New Haven
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Dear Bill,

Let me try to explain the presumable form of the L-function of Shimura's varieties for groups that are essentially products of GL(2)'s—for example, the Hilbert modular groups and their subgroups. I will have to take for granted some of the things I am working out with Labesse. When these things are written out in a readable form, I will send them to you. Of course to prove that the L-functions actually have the expected form, one will have to proceed as in Ihara's paper. I have thought over this a little in the general case, i.e. also for higher-dimensional groups—my calculations have been tentative and only partially reasonable. I have to think things out more clearly. I only want to draw your attention to the fact that things seem to work out much neater when one does everything strictly adelically. Then not only do Tate's results and formulae for class numbers coming from properties of the Tamagawa number seem to play a role, but also the formulae of MacDonald for spherical functions. It seems that one has to know the value of the Selberg integrals for spherical functions whose Fourier transforms are given and MacDonald's results are a first step. In any case that comes later. Let me explain the simple case now.

Let F be a finite algebraic extension of $\mathbf{Q}$ and let K be a Galois extension of $\mathbf{Q}$ containing F. Let δ be a subset of $\mathfrak{G}(K/\mathbf{Q})$ such that $\mathfrak{G}(K/\mathbf{Q})\delta = \delta$ and let, as usual, F' be the fixed field of

$$\left\{ \sigma \in \mathfrak{G}(K/\mathbf{Q}) \;\middle|\; \delta\sigma = \delta \right\}$$

If K is replaced by $K_1 \supseteq K$ then δ is replaced by its inverse image δ_1 in $\mathfrak{G}(K_1/\mathbf{Q})$. F' does not change so we may enlarge K at will. Let B and B' be $F^\times$ and $F'^\times$ considered as algebraic groups over $\mathbf{Q}$. As in Shimura there is a map $\varphi : B' \to B$ and hence a map $B'_{\mathbf{A}} = I_{F'} \to B_{\mathbf{A}} = I_F$ ($\mathbf{A} = \mathbf{A}(\mathbf{Q})$). This induces a map $C_{F'} \to C_F$ of idele class groups.

Let E be a quadratic extension of F associated to a subgroup U of C_F containing $\varphi(C_{F'})$. We may suppose $E \subseteq K$.

There is a map $\tau_{K/F'}$ of $W_{K/F'}$ (Weil group) onto $C_{F'}$. **[2]** If $w_1, \ldots, w_r$ is a set of representatives for $W_{K/K}\backslash W_{K/F'}$ and if $w_i w = a_i(w) w_k$, $w \in W_{K/F'}$, then

$$\tau_{K/F'}(w) = \prod_i a_i(w).$$

If S is the disjoint union

$$\bigcup_j \sigma_j \mathfrak{G}(K/F')$$

and if $v_j \in W_{K/\mathbf{Q}}$ maps to σ_j then $\varphi(a)$, $a \in C_{F'}$, is

$$\prod_j v_j a v_j^{-1}$$

But

$$v_j w_i w = v_j a_i(w) v_j^{-1} v_j w_k$$

so

$$\varphi\big(\tau_{K/F'}(w)\big) = \prod_\sigma a_\sigma(w).$$

Here one chooses for each σ in S a w_σ mapping to σ and sets

$$w_\sigma w = a_\sigma(w) w_\tau \qquad w \in W_{K/F'}.$$

There is also a map $\tau_{K/F} : W_{K/F} \to C_F$. The inverse image of U is $W_{K/E}$. Let χ be the non-trivial character of $U\backslash C_F$ or of $W_{K/E}\backslash W_{K/F}$ or of $\mathfrak{G}(K/E)\backslash\mathfrak{G}(K/F)$. These quotient groups are all the same. Let $\{\sigma_i\}$ be the set of representatives of $\mathfrak{G}(K/F)\backslash S$. If $\sigma \in \mathfrak{G}(K/F')$ let $\sigma_i\sigma = a_i(\sigma)\sigma_k$. I claim that

$$\prod_i \chi\big(a_i(\sigma)\big) = 1 \qquad a_i(\sigma) \in \mathfrak{G}(K/F).$$

Let w_i in $W_{K/F}$ map to σ_i and let w map to σ. Let

$$w_i w = a_i(w) w_k$$

Then

$$\prod_i \chi\big(a_i(\sigma)\big) = \prod_i \chi\big(a_i(w)\big) = \prod_i \chi\Big(\tau_{K/F}\big(a_i(w)\big)\Big).$$

In the last member of this equation χ is a character of C_F. If v_j is a set of representatives for $W_{K/K}\backslash W_{K/F}$, $v \in W_{K/F}$, and

$$v_j v = b_j(v) v_\ell$$

then

$$\tau_{K/F}(v) = \prod_j b_j(v).$$

Thus

$$\prod_i \tau_{K/F}\big(a_i(w)\big) = \prod_i \prod_j b_j\big(a_i(w)\big)$$

But

$$v_j w_i w = v_j a_i(w) w_k = b_j\big(a_i(w)\big) v_\ell w_k.$$

Thus if we take $\{\, w_\sigma \mid \sigma \in S \,\} = \{v_j w_i\}$ we see that

$$\prod_i \tau_{K/F}\big(a_i(w)\big) = \varphi\big(\tau_{K/F'}(w)\big)$$

lies in U. This proves the assertion. It will be applied later.

[**3**] Let D be a quaternion algebra (perhaps split) over F and let $\widetilde{G}$ be the group $D^\times$. Let G be the inverse image of $\varphi(B') = A$ with respect to the map $\widetilde{G} \xrightarrow{\text{Norm}} B$. If $C = B'/\varphi(B)$ we have

$$1 \longrightarrow G \longrightarrow \widetilde{G} \longrightarrow C \longrightarrow 1 \quad .$$

Let $\widehat{C}_0$ be the connected component of the associated (or dual) group to C. Then

$$1 \longrightarrow \widehat{C}_0 \longrightarrow \widehat{\widehat{G}} \longrightarrow \widehat{G} \longrightarrow 1$$

is also exact. $\widehat{\widehat{G}}$ is the split extension of

$$\widehat{\widehat{G}}_0 = \prod_{\varphi \in \mathfrak{G}(K/F)\backslash \mathfrak{G}(K/\mathbf{Q})} \mathrm{GL}(2, \mathbf{C})$$

by $\mathfrak{G}(K/\mathbf{Q})$ where

$$\sigma(a_\varphi)\sigma^{-1} = (a'_\varphi) \qquad a'_\varphi = a_{\varphi\sigma}.$$

