

October 25, 1972

Borel,

The following approach might work to yield a direct sum decomposition. Of course, you may not like it.  $G$  real reductive group.  $\Gamma$ -arithmetic subgroup. Slowly increasing is taken to mean that the condition of the first definition of p. 7 of H.C.—Automorphic Forms on s-s Lie groups is satisfied.  $V_G$  space of  $C^\infty$ -functions on  $\Gamma \backslash G$  such that  $X_\psi$  is slowly increasing for all  $X$  in universal enveloping algebra *at a rate*, i.e., *with an  $r$  independent of  $X$  (an important condition)*  $P$  (provided with  $\Lambda$ ,  $M \supseteq A$ , and  $N$ ) cuspidal subgroup

$$\varphi_p(g) = \int_{\Gamma \cap N \backslash N} \varphi(n g) dn$$

Set

$$\varphi_p(m : k) = \varphi_p(mk)$$

**Lemma 1.** *If  $\varphi \in V_G$  then  $\varphi_p(\cdot : k) \in V_H$  uniformly in  $k$ .*

Let

$$\varphi(n : m : k) = \varphi(nmk)$$

Let  $M = \Lambda M_0$ . Let  $X$  be in  $n$ . Using HC's notation

$$\varphi(nam_0k; \text{Ad } k \text{ Ad } m_0(\lambda)) = \varphi(n, \text{Ad } nX, am_0, k)$$

If  $X$  is a root vector  $X_\alpha$  this is

$$\alpha(a)\varphi(n, X : am_0 : k)$$

Using this fact one should not have too much trouble deducing:

**Lemma 2.** *For  $m_0$  in a compact set  $\Omega$ ,  $\varphi(n : am_0 : k) - \varphi_p(n : am_0 : k)$  is rapidly decreasing as, a function of  $a$ , as all  $\alpha(a) \rightarrow \alpha$ , uniformly for  $n \in N$ ,  $k \in K$ , and  $m \in \Omega$ .  $\varphi$  is assumed to lie in  $V_G$ . N.B.—here I use that the rate is independent of  $X$ .*

**Lemma 3.** *If  $\varphi_p(g) \equiv 0$  for  $P \neq G$  and  $\varphi \in V$  then  $\varphi$  is rapidly decreasing.*

Let's write  $\varphi \perp P$  if  $\varphi_p(am_0 : k)$  is orthogonal to all cusp forms for  $M_0$  for all  $a$  and all  $k$ .

**Lemma 4.** *If  $\varphi \perp P$  for all cuspidal  $P$  then  $\varphi$  is 0.*

**Lemma 5.** *Suppose  $\varphi$  is a function on  $N(\Gamma \cap P) \backslash G$ ,  $X_\varphi$  is slowly increasing for all  $X$ , and  $\varphi$  vanishes outside of  $NA_t M_0 k$ . Then*

$$\sum_{\Gamma \cap P \backslash \Gamma} \varphi(\gamma_g)$$

*lies in  $V_G$ .*

The first lemma is standard in the theory of Eisenstein series. The second is also implicit in that theory. If  $\mathfrak{P}$  is a class of associative cuspidal subgroups let  $V_G(\mathfrak{P})$  be the set of all  $\varphi$  in  $V_G$  such that  $\varphi \perp P$  if  $P \notin \mathfrak{P}$ .

By Lemma 4, the sum

$$\sum_{\mathfrak{P}} V_G(\mathfrak{P})$$

is direct.

**Theorem.**

$$V_G = \bigoplus_{\mathfrak{P}} V_G(\mathfrak{P})$$

We have to show that  $V$  is contained in the right hand side. We use induction on rank  $\Gamma$ , i.e. in  $\dim A$  for minimal  $P$ . If this is 0 there is no problem.

If  $P$  is cuspidal and  $P_0$  a class of associate cuspidal subgroups for  $M$  (or  $M_0$  if you like) there is a  $\mathfrak{P}$  containing  $\mathfrak{P}_0$ .

**Lemma 6.** *If the  $\varphi$  of Lemma 5 lies in  $V_M(\mathfrak{P}_0)$ , i.e. if  $\varphi(am_0k)$  lies in  $V_{M_0}(\mathfrak{P}_0)$  for each  $a$  and  $k$ , then*

$$\sum_{\Gamma \cap P \backslash \Gamma} \varphi(\gamma_y)$$

*lies in  $V_G(\mathfrak{P})$ .*

Proved as for Eisenstein series presumably, of Lemma 4.4 of my notes.

