly decreasing as, a function of a, as all $\alpha(a) \to \alpha$, uniformly for $n \in N$, $k \in K$, and $m \in \Omega$. φ is assumed to lie in V_G. N.B.—here I use that the rate is independent of X.*

Lemma 3. *If $\varphi_p(g) \equiv 0$ for $P \neq G$ and $\varphi \in V$ then φ is rapidly decreasing.*

Let's write $\varphi \perp P$ if $\varphi_p(am_0 : k)$ is orthogonal to all cusp forms for M_0 for all a and all k.

Lemma 4. *If $\varphi \perp P$ for all cuspidal P then φ is 0.*

Lemma 5. *Suppose φ is a function on $N(\Gamma\cap P)\backslash G$, X_φ is slowly increasing for all X, and φ vanishes outside of NA_tM_0k. Then*

$$\sum_{\Gamma\cap P\backslash\Gamma} \varphi(\gamma_g)$$

lies in V_G.

The first lemma is standard in the theory of Eisenstein series. The second is also implicit in that theory. If $\mathfrak{P}$ is a class of associative cuspidal subgroups let $V_G(\mathfrak{P})$ be the set of all φ in V_G such that $\varphi \perp P$ if $P \notin \mathfrak{P}$.

By Lemma 4, the sum

$$\sum_{\mathfrak{P}} V_G(\mathfrak{P})$$

is direct.

Theorem.

$$V_G = \bigoplus_{\mathfrak{P}} V_G(\mathfrak{P})$$

We have to show that V is contained in the right hand side. We use induction on rank Γ, ie in $\dim A$ for minimal P. If this is 0 there is no problem.

If P is cuspidal and P_0 a class of associate cuspidal subgroups for M (or M_0 if you like) there is a $\mathfrak{P}$ containing $\mathfrak{P}_0$.

Lemma 6. *If the φ of Lemma 5 lies in $V_M(\mathfrak{P}_0)$, ie if $\varphi(am_0k)$ lies in $V_{M_0}(\mathfrak{P}_0)$ for each a and k, then*

$$\sum_{\Gamma\cap P\backslash\Gamma} \varphi(\gamma_y)$$

lies in $V_G(\mathfrak{P})$.

Proved as for Eisenstein series presumably, of Lemma 4.4 of my notes.

Let L be $L^2(\Gamma\backslash G)$ and define L^∞ accordingly. We have, from the elements of the theory of Eisenstein series, a decomposition

$$L = \bigoplus L(\mathfrak{P})$$

and hence

$$L^\infty = \bigoplus L^\infty(\mathfrak{P})$$

Lemma 7. *$L^\infty \subseteq V_G$ and $L^\infty(\mathfrak{P}) = L^\infty \cap V_G(\mathfrak{P})$*

I think that the first statement can be deduced easily from Sobolev's lemma. The second is then clear.

To prove the theorem we need a technical result which I describe but do not prove—it is very similar to your partition of unity argument.

Let q be the rank of Γ. For $0 \leqslant i \leqslant q$, let X_i be a set of representations for conjugacy classes of cuspidal subgroups of rank i. For each P in X_q choose a Seigel domain

$$S(P) = S(P, c, \omega_p) = \left\{\, sak \;\middle|\; s \in \omega_p,\ a \in A_p,\ k \in K, \alpha(a) > c \;\forall \text{ simple roots } \alpha \text{ of } P \,\right\}$$

whose union, over X_q, covers $\Gamma\backslash G$.

We choose by induction downwards from q, for every i, $0 \leqslant i \leqslant q$, positive constants t_i and u_i, and for each P in X_i two compact sets $\mu_p \subseteq \operatorname{Int}\nu_p$ in M_p so that

$$(\Gamma \cap N_p)\mu_P = N_p\mu_p$$
$$(\Gamma \cap N_p)\nu_P = N_p\nu_p$$

and so that if

$$S(P, \mu_p) = \left\{\, sak \;\middle|\; s \in \mu_p, k \in K, a \in A_p, \alpha(a) > t_{i+1} \;\forall\alpha \,\right\}$$

and $S(P, \nu_p)$ is defined in the same way with $t_i + 1$ replaced by t_i, the following two conditions are satisfied.

(i) If $P \subset X_i$ and $P' \subset X_{i'}$ with $i' \geqslant 1$, then

$$\gamma S(P, \nu_p) \cap S(P', \nu_{p'}) \neq \varnothing$$

$\Longrightarrow$ $\gamma P \gamma^{-1}$ belongs to P' in the sense of my Eisenstein series p. 2.10—i.e. $\gamma P \gamma^{-1} \supseteq P'$.

(ii) If $y \in S(P, c, \omega_p)$ for some P in X_q and if $\xi_\alpha\big(u(g)\big) > u_1$ for at least i simple roots of P then there is an $i' \geqslant 1$, a P' in $X_{i'}$, and a γ in Γ so that $\gamma P' \gamma^{-1}$ belongs to P and $g \in \gamma S(P, \mu_p)$.

Suppose $\varphi \in V_G$. We prove by induction upwards that for each $i \geqslant 0$ there is a φ' congruent to φ modulo $\sum V_G(\mathfrak{P})$ so that φ' is rapidly decreasing in $S(P, \mu_p)$ if rank $P = i$. For $i = 0$, $S(P, \mu_p)$ is compact and there is no problem. In general choose the function λ_p so that $X\lambda_p$ is bounded for all X so that λ_p is 1 in $S(P, \mu_p)$ and 0 off $S(P, \nu_p)$. λ_p is a function on $N_p(\Gamma \cap P)\backslash G$. Set

$$\varphi^{i+1} - \overline{\varphi}^i - \sum_{P \in X_{i+1}} \sum_{\Gamma \cap P \backslash \Gamma} \lambda_p(\gamma_g)\gamma_p'(\gamma_g)$$

If I assume the theorem is valid for groups of rank smaller than Γ, then I can apply Lemma 6 to see that the sum on the right is in $\sum V_G(\mathfrak{P})$. Thus $\varphi^{i+1} \equiv \varphi^i \equiv \varphi \pmod{V_G(\mathfrak{P})}$. φ^{i+1} is by condition (i) and Lemma 2 rapidly decreasing in $S(P, \mu_p)$ if rank $P = i + 1$.

Condition (i) and the induction assumption can probably be used to show without too much difficulty that this is also true if rank $P \leqslant i$. This takes care of the induction.

Now consider φ^q. φ^q is rapidly decreasing so is X_{φ^q}. This is a consequence of Lemma 3. Hence $\varphi^q \subset L^\infty = \bigoplus L^\infty(\mathfrak{P}) \subseteq \sum V_G(\mathfrak{P})$ by Lemma 7. With this we are done.

Does this argument sound alright to you?

R.L.

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