Consider the representation ρ_0 of $\widehat{\widehat{G}}_0 \times \mathfrak{G}(K/F') \subseteq \widehat{\widehat{G}}$ on

$$\bigotimes_{\varphi \in \mathfrak{G}(K/F)\backslash S} V_\varphi$$

where V_φ is the space of column vectors of length 2 such that

$$\rho_0\big((a_\varphi)\big) = \bigotimes_{\varphi \in \mathfrak{G}(K/F)\backslash S} a_\varphi \qquad (a_\varphi) \in \widehat{\widehat{G}}_0$$

and

$$\rho_0(\sigma)\Big(\bigotimes v_\varphi\Big) = \bigotimes v'_\varphi \qquad v'_\varphi = v_{\varphi\sigma}$$

for $\sigma \in \mathfrak{G}(K/F')$. Let

$$\rho = \mathrm{Ind}\Big(\widehat{\widehat{G}}_0, \widehat{\widehat{G}}_0 \times \mathfrak{G}(K/F'), \rho_0\Big)$$

As I mentioned to you in Vancouver one can define ρ in general, but I prefer to introduce it in this way. As a matter of fact I should show that ρ is trivial on $\widehat{C}_0$ and thus a representation of $\widehat{G}$. I do this now.

Let me think of B'', the algebraic group over $\mathbf{Q}$ associated to $K^\times$, as the group obtained from

$$\prod_{\tau \in \mathfrak{G}(K/\mathbf{Q})} G_m$$

by the action

$$\sigma\big((\alpha_\tau)\big) = (\alpha'_\tau) \qquad \alpha'_\tau = \alpha_{\tau\sigma}.$$

Then B is

$$\big\{\, (\alpha_\tau) \;\big|\; \alpha_{\sigma\tau} = \alpha_\tau \ \forall \tau \in \mathfrak{G}(K/\mathbf{Q}),\ \sigma \in \mathfrak{G}(K/F) \,\big\}$$

and B' is obtained in a similar way. The map $\varphi : B' \to B$ sends $(\alpha_\tau) \to (\beta_\tau)$ where **[4]**

$$\beta_\tau = \prod_{S/\mathfrak{G}(K/F')} \alpha_{\sigma^{-1}\tau}.$$

Because $C = \varphi(B')\backslash B$ and $B \subseteq D$ I may think of $\Lambda(D)$, the lattice of characters of D, as

$$\bigoplus_{\tau \in \mathfrak{G}(K/\mathbf{Q})} \mathbf{Z},$$

of $\Lambda(B)$ as the quotient of this by

$$\left\{ (m_\tau) \;\middle|\; \sum_{\sigma \in \mathfrak{G}(K/F)} m_{\sigma\tau} = 0 \ \forall \tau \right\}.$$

To get $\Lambda(C)$ I divide this last group into

$$\left\{ (m_\tau) \;\middle|\; \sum_{\sigma\in S} m_{\sigma\tau} = 0 \;\forall\tau \right\}.$$

I think of $\widehat{\Lambda}(B) = \operatorname{Hom}\bigl(\Lambda(B), \mathbf{Z}\bigr)$ as

$$\bigoplus_{\varphi\in\mathfrak{G}(K/F)\backslash\mathfrak{G}(K/\mathbf{Q})} \mathbf{Z}.$$

Then

$$\bigoplus_{\tau\in\mathfrak{G}(K/\mathbf{Q})} m_\tau \times \bigoplus_{\varphi\in\mathfrak{G}(K/F)\backslash\mathfrak{G}(K/\mathbf{Q})} n_\varphi \to \sum_\varphi \sum_{\tau\to\varphi} m_\tau n_\varphi.$$

$\widehat{\Lambda}(C)$ is the quotient of $\widehat{\Lambda}(B)$ by the elements vanishing on $\Lambda(C)$. $\widehat{B}_0$ is the centre of $\widehat{\widehat{G}}_0$ and the restrictions of the weights of ρ to $\widehat{B}_0$ are $(m'_\varphi) \in \widehat{\Lambda}(B)$ with $m'_\varphi = m_{\varphi\tau}$, $\tau \in \mathfrak{G}(K/\mathbf{Q})$ and with (m_φ), the restriction of a weight of ρ_0 to $\widehat{B}_0$, given by $m_\varphi = 0$, $\varphi \notin \mathfrak{G}(K/F)\backslash S$, $m_\varphi = 1$, $\varphi \in \mathfrak{G}(K/F)\backslash S$. These weights vanish on $\Lambda(C)$ as required. Thus they are annihilated by $\widehat{C}_0$ and ρ is a representation of $\widehat{G}$.

I return to the main line of the discussion. Let $H = H(E)$ be $E^\times$ considered as an algebraic group over $\mathbf{Q}$. Then

$$\widehat{H} = \widehat{H}_0 \times \mathfrak{G}(K/\mathbf{Q})$$

where

$$\widehat{H}_0 = \prod_{\psi\in\mathfrak{G}(K/E)\backslash\mathfrak{G}(K/\mathbf{Q})} \mathrm{GL}(1,\mathbf{C})$$

and

$$\sigma(a_\psi)\sigma^{-1} = a'_\psi \qquad a'_\psi = a_{\psi\sigma}.$$

There is a map μ of $\widehat{H}$ into $\widehat{\widehat{G}}$ which is the identity on $\mathfrak{G}(K/\mathbf{Q})$ and sends

$$(a_\psi) \to \prod_{\varphi\in\mathfrak{G}(K/E)\backslash\mathfrak{G}(K/\mathbf{Q})} \begin{pmatrix} a_{\psi_1(\varphi)} & 0 \\ 0 & a_{\psi_2(\varphi)} \end{pmatrix}.$$

$\psi_1(\varphi)$ and $\psi_2(\varphi)$ are the two ψ's, taken in an arbitrary but fixed order, which project to φ. If a section $\varphi \to \psi(\varphi)$ of $\mathfrak{G}(K/F)\backslash\mathfrak{G}(K/\mathbf{Q}) \to \mathfrak{G}(K/E)\backslash\mathfrak{G}(K/\mathbf{Q})$ is given, let $\psi(\varphi) = \psi_{i(\varphi)}(\varphi)$. [**5**] If

$$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

then

$$\bigotimes v_{i(\varphi)} \in \bigotimes_\varphi V_\varphi.$$

The vectors obtained by letting $\varphi \to \psi(\varphi)$ vary over all sections yield a basis of $\bigotimes_\varphi V_\varphi$. The whole point of our preceding discussion is the following claim.