Let  $L$  be  $L^2(\Gamma \backslash G)$  and define  $L^\infty$  accordingly. We have, from the elements of the theory of Eisenstein series, a decomposition

$$L = \bigoplus L(\mathfrak{P})$$

and hence

$$L^\infty = \bigoplus L^\infty(\mathfrak{P})$$

**Lemma 7.**  $L^\infty \subseteq V_G$  and  $L^\infty(\mathfrak{P}) = L^\infty \cap V_G(\mathfrak{P})$

I think that the first statement can be deduced easily from Sobolev's lemma. The second is then clear.

To prove the theorem we need a technical result which I describe but do not prove—it is very similar to your partition of unity argument.

Let  $q$  be the rank of  $\Gamma$ . For  $0 \leq i \leq q$ , let  $X_i$  be a set of representations for conjugacy classes of cuspidal subgroups of rank  $i$ . For each  $P$  in  $X_q$  choose a Siegel domain

$$S(P) = S(P, c, \omega_p) = \{ sak \mid s \in \omega_p, a \in A_p, k \in K, \alpha(a) > c \ \forall \text{ simple roots } \alpha \text{ of } P \}$$

whose union, over  $X_q$ , covers  $\Gamma \backslash G$ .

We choose by induction downwards from  $q$ , for every  $i$ ,  $0 \leq i \leq q$ , positive constants  $t_i$  and  $u_i$ , and for each  $P$  in  $X_i$  two compact sets  $\mu_p \subseteq \text{Int } \nu_p$  in  $M_p$  so that

$$(\Gamma \cap N_p)\mu_p = N_p\mu_p$$

$$(\Gamma \cap N_p)\nu_p = N_p\nu_p$$

and so that if

$$S(P, \mu_p) = \{ sak \mid s \in \mu_p, k \in K, a \in A_p, \alpha(a) > t_{i+1} \ \forall \alpha \}$$

and  $S(P, \nu_p)$  is defined in the same way with  $t_i+1$  replaced by  $t_i$ , the following two conditions are satisfied.

(i) If  $P \subset X_i$  and  $P' \subset X_{i'}$  with  $i' \geq 1$ , then

$$\gamma S(P, \nu_p) \cap S(P', \nu_{p'}) \neq \emptyset$$

$\implies \gamma P \gamma^{-1}$  belongs to  $P'$  in the sense of my Eisenstein series p. 2.10—i.e.,  $\gamma P \gamma^{-1} \supseteq P'$ .

(ii) If  $y \in S(P, c, \omega_p)$  for some  $P$  in  $X_q$  and if  $\xi_\alpha(u(g)) > u_1$  for at least  $i$  simple roots of  $P$  then there is an  $i' \geq 1$ , a  $P'$  in  $X_{i'}$ , and a  $\gamma$  in  $\Gamma$  so that  $\gamma P' \gamma^{-1}$  belongs to  $P$  and  $g \in \gamma S(P, \mu_p)$ .

Suppose  $\varphi \in V_G$ . We prove by induction upwards that for each  $i \geq 0$  there is a  $\varphi'$  congruent to  $\varphi$  modulo  $\sum V_G(\mathfrak{P})$  so that  $\varphi'$  is rapidly decreasing in  $S(P, \mu_p)$  if  $\text{rank } P = i$ . For  $i = 0$ ,  $S(P, \mu_p)$  is compact and there is no problem. In general choose the function  $\lambda_p$  so that  $X\lambda_p$  is bounded for all  $X$  so that  $\lambda_p$  is 1 in  $S(P, \mu_p)$  and 0 off  $S(P, \nu_p)$ .  $\lambda_p$  is a function on  $N_p(\Gamma \cap P) \backslash G$ . Set

$$\varphi^{i+1} - \overline{\varphi}^i - \sum_{P \in X_{i+1}} \sum_{\Gamma \cap P \backslash \Gamma} \lambda_p(\gamma_g) \gamma'_p(\gamma_g)$$

If I assume the theorem is valid for groups of rank smaller than  $\Gamma$ , then I can apply Lemma 6 to see that the sum on the right is in  $\sum V_G(\mathfrak{P})$ . Thus  $\varphi^{i+1} \equiv \varphi^i \equiv \varphi \pmod{\sum V_G(\mathfrak{P})}$ .  $\varphi^{i+1}$  is by condition (i) and Lemma 2 rapidly decreasing in  $S(P, \mu_p)$  if  $\text{rank } P = i+1$ .

Condition (i) and the induction assumption can probably be used to show without too much difficulty that this is also true if  $\text{rank } P \leq i$ . This takes care of the induction.

Now consider  $\varphi^q$ .  $\varphi^q$  is rapidly decreasing so is  $X_{\varphi^q}$ . This is a consequence of Lemma 3. Hence  $\varphi^q \subset L^\infty = \bigoplus L^\infty(\mathfrak{P}) \subseteq \sum V_G(\mathfrak{P})$  by Lemma 7. With this we are done.

Does this argument sound alright to you?

R.L.

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