October 25, 1972

Borel,

I justify my observation to the effect that:

If G is semi-simple, π irreducible unitary, $\Lambda - g$ the highest weight of $\widetilde{\mu}$ $\left(2g = \sum_{\alpha>0} \alpha\right)$, G/K a bounded symmetric domain, and Λ sufficiently non-singular, then $H^q(\pi,\mu) = 0$ unless $q = \frac{1}{2}\dim(G/K)$, π is a member $\pi_s\Lambda$, $s \in \Omega_{\mathbf{C}}$, of the Weyl group when $H^q(\pi,\mu) \cong \mathbf{C}$.

I first reinterpret some results of Matsushima-Murakami (Osaka J. Math 1965) representation theoretically. s unitary, acts on U, U^∞ infinitely differentiable vectors. $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ as usual. The complex structure on G/K determines a decomposition $\mathfrak{m}' = \mathfrak{m}'_+ + \mathfrak{m}'_-$ if $\mathfrak{m}' = \mathrm{Hom}_{\mathbf{R}}(\mathfrak{m}, \mathbf{C})$. $C^n(U^\infty,\mu)$ is the set of K-invariant elements in $U^\infty \otimes L(\mathbf{C}) \otimes \Lambda^n \mathfrak{m}'$. Thus

$$C^n(U^n,\mu) = \bigoplus_{p+q=n} C^{p,q}(U^\infty,\mu)$$

with

$$C^{p,q}(U^\infty,\mu) = U^\infty \otimes L(\mathbf{C}) \otimes \Lambda^p \mathfrak{m}'_+ \otimes \Lambda^q \mathfrak{m}'_-.$$

If $s = \pi$ is unitary, irreducible, $\pi(\omega) = \mu(\omega)$ then $C^n(U^\infty,\mu) = H^n(\pi,\mu)$ and there is a Hodge decomposition.

Assume now, it is the interesting case, that $s = \pi$, $\pi(\omega) = \mu(\omega)$. Fix a Cartan subalgebra $\mathfrak{A}$ of $\mathfrak{G}$ contained in k and an order on the roots so that the annihilator of $\mathfrak{m}'_+$ in $\mathfrak{m} \otimes \mathbf{C}$ is spanned by the positive roots. According to Proposition 10.1 of Matsushima-Murakami the space $L(\mathbf{C}) \otimes \Lambda \mathfrak{m}'_+$ is a direct sum

$$\left(\bigoplus_{\omega \in \Omega'} X_\omega\right) \oplus X$$

of spaces invariant under K. X is of no importance.

$$\Omega' = \left\{\, \omega \in \Omega \;\middle|\; \omega^{-1} \text{ takes positive compact roots to positive roots} \,\right\}.$$

The lowest weight of the representation of k on X_ω is $-\omega \wedge +g$. As in their paragraph 11 one deduces the very strong vanishing theorem:

$H^{p,q}(\pi,\mu) = 0$ if $p + q \neq \pm \dim G/K$ provided $(\Lambda,\alpha) > (q,\alpha)$ for all positive roots α.

My notation (in particular my Λ) differs from theirs. To treat the case $p + q = N$, I reformulate things in the language of people like Okamoto and Schmid although this is probably ultimately unnecessary.

I mention first a general type of cohomology problem which might interest you. It has been studied by M.S. Osborne. G : real semi-simple (or reductive), $\mathfrak{g}$ its Lie algebra, $\mathfrak{h}$ a reductive subalgebra, the Lie algebra of H, $\mathfrak{p}$ a parabolic subalgebra of $\mathfrak{g} \otimes \mathbf{C}$, η: nilpotent radical of $\mathfrak{p}$.

William Casselman rewrote this letter of Langlands by hand. This typed version is based on Casselman's handwritten version, not the original one by Langlands.

We suppose $\mathfrak{h} \otimes \mathbf{C}$ is the reductive part of $\mathfrak{p}$. s irreducible, admissible representation of G on Banach space U. U^∞ defined in obvious manner. $\mathfrak{g} \otimes \mathbf{C}$ acts on U^∞. Consider Lie algebra cohomology of $\mathfrak{n}$ on U^∞. $C^q(\mathfrak{n}, U^\infty)$ standard cochain complex. Semi-norm $\|Xu\|$, $X \in U(\mathfrak{g})$ gives topology on U^∞ and hence on $C^q(\mathfrak{n}, U^\infty)$. Are coboundaries closed? Then $H^q(\mathfrak{n}, U^\infty)$ is a Banach space on which $\mathfrak{h}$ acts. Does it have a finite composition series? If so, what is the relationship between s and the representations in the composition series?

In any case Okamoto-Ozeki (Osaka J. Math.) and Okamoto-Narasimhan (Annals) study these for our original $\mathfrak{g}$ when $\mathfrak{h} = \mathfrak{a}$, the Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{k}$ and $\mathfrak{n} = \mathfrak{m}_+$ with $s = \pi$ unitary. They study rather, if τ_λ is an irreducible representation of K on V_λ with highest weight λ,

$$\mathrm{Hom}_K\big(V_\lambda, H^q(\mathfrak{m}_+, U^\infty)\big)$$

which is the cohomology of the complex

$$\mathrm{Hom}_K\big(V_\lambda, C^q(\mathfrak{m}_+, U^\infty)\big)$$

or, if $\widetilde{V}_\lambda = \mathrm{Hom}_{\mathbf{C}}(V_\lambda, \mathbf{C})$, of the K-invariant elements in

$$\widetilde{V}_\lambda \otimes C^q(\mathfrak{m}_+, U^\infty).$$

In Okamoto-Ozeki, a Lapacian $\square$ is introduced. On this complex it equals a scalar