Claim. *$\rho \circ \mu$ is in a canonical way the direct sum of two representations of the same degree.*

μ takes $\widehat{H}_0 \times \mathfrak{G}(K/F')$ to $\widehat{\widehat{G}}_0 \times \mathfrak{G}(K/F')$ and

$$\rho \circ \mu = \mathrm{Ind}\Big(\widehat{H}, \widehat{H}_0 \times \mathfrak{G}(K/F'), \rho_0 \circ \mu\Big).$$

It is enough to prove that $\rho_0 \circ \mu$ is the direct sum of two representations of the same degree. $\rho_0 \circ \mu$ on $\widehat{H}_0$ takes $\mathbf{C}\big(\bigotimes v_{i(\varphi)}\big)$ to $\mathbf{C}\big(\bigotimes v_{i(\varphi)}\big)$. Let V^+ be the span of $\bigotimes v_{i(\varphi)}$ such that $i(\varphi) = 1$ for an even number of φ and let V^0 be the span of the other $\bigotimes v_{i(\varphi)}$. The whole point of the initial discussion was to show that V^+ and V^- are invariant under $\mathfrak{G}(K/F')$. Thus they are invariant under $\widehat{H}_0 \times \mathfrak{G}(K/F')$. This proves the claim.

Let R be the intersection of $G_{\mathbf{R}}$ with

$$B_{\mathbf{R}} \subseteq \text{centre of } \widetilde{G}_{\mathbf{R}}$$

and suppose F is totally real (cf. next page). The representation on the space of cusp forms on $L_2(RG_{\mathbf{Q}}\backslash G_{\mathbf{A}})$ is a direct sum of $\pi = \bigotimes_v \pi_v$. The product is over valuations of $\mathbf{Q}$. Two irreducible representations π_v, $\pi_{v'}$ of $G_{\mathbf{Q}_v}$ are said to be L-indistinguishable (I sometimes call them arithmetically indistinguishable) if they differ only by an automorphism of $\widetilde{G}_{\mathbf{Q}_v}$. Thus $\pi'_v(g) \sim \pi_v(hgh^{-1})$, $h \in \widetilde{G}_{\mathbf{Q}_{v'}}$. π is said to be stable if whenever $\pi' = \bigotimes \pi'_v$, where π'_v and π_v are L-indistinguishable for all v and equivalent for almost all v, π and π' occur with the same multiplicity. In general π and π' are said to be L-indistinguishable. Let $\mathfrak{I}$ be the collection of classes of L-indistinguishable representations occurring in the space of cusp forms. The representation in this space is the sum of three subrepresentations. The first one is **[6]**

$$\bigoplus_{s \in \mathfrak{I}} n(s) \bigoplus_{\pi \in S} \pi \qquad n(s) \in \mathbf{Z},\ n(s) \geqslant 0.$$

Thus in the first one two L-indistinguishable representations occur with the same multiplicity.

I notice that on the previous page I didn't describe G completely. Take F and $\overline{\mathbf{Q}}$, and hence $\overline{F}$ to be subfields of $\mathbf{C}$. Let S be the set of σ in $\mathfrak{G}(K/\mathbf{Q})$ for which $D \times_{F,0} \mathbf{R}$ splits.

If E is a quadratic extension of F and $\widetilde{\theta}$ a character of $E^\times\backslash\Gamma_E$ then for each v we have a character $\widetilde{\theta}_v$ of $E_v^\times$. $E_v = E \otimes_F F_v$. Let ψ be a non-trivial character of $\mathbf{Q}\backslash\mathbf{A}$, from $\psi_{F/\mathbf{Q}}$, and let ψ_v be the corresponding character of F_v. If v splits in E then $\widetilde{\theta}_v$ is really two characters μ_v and ν_v of $F_v^\times$ and we take $\pi_{\widetilde{\theta}_v}$ to be the element $\rho(\mu_v, \nu_v)$ of the principal series. $\pi_{\widetilde{\theta}_v}$ is a representative of $D_v^\times$. If v does not split in E we can still define $\pi_{\widetilde{\theta}_v}$. If $\mathrm{Nm}\, E_v^\times \neq \mathrm{Nm}\, D_v^\times$ (i.e. if v is non-archimedean or D_v is split) and if

$$D_v^+ = \big\{\, a \in D_v^\times \;\big|\; \mathrm{Nm}\, a \in \mathrm{Nm}\, E_v^\times \,\big\}$$

the restriction of $\pi_{\widetilde{\theta}_v}$ to D_v^+ is, as in for example Jacquet-Langlands, the direct sum of $\pi^+_{\widetilde{\theta}_v}$ and $\pi^-_{\widetilde{\theta}_v}$. The pair $(\pi^+_{\widetilde{\theta}_v}, \pi^-_{\widetilde{\theta}_v})$ is determined by $\widetilde{\theta}_v$ alone, but the order depends on the choice of ψ_v. (N.B.—these statements have not been verified for a non-split algebra over a non-archimedean field. Moreover for a non-split algebra $\pi_{\widetilde{\theta}_v}$ is only defined when the corresponding representation of the Weil group is irreducible.) If v splits set $D_v^+ = D_v^\times$, $\pi^+_{\widetilde{\theta}_v} = \pi_{\widetilde{\theta}_v}$. But $\pi^-_{\widetilde{\theta}_v}$ is now not defined.

Suppose $A = \varphi(B')$, w is a place of $\mathbf{Q}$, and

$$\prod_{v|w} \mathrm{Nm}\, D_v^\times \supseteq \prod_{v|w} \mathrm{Nm}\, E_w^\times \supseteq A_w \cap \prod_{v|w} \mathrm{Nm}\, D_v^\times.$$

Then

$$G_{\mathbf{Q}_w} \subseteq \prod_{v|w} D_v^+$$

and the restriction of $\prod_{v|w} \pi_{\tilde{\theta}_v}^{\delta(v)}$, $\delta(v) = \pm 1$, is defined—provided of course that all of $\pi_{\tilde{\theta}_v}^{\delta(v)}$ make sense.

We extend w to a place w of K and regard F_v as a subfield of K_w. θ_v determines a representation σ_v of W_{K_w/F_w} in $\mathrm{GL}(2, \mathbf{C})$. There is a map of $\mathfrak{G}(K/F)\backslash\mathfrak{G}(K/\mathbf{Q})/\mathfrak{G}(K_w/\mathbf{Q}_w)$ to the set of v which sends g to $v(x) = w(x^g)$. If $v(g) = v(g')$ then the map $x^g \to x^{g'}$ on F^g can be extended to a map from $\overline{F^g} \subseteq K_w$ [**7**] to $\overline{F^{g'}}$. Thus $\exists u \in \mathfrak{G}(K_w/\mathbf{Q}_w)$ so that $g' = gu$ on F. Thus the map is 1:1. If $v = v(\varphi)$ the map $x \to x^\varphi$ extends to an isomorphism $F_v \simeq F_\varphi$, the closure of F^φ in K_w. We take an isomorphism $i_\varphi : W_{K_w/F_\varphi} \subseteq W_{K_w/\mathbf{Q}_w} \simeq W_{K_w/F_v}$. This isomorphism is determined up to an inner automorphism by an element of $F_v^\times$. We may in fact suppose $i_{\varphi'}(w) = i_\varphi(vwv^{-1})$ if $\varphi' = \varphi\sigma$ and $v \to \sigma$. If $v = v(g)$ then

$$\mathfrak{G}(K_w/\mathbf{Q}_w) \cap g^{-1}\mathfrak{G}(K/F)g \simeq \mathfrak{G}(K_w/F_v).$$

I define a map

$$W_{K_w/\mathbf{Q}_w} \to \left(\prod_{\varphi \in \mathfrak{G}(K/F)\backslash\mathfrak{G}(K/F)g\mathfrak{G}(K_w/\mathbf{Q}_w)} \mathrm{GL}(2, \mathbf{C}) \right) \times \mathfrak{G}(K_w/\mathbf{Q}_w)$$

which has the obvious value in the second component by choosing for each φ a gw_φ in $gW_{K_w/\mathbf{Q}_w}$ representing it and letting

$$w_\varphi w = a_\varphi(w) w_{\varphi'} \qquad a_\varphi(w) \in W_{K_w/F_{\varphi_0}},\ \varphi_0 = \mathfrak{G}(K/F)\backslash\mathfrak{G}(K/\mathbf{Q})g$$

and, if $w \to \sigma$, mapping

$$w \to \prod_\varphi \sigma_v\Big(i_{\varphi_0}\big(a_\varphi(w)\big)\Big) \times \sigma.$$

Since

$$a_\varphi(w_1) a_{\varphi'}(w_2) = a_\varphi(w_1 w_2)$$

and

$$\sigma_1\big(a_\sigma(w_1)\big)\sigma_1^{-1} = \big(a_{\varphi'}(w_2)\big)$$

this map is a homomorphism. Putting the maps for the various double cosets we obtain

$$\widetilde{\sigma}_w : \mathfrak{G}(K_w/\mathbf{Q}_w) \to \widehat{\widehat{G}}.$$

Apart from inner automorphisms with respect to elements in the connected component $\widehat{\widehat{G}}_0$ of $\widehat{\widehat{G}}$ this mapping is independent of the choices made in its definition. It yields an element of $H^1(W_{K_w/\mathbf{Q}_w}, \widehat{\widehat{G}}_0)$.

Now some simple remarks are in order. Suppose $V \subset W$ are two groups with $[W : V]$ finite and H is a group on which V operates. Form

$$\prod_{\varphi \in V\backslash W} H = G$$

on which w operates by

$$w(h_\varphi)w^{+1} = a_\varphi(w) h_{\varphi'} a_\varphi(w)^{-1} \qquad \varphi' = \varphi w$$

if $\{w_\varphi\}$ is a set of coset representatives and

$$w_\varphi w = a_\varphi(w) w_{\varphi'}.$$

The notation is bad but is the way it is because I am thinking of semi-direct products.

[**8**] It is clear that

$$H^0(W,G) \simeq H^0(V,H)$$

by means of the map

$$g = (h_\varphi) \to h_{\varphi_0}$$

where φ_0 is the coset of V. It is also easy to see that the same projection yields

$$H^1(W,G) \simeq H^1(V,H) \qquad \text{(as sets)}$$

The reverse map sends

$$v \to h(v)$$

to

$$w \to \big(h_\varphi(w)\big)$$

where

$$h_\varphi(w) = h\big(a_\varphi(w_1)\big).$$

Notice

$$\begin{aligned} h\big(a_\varphi(w_1)\big) w_1 h\big(a_\varphi(w_1)\big) w_1^{-1} &= h\big(a_\varphi(w_1)\big) a_\varphi(w_1) h\big(a_{\varphi'}(w_2)\big) a_\varphi(w_1)^{-1} \\ &= h\big(a_\varphi(w_1 w_2)\big). \end{aligned}$$

Note if

$$v \to h h(v) v h^{-1} v^{-1}$$

then

$$w \to (h) h(w) w (h) w^{-1}$$

(h) means all components are equal.

Suppose

$$h_{\varphi_0}(v) = g_{\varphi_0}(v) \qquad \forall v \in V.$$

Is there a family (h_φ) so that

$$h_\varphi h_\varphi(w) a_\varphi(w) h_{\varphi'}^{-1} a_\varphi(w)^{-1} = g_\varphi(w) \qquad \forall w \in W$$

or

$$h_\varphi h_\varphi(w) a_\varphi(w) = g_\varphi(w) a_\varphi(w) h_\varphi.$$

Take

$$h_\varphi = g_\varphi(w_\varphi^{-1}) h_\varphi(w_\varphi^{-1})^{-1}.$$

Then the question becomes

$$h_\varphi(w_\varphi^{-1})^{-1} h_\varphi(w) a_\varphi(w) h_{\varphi'}(w_{\varphi'}^{-1}) \stackrel{?}{=} g_\varphi(w_\varphi^{-1})^{-1} g_\varphi(w) a_\varphi(w) g_{\varphi'}(w_{\varphi'}^{-1}).$$

The left side is

$$h_\varphi(w_\varphi^{-1})^{-1} h_\varphi(w w_{\varphi'}^{-1}) a_\varphi(w)$$

because

$$h_\varphi(w) a_\varphi(w) h_{\varphi'}(w_{\varphi'}^{-1}) a_\varphi(u)^{-1} = h_\varphi(w w_{\varphi'}^{-1}).$$

Also

$$w_\varphi w w_{\varphi'}^{-1} = a_\varphi(w)$$

[9] so we can manipulate the left side further to obtain

$$a_\varphi(w)w_{\varphi_0}^{-1}h_{\varphi_0}(w_{\varphi'}w^{-1}w_\varphi)^{-1}w_{\varphi_0}.$$

By assumption one obtains the same result with h replaced g as required. Thus the map is indeed an isomorphism. Our isomorphisms are clearly compatible with sequences

$$1 \longrightarrow H^0(V,G') \longrightarrow H^0(V,G) \longrightarrow H^0(V,G'') \longrightarrow H^1(V,G') \longrightarrow H^1(V,G) \longrightarrow H^1(V,G'') \quad .$$

Moreover no further difficulties are caused by the imposition of topological conditions.