$$\frac{1}{2}\big\{(\lambda + g, \lambda + g) - (g, g) - \pi(\omega)\big\}$$

if $\omega =$ Casimir operator. Thus the cohomology groups are 0 or the boundary operator is 0. Comparing this with the results of Matsushima-Murakami we see that $H^{p,q}(U^\infty, \mu)$ is isomorphic to

$$(*) \qquad \bigoplus_{\substack{\omega \in \Omega' \\ \eta(\omega) = p}} \mathrm{Hom}_K\big(V_{\lambda(\omega), H^q(\mathfrak{m}_+, U^\infty)}\big)$$

where $\lambda(\omega) = \omega \wedge -g$. $\eta(\omega)$ is defined in Matsushima-Murakami. Note these results (in §§5, 11, 12) imply in particular that if $\pi(\omega) = \mu(\omega)$ and as on p. 1 of this letter

$$L(\mathbf{C}) \otimes \Lambda\mathfrak{m}'_+ = \left(\bigoplus_{\omega \in \Omega'} X_\omega\right) \oplus X$$

then there are no K-invariant elements in $U^\infty \otimes X \otimes \Lambda\mathfrak{m}^-_-$.

Schmid however considers the general problem above with $\mathfrak{h} = \mathfrak{a}$ (as above), $\mathfrak{n}$ chosen to contain $\mathfrak{m}_+$, the subalgebra spanned by positive roots.

The groups $H^q(\mathfrak{m}_+, U^\infty)$ are finite-dimensional. Let $\mathfrak{n}_0 = \mathfrak{n} \cap (\mathfrak{k} \otimes \mathbf{C})$. According to my notes (not always reliable) a spectral sequence argument + finite-dimensional Borel-Weil shows that

$$H^n(\mathfrak{n}, U^n) \cong \bigoplus_{p+q=n} H^p\big(\mathfrak{n}_0, H^q(\mathfrak{m}_+, U^\infty)\big)$$

This must be in any case a generally accepted fact. Let T be the Cartan subgroup with Lie algebra $\mathfrak{a}$, and let λ be a highest weight of $\mathfrak{a}$ with respect to k (not $\mathfrak{g}$). Let W_λ be the corresponding 1-dimensional representation of T. By Borel-Weil again

$$\mathrm{Hom}_T\big(W_\lambda, H^n(\mathfrak{n}, U^\infty)\big) \cong \bigoplus_{\lambda_i = \lambda} \mathbf{C}$$

if as a K-module

$$H^q(\mathfrak{m}_+, U^\infty) \cong \bigoplus_i V_{\lambda_i}$$

But Schmid (Annals '71) shows that if λ is sufficiently regular the left-side is 0 unless $\pi = \pi_{\lambda+g}$ in the discrete series. If Λ is sufficiently regular so are the $\lambda(\omega)$ occurring in $(*)$ and $\lambda(\omega) + g = \omega\Lambda$. Retracing our steps backward from Schmid to Matsushima-Murakami via Okamoto-Ozeki, we get all the required vanishing.

To completely justify my assertion I have only to show that for sufficiently non-singular Λ, if $\pi = \pi_{\omega\Lambda}$ the $H^n(\pi, \mu) \cong \mathbf{C}$.

It may be over-elaborate but I use the weak form of Blattner's conjecture has given in Schmid (Rice Studies) Theorem 2 to compute

$$(**) \qquad \mathrm{Hom}_K\Big(\widetilde{L}(\mathbf{C}) \otimes \Lambda^p\mathfrak{m}_+ \otimes \Lambda^q\mathfrak{m}_-, U^\infty\Big)$$

where

$$\widetilde{L}(\mathbf{C}) = \mathrm{Hom}\big(L(\mathbf{C}), \mathbf{C}\big)$$

Suppose $s \in \Omega_{\mathbf{R}}$, the real Weyl group, and

$$s\mu + s\rho_K - (\omega\Lambda + \rho_p) = \sum \delta$$

where δ are positive with respect to an order putting $\omega\Lambda$ in the positive chamber. With respect to this order ρ_K and ρ_p have the same meaning as in Schmid, so $\rho = \rho_K + \rho_p = \omega g$. If, with respect to this order, μ is the highest weight of a representation occurring in $(**)$ (the first term) then $s\mu$ is a weight and

$$s\mu = \omega\Lambda - \omega g = \sum \gamma + \sum_{i=1}^{p} \alpha_i + \sum_{j=1}^{q} \beta_j$$

where γ are positive in this new order, α_i are positive and distinct in the old order, β_i are distinct and negative in the old order. Since $\omega g = \rho$ and $s\mu = \omega\Lambda + \rho_p - s\rho_K + \sum \delta$ we have

$$\sum \delta + \sum \gamma + \{\rho_K + s\rho_p\} + \Big\{2\rho_p - \sum \alpha_i - \sum \rho_i\Big\} = 0.$$

The two sums in brackets are sums of roots positive in the new order. Thus the sums over γ and δ are 0 and $s\rho_K = s$ so that $s = 1$. Moreover

$$2\rho_p = \sum \alpha_i + \sum \beta_i$$

This determines μ uniquely. It is equal to

$$\omega\Lambda - \omega g + 2\rho_p = \omega\Lambda + \rho_p - \rho_K$$

Thus there is at most one λ such that τ_λ occurs in the representation $\widetilde{L}(\mathbf{C}) \otimes \Lambda^p\mathfrak{m}_+ \otimes \Lambda^q\mathfrak{m}_-$ for which

$$\mathrm{Hom}_K(V_\lambda, U^\infty) \neq 0$$

and for it the Hom is of dimension 1.

I hope that with this my work is justified.

Compiled on July 30, 2024.