For each double coset

$$\alpha \in \mathfrak{G}(K/F)\backslash\mathfrak{G}(K/\mathbf{Q})/\mathfrak{G}(K_w/\mathbf{Q}_w)$$

let

$$G_\alpha = \prod_{\varphi\in\alpha} \mathrm{GL}(2,\mathbf{C}).$$

$\mathfrak{G}(K_w/\mathbf{Q}_w)$ operates on G_α as above and

$$H^1(W_{K_w/\mathbf{Q}_w}, \widehat{\widehat{G}}) = \prod_\alpha H^1(W_{K_w/\mathbf{Q}_w}, G_\alpha)$$
$$H^1(W_{K_w/\mathbf{Q}_w}, \widehat{B}) = \prod_\alpha H^1(W_{K_w/\mathbf{Q}_w}, Z_\alpha)$$

where Z_α is formed like G_α except that $\mathrm{GL}(2,\mathbf{C})$ is replaced by $\mathrm{GL}(1,\mathbf{C})$. To check

$$H^1(W_{K_w/\mathbf{Q}_w}, \widehat{B}) \hookrightarrow H^1(W_{K_w/\mathbf{Q}_w}, \widehat{\widehat{G}}).$$

I have only to look at each of the factors. By the above considerations we can replace $K_w/\mathbf{Q}_w$ by K_w/F_v, G_α by $\mathrm{GL}(2,\mathbf{C})$ and Z_α by $\mathrm{GL}(1,\mathbf{C}) \hookrightarrow \mathrm{GL}(2,\mathbf{C})$. We have only to observe that

$$\mathrm{Hom}\big(W_{K_w/F_v}, \mathrm{GL}(1,\mathbf{C})\big) \hookrightarrow \mathrm{Hom}\big(W_{K_w/F_v}, \mathrm{GL}(2,\mathbf{C})\big).$$

Suppose $\left\{\widetilde{\theta}_v|v|_w\right\}$ and $\left\{\widetilde{\theta}'_v|v|_w\right\}$ are given so that $\widetilde{\sigma}_w$ and $\widetilde{\sigma}'_w$ are defined. They yield, upon projection, the same map into $\widehat{G}$ (up to inner automorphisms from the connected component) if and only if the corresponding cocycles in $H^1(W_{K_w/\mathbf{Q}_w}, \widehat{G})$ are equal.

Have

$$\begin{array}{ccccc} H^1(W,\widehat{G}) & \longrightarrow & H^1(W,\widehat{\widehat{G}}) & \longrightarrow & H^1(W,\widehat{G}) \\ & \searrow & \nearrow & & \\ & & H^1(W,\widehat{B}) & & \end{array}$$

$W = W_{K_w/\mathbf{Q}_w}$. N.B. $\widehat{B}$ is central.

The two cycles differ by an element of $H^1(W,\widehat{B})$ if and **[10]** only if there is a family $\{\mu_v|v|_w\}$ of characters of $F_v^\times$ so that

$$\sigma'_v = \mu_v(\mathrm{Nm}\, g)\sigma_v.$$

The question is when this comes from an element of $H^1(W,\widehat{C})$. We have

$$1 \longrightarrow \widehat{C} \longrightarrow \widehat{B} \longrightarrow \widehat{A} \longrightarrow 1$$

and thus

$$H^1(W,\widehat{C}) \longrightarrow H^1(W,\widehat{B}) \longrightarrow H^1(W,\widehat{A}) \quad .$$

Thus it comes from an element of $H^1(W,\widehat{C})$ if and only if its image in $H^1(W,\widehat{A})=0$. But have pairings

$$H^1(W,\widehat{B}) \simeq \text{Dual of } B_{\mathbf{Q}_w}$$
$$H^1(W,\widehat{A}) \simeq \text{Dual of } A_{\mathbf{Q}_w}.$$

(If you don't believe this, see "Representations of Abelian algebraic groups".) The condition is then that the element of the dual vanish on $A_{\mathbf{Q}_w}$. This turns out to be precisely the condition that the restrictions of $\prod \pi_{\widetilde{\theta}_v}^{\delta(v)}$ and $\prod \pi_{\widetilde{\theta}'_v}^{\delta(v)}$ to $G_{\mathbf{Q}_w}$ be equivalent. We take any one of these representations to be $\pi(\sigma_w)$, if σ_w is the restriction of $\widetilde{\sigma}_w$. Its signature is $\{\delta(v)|v|_w\}$. Notice that $\pi(\sigma_w)$ is not uniquely determined before its signature is given.

Let $H=H(E)$ (conflicts with earlier notation) be the algebraic group $N^{-1}(A)$ where N is the norm map $E \to B$. We have assumed that $N(H_{\mathbf{Q}_w})=A_{\mathbf{Q}_w}$. The collection $\{\widetilde{\theta}_v\}$ determines a character of $\prod_{v|w} E_v^\times$ and hence a character θ_w of the subgroup $H_{\mathbf{Q}_w}$. If $\{\widetilde{\theta}_v\}$ and $\{\widetilde{\theta}'_v\}$ determine the same character there is a collection $\{\mu_v\}$ so that

$$\widetilde{\theta}'_v = \widetilde{\theta}_v N_{E_v/F_v}\mu_v$$

The collection $\{\mu_v\}$ determines a character of $B_{\mathbf{Q}_w}$ which must vanish on $N(H_{\mathbf{Q}_w})=A_{\mathbf{Q}_w}$. Thus $\widetilde{\sigma}_w$ and $\widetilde{\sigma}'_w$ yield the same σ_w. If s is the non-trivial element of $\mathfrak{G}(E/F)$ and $\theta_w^s(\alpha)=\theta_w(\alpha^s)$ then θ_w and θ_w^s also yield the same σ_w and of course the same representations $\pi(\sigma_w)$.

I have a now to confess that there are certain cases in which the above definition of the $\pi(\sigma_w)$ is incorrect. (cf also p. 14 $*$) I will describe those in a moment. Suppose σ_w is obtained from $\widetilde{\sigma}_w = \bigotimes \sigma_v$ and σ_v is obtained from the character $\widetilde{\theta}_v$ of E_v. E_v is uniquely determined by **[11]** σ_v except in the following cases.

(i) E_v (non-split) and $\widetilde{\theta}_v = N_{E_v/F_v}\mu_v$ is confounded with E_v split and $\widetilde{\theta}_v=(\mu_v,\epsilon_v\mu_v)$.

(ii) Suppose K_v/F_v is an extension of degree 4 with Galois group $\mathbf{Z}_2\oplus\mathbf{Z}_2$, L_v is an intermediate quadratic extension, the non-trivial element of $\mathfrak{G}(L_v/F_v)$ and η_v a character of $L_v^\times$ so that

$$\eta_v(x^s x^{-1}) = \mu_v(x) \qquad \forall x \in L_v^\times$$

if μ_v is the character associated to K_v/L_v. Then given any other intermediate quadratic extension E_v there exists $\widetilde{\theta}_v$ so that

$$\operatorname{Ind}(W_{L_v/F_v}, W_{L_v/L_v}, \eta_v) \simeq \operatorname{Ind}(W_{E_v/F_v}, W_{E_v/E_v}, \widetilde{\theta}_v).$$

Except in these two exceptional cases E_v is determined by $\pi(\sigma_v)=\pi_{\widetilde{\theta}_v}$ by the condition that if $U \subseteq F_v^\times$ is open and of finite index then the restriction of $\pi_{\widetilde{\theta}_v}$ to

$$\left\{\, g \in D_v^\times \;\middle|\; \operatorname{Nm} g \in U \,\right\}$$

is reducible $\iff$ $U \subseteq \operatorname{Nm} E_v^\times$. For GL(2) you know this from your own work and, for example, Th. 4.6 of Jacquet-Langlands. For a division algebra it still needs a proof if the field is non-archimedean. In the first exceptional case the non-split E_v is determined by this condition. In the second the field K_v, which is determined by σ_v, is determined by the condition that the restriction of $\pi_{\widetilde{\theta}_v}$ to

$$\left\{\, g \in D_v^\times \;\middle|\; \operatorname{Nm} g \in U \,\right\}$$

splits into four irreducible parts if and only if $U \subseteq \operatorname{Nm} K_v^\times$. By the way once E_v is given the pair consisting of θ_v and its conjugate should be determined.

In the second exceptional case (and in fact in all cases) $\pi(\sigma_w)$ is any one of the irreducible components of the restriction of $\bigotimes \pi(\sigma_v)$ to $G_{\mathbf{Q}_w}$. However this change will not be very important to us except when there are two quadratic extensions E and E' so that $\prod_v \operatorname{Nm} E_v^\times$ and $\prod_v \operatorname{Nm} E_v'^\times$ both contain $A_{\mathbf{Q}_w}$. If K is the composite then $\prod_v \operatorname{Nm} K_v^\times \supseteq A_{\mathbf{Q}_w}$. Take a v for which $[F_v^\times : \operatorname{Nm} K_v^\times] = 4$. There are three intermediate quadratic fields labeled 1, 2, 3, and four irreducible components **[12]** of the restrictions of $\pi_{\widetilde{\theta}_v^i}$, $i = 1, 2, 3$ to $\{\, g \mid \operatorname{Nm} g \in \operatorname{Nm} K_v^\times \,\}$. In the following matrix the $+$ at for example $1, 1$ indicates that the first of the four representations is a component of $\pi^+_{\widetilde{\theta}^1}$ and not of $\pi^-_{\widetilde{\theta}^1}$. The matrix is

	1	2	3	4
1	+	+	−	−
2	+	−	+	−
3	+	−	−	+

Remember ψ_v is given (it is a character of F_v^+.) The first two rows of the matrix reflect merely the choice of the labels 1, 2, 3, 4. The last does not. The important point is that there is a column with only plus signs. That this is so is independent of the choice of ψ_v and needs to be proved. I have not yet proved it; but it is the only possibility that makes sense globally. It also makes sense in terms of the character formulae of Sally + Shalika.

Let me now work globally and take a quadratic extension E of F so that

$$NI_E \subseteq I_F \qquad \text{(cf p. 14 *)}$$

contains $A_{\mathbf{A}}$. I consider those θ, characters of $H(E)_{\mathbf{Q}} \backslash H(E)_{\mathbf{A}}$, such that $\theta \neq \theta^s$, s the non-trivial element of $\mathfrak{G}(E/F)$. If θ is the restriction of $\widetilde{\theta}$, this means

$$\widetilde{\theta}(x^s x^{-1}) \not\equiv \mu(N_{E/F} x)$$

for any character μ of I_F trivial on $A_{\mathbf{A}}$. If for any given w, and the θ_w determined by θ, $\pi(\sigma_w)$ is introduced in the correct way it is still possible to introduce the signature of $\pi(\sigma_w)$—it is just the signature of the wrong $\pi(\sigma_w)$ of which the right one is a component. The signature depends not only on σ_w but also on E and on the given choice of a character of $\mathbf{A}/\mathbf{Q}$. Let $S^*(\theta, \theta^s)$ be the set of all $\pi = \bigotimes \pi(\sigma_w)$ for which the signature of $\pi(\sigma_w)$ is $(1, \ldots, 1)$ for almost all w and for which

$$\prod_v \delta(v) = 1.$$

The product is over all valuations of F. The second part of the representation on the space of cusp forms introduced earlier is

$$\bigoplus_E \bigoplus_{\{\theta, \theta^s\}} \sum_{\pi \in S^*(\theta, \theta^s)} \pi$$

The sum is over those E and θ satisfying the previous conditions. **[13]** Moreover θ is to be trivial on $H(E)_{\mathbf{R}} \cap B_{\mathbf{R}} \subseteq \widetilde{G}_{\mathbf{R}}$.

Without changing the conditions on E and retaining the last condition on θ we now suppose that $\theta = \theta^s$ but that θ is not the restriction of $\widetilde{\theta} = N_{E/F}\mu$. Then

$$\widetilde{\theta}(x^s x^{-1}) = \mu(N_{E/F} x).$$

Because an element of F which is locally a norm is globally a norm we may suppose it is a character of $F^\times\backslash I_F$. If $x \in I_F$ then

$$1 = \widetilde{\theta}(x^s x^{-1}) = \mu(x^2)$$

so $\mu^2 = 1$ and μ determines a quadratic extension of F different from E (because $\mu(N_{E/F}x) \not\equiv 1$). Let the composite be K. K has three quadratic subfields E^1, E^2, E^3 and to each of these is associated a θ' so that $\theta' = (\theta^i)^{s_i}$ and so that

$$\operatorname{Ind}(W_{E'/F}, W_{E'/E'}, \widetilde{\theta}') = \widetilde{\sigma}$$

is independent of i. E and $\widetilde{\theta}$ are one of the three pairs. In any case the third part of the representation is going to involve a sum over these tuples $(E^1, E^2, E^3, \theta^1, \theta^2, \theta^3)$. With that explained let me tell you what a given summand is. Suppose $\pi = \bigotimes \pi(\sigma_w)$ is given, where $\widetilde{\sigma}$ has local components $\widetilde{\sigma}_v$ and σ_w is the restriction of $\bigotimes_{v|w} \widetilde{\sigma}_v$. To each place v we assign a column of the following form:

either	or	or	or
$+$	$+$	$-$	$-$
$-$	$+$	$+$	$-$
$-$	$+$	$-$	$+$

If $[NK_v^\times : F_V^\times] = 4$ I have explained this column on the previous page. If $[NK_v^\times : F_v^\times] = 1$ we take the column of $+$. If $[NK_v^\times : F_v^\times] = 2$ then one of $[E'_v : F_v]$ is 1 and the others are 2. We put a plus at the spot where it is 1. Of course if i_1 and i_2 are the other two spots ${}_E i_1 \simeq {}_E i_2$. The signature $\delta(v)$ of σ_w at $v|w$ is defined with respect to both extensions and is the same for both. In these rows we put this signature. Thus π has, corresponding to the three rows, three signatures $\delta^i(v)$, v a valuation of F, $i = 1, 2, 3$. We take those π for which all but a finite number of $\delta^1(v)$, $\delta^2(v)$, and $\delta^3(v)$ are 1 and consider **[14]**

$$\delta^i = \prod_v \delta(v).$$

We have

$$\begin{matrix} \delta^1 = + & & + & & - & & - \\ \delta^2 = + & \text{or} & - & \text{or} & + & \text{or} & - \\ \delta^3 = + & & - & & - & & + \end{matrix}$$

The last component is a sum over $\{E^1, E^2, E^3, \theta^1, \theta^2, \theta^3\}$ of the direct sum of those π's of the above type for which

$$\begin{matrix} \delta^1 & & + \\ \delta^2 & = & +. \\ \delta^3 & & + \end{matrix}$$

Before I go on to the L-functions of Shimura varieties let me repeat that the proofs of the things I have just described are not yet written up, in fact some details are still missing, and that when quaternion algebras over non-archimedean fields are involved there are large gaps. As usual we are going to ignore trivial parts of the L-function.

The third summand of the representation in the space of cusp forms plays no role so we may forget it. The first and second summands yield two factors of the L-function.

$*$ I have first to correct a mistake made above. The condition on E is not that NI_E should contain $A_{\mathbf{A}}$ but that $\operatorname{Nm} D_F^\times \operatorname{Nm} I_E$ should contain $A_{\mathbf{A}} \cap \operatorname{Nm} D_{\mathbf{A}}$. Besides E has to be imbeddable in D. This requires some change in the local discussion. We can no longer require that

$$\prod_{v|w} \operatorname{Nm} E_v^\times \supseteq A_w \cap \prod_{v|w} \operatorname{Nm} D_v^\times.$$

This condition dropped we have to reconsider $\bigotimes_{v|w} \pi_{\widetilde{\theta}_v}^{\delta(v)} \backslash G_w$. In fact the signature can be regarded as an element of $\prod_{v|w} F_v^\times / \prod_{v|w} \operatorname{Nm} E_v^\times$. To get representations whose restriction to G_w makes sense we must sum

$$\sum \bigotimes_{v|w} \pi_{\widetilde{\theta}_v}^{\delta(v)}$$

where the sum is over all $\delta(v)$ in a coset of $A_w \prod_{v|w} \operatorname{Nm} E_v^\times$ intersected with $\prod_{v|w} \operatorname{Nm} D_v^\times$. These are our $\pi(\sigma_w)$. Associated to a given $\pi(\sigma_w)$ is a signature which is now a coset of $A_w \prod_{v|w} \operatorname{Nm} E_v^\times$. The global condition in the second contribution to the trace is that this
[15] coset lie in $F^\times \operatorname{Nm} I_E \supseteq A_{\mathbf{A}} \operatorname{Nm} I_E$. For the third case there are three signatures but otherwise the condition is the same. We only take those for which all three signatures lie in $F^\times \operatorname{Nm} I_E$. However as I said the third part of the representation plays no role for the L-function.

I can't resist however first formulating some of the previous remarks in a way that has meaning for a general group. (This is a digression.) Let $L \supseteq \mathbf{Q}$. From

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G & \longrightarrow & \widetilde{G} = D^\times & \longrightarrow & C & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & & & \\ 1 & \longrightarrow & H(E) & \longrightarrow & \overline{\partial}(E) & \longrightarrow & C & \longrightarrow & 1 \end{array}$$

and $\qquad \partial(E) = E^\times$

together with

$$H^1(L, \widetilde{G}) = 0$$
$$H^1\Big(L, \overline{\partial}(E)\Big) = 0$$

we see that

$$H^1(L, G) = \operatorname{Nm} D_L^\times \backslash C_L$$
$$H^1\big(L, H(E)\big) = \operatorname{Nm} E_L^\times \backslash C_L.$$

Thus the kernel of

$$H^1\big(L, H(E)\big) \to H^1(L, G)$$

is

$$\operatorname{Nm} D_L^\times A_L / \operatorname{Nm} E_L^\times A_L \subseteq F_L^\times / \operatorname{Nm} E_L^\times A_L$$

(The notation is dreadful.) Thus we see, taking $L = \mathbf{Q}_w$, that the signature is an arbitrary element of

$$\text{Kernel} : H^1\big(\mathbf{Q}_w, H(E)\big) \to H^1(\mathbf{Q}_v, G).$$

Globally we have

$$\begin{array}{ccc} H^1\big(\mathbf{Q}, H(E)\big) & \longrightarrow & \prod_w H^1\big(\mathbf{Q}_w, H(E)\big) \\ \downarrow & & \downarrow \\ H^1(\mathbf{Q}, G) & \longrightarrow & \prod_w H^1(\mathbf{Q}_w, G) \end{array}$$

Taking kernels of the vertical maps we have

$$\operatorname{Nm} E^\times_{\mathbf{Q}} A_{\mathbf{Q}} \backslash \operatorname{Nm} D^\times_{\mathbf{Q}} A_{\mathbf{Q}} \to \prod_w \operatorname{Nm} E^\times_{\mathbf{Q}_w} A_{\mathbf{Q}_w} \backslash \operatorname{Nm} D^\times_{\mathbf{Q}_w} A_{\mathbf{Q}_w}.$$

The image is

$$A_{\mathbf{A}} \operatorname{Nm} I_E \operatorname{Nm} D^\times_{\mathbf{Q}}.$$

Observe that $\operatorname{Nm} D^\times_{\mathbf{A}} \cap F^\times = \operatorname{Nm} D^\times_{\mathbf{Q}}$. (Here $D^\times$ functions as an algebraic group over $\mathbf{Q}$—sorry for notation $D_F = D_{\mathbf{Q}}$!!) **[16]** Thus

$$\begin{aligned}\operatorname{Nm} I_E F^\times \backslash I_F &= \operatorname{Nm} I_E F^\times \backslash \operatorname{Nm} D^\times_{\mathbf{A}} F^\times \\ &= \operatorname{Nm} I_E \operatorname{Nm} D^\times_{\mathbf{Q}} \backslash \operatorname{Nm} D^\times_{\mathbf{A}}\end{aligned}$$

is of order 2. The global condition on E is merely that

$$A_{\mathbf{A}} \operatorname{Nm} I_E \operatorname{Nm} D^\times_{\mathbf{Q}} \neq \operatorname{Nm} D^\times_{\mathbf{A}} A_{\mathbf{A}}.$$

When the equality sign holds the above image consists of only one element. The condition on the global signature, an element of the kernel in $\prod_w H^1(\mathbf{Q}_w, H(E))$, that the representation should occur is that it lie in the image of the kernel in $H^1(\mathbf{Q}, H(E))$.

Some of the things I have just talked about have interpretations for a general group G, over a global field F (a new F). First of all consider a local field F. Take a torus T and consider imbeddings $T \xrightarrow{\varphi} G$, whose images are Cartan subgroups. φ is defined over F. φ and φ' are of course equivalent if $\exists g \in G_F$ and $\varphi'(t) = g\varphi(t)g^{-1}$. We call φ and φ' stably equivalent if $\exists g \in G_{\overline{F}}$ so that $\varphi'(t) = g\varphi(t)g^{-1}$. Then $\varphi^{-1}(g^\sigma g^{-1})$ is a 1-cocycle. The equivalence classes of stably equivalent imbeddings may be identified with

$$(*) \qquad \text{Kernel} : H^1(F, T) \to H^1(F, G).$$

What we have above is the following: We associate to certain representations a finite collection, each element of the collection consisting of a class of stably equivalent imbeddings—there were none, one, or three elements in the collection. The global condition was to be imposed for a finite collection class of stable equivalence classes of global imbeddings.

I observe that, for a real group, and a T with $T_{\mathbf{R}}$ compact the kernel of $(*)$ is the quotient $\Omega_{\mathbf{R}} \backslash \Omega_{\mathbf{C}}$ (real and complex Weyl groups) and that a set of L-indistinguishable representations in the discrete series is parametrized by the same homogeneous space!! Be that as it may, the digression is over.

[17] The first component of the representation was a sum over $\mathfrak{I}$. Let $\mathfrak{I}_0 \subseteq \mathfrak{I}$ be the collection of those $s \in \mathfrak{I}$ for which $\pi = \bigotimes \pi_w \in s$ has for π_∞ the restriction of $\bigotimes \widetilde{\pi}_v$, where $\widetilde{\pi}_v$, which is trivial on $F^\times_v \subseteq D^\times_v$, is trivial if D_v does not split and is the first member of the discrete series if it does. Let π^0_∞ be a fixed representation of this type. Let $\pi^f = \bigotimes_{w \text{ finite}} \pi_w$. It is a representation of $G_{\mathbf{A}^f}$. Let K be a compact open subgroup of $G_{\mathbf{A}^f}$ and let $m(s, K)$, $s \in \mathfrak{I}_0$, be the multiplicity with which the trivial representation of K occurs in

$$\bigoplus_{\substack{\pi \in s \\ \pi_\infty = \pi^0_\infty}} \pi^f.$$

If ρ is the representation of $\widehat{G}$ introduced earlier the contribution of the first component of the representation to the L-function of the variety over F' associated by Shimura to K is presumably

$$\prod_{s\in\mathfrak{I}_0} L(s,\pi,\rho)^{m(s,K)}.$$

Here π is any element of s. The L-function, which of course has at the present time only been defined up to finitely many factors of the Euler product, is presumably independent of π.

The second part of the representation was a sum over E and $\{\theta,\theta^s\}$. We are now only interested in those E which are totally imaginary and those pairs $\{\theta,\theta^s\}$ so that for every $v|\infty$ at which D_v splits

$$\{\widetilde{\theta}_v,\widetilde{\theta}^s_v\} = \left\{ z \to \frac{z}{|z|},\ z \to \frac{\overline{z}}{|z|} \right\}$$

and so that when $v|\infty$ and D_v does not split $\widetilde{\theta}_v$ is trivial. Let R be the set of $v|\infty$ which split D. The signature of a π_∞ may be represented by a map $R \to \{+1,-1\}$—of course two maps may, as we now know, represent the same signature. Since we have fixed a holomorphic structure this determines holomorphic and anti-holomorphic representations and hence a distinguished signature $v \to \eta_v$.

We can write the second part as

$$\bigoplus_E \bigoplus_{\{\theta,\theta^s\}} \bigoplus_\omega \left\{ \bigoplus_{\pi\in S^+(\theta,\theta^s,\omega)} \pi \right\}.$$

Here ω runs over all representations of $G_{\mathbf{A}^f}$ which can occur in π^f and

$$S^+(\theta,\theta^s,\omega) = \left\{ \pi \in S^+(\theta,\theta^s) \;\middle|\; \pi^f = \omega \right\} \qquad \pi^f = \bigotimes_{w \text{ finite}} \pi_w.$$

If $\pi \in S^+(\theta,\theta^s,\omega)$ it has a partial signature $v \to \delta_v$, $v \in \mathbf{R}$. $\prod_{v\in R}\delta_v = \delta(\theta,\theta^s,\omega)$ depends only on θ, θ^s, ω. Let $m(\omega,K)$, K **[18]** compact open subgroup of $G_{\mathbf{A}^f}$, be the multiplicity with which the trivial representation of K occurs in ω. Let $H = H(E)$ and $J = J(E)$ be as above. θ, coming from $\widetilde{\theta}$, gives rise to (another K)

$$\begin{array}{ccccc} W_{K/\mathbf{Q}} & \xrightarrow{\varphi} & \widehat{H} & \xrightarrow{\mu} & \widehat{G} \\ & \searrow^{\widetilde{\varphi}} & \uparrow & & \uparrow \\ & & \widehat{J} & \xrightarrow{\mu} & \widehat{\widehat{G}} \end{array}$$

The bottom μ was defined at the beginning of the letter. The μ at the top is defined by commutativity. $\psi = \rho\circ\mu\circ\widetilde{\varphi} = \rho\circ\mu\circ\varphi$ is the direct sum of two representations ψ^+ and ψ^-. The contribution of the second part of the representation to the L-function will be (probably!)

$$\prod_{\left\{ E,\{\theta,\theta^s\},\omega \;\middle|\; S^+(\theta,\theta^s,\omega)\neq\varnothing \right\}} L(s,\psi^\delta)^{m(\omega,K)} \qquad \delta = + \text{ or } -.$$

What we have to decide however is what the sign of δ should be. F' is given as a subfield of $\mathbf{C}$. Let v_0 be the corresponding real place. Any element $\pi \in S^+(\theta,\theta^s,\omega)$ contributes forms of type p, q where $p+q = r$ (Number of elements in R) and $(-1)^{p-q} = \delta(\theta,\theta^s,\omega)/\eta$, $\eta = \prod_{v\in R}\eta_v$. ψ^+ and ψ^- are the two representations whose restrictions to $W_{K_{v_0}/F'_{v_0}}$ $(K \supseteq E)$

are

$$\bigoplus_{\substack{p+q=r \\ p-q\equiv j \pmod 2}} \frac{z^D\overline{z}^q}{z^r} \qquad j = 0 \text{ or } 1.$$

We choose ψ^δ so that it is one for which $(-1)^j = \delta(\theta, \theta^s, \omega)/\eta$.

If you have any comments, I'd sure like to hear them.

Yours,
Bob Langlands

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