

November 16, 1972

Dear Bill,

Thanks for your preprints, which came at a very opportune time. I was about to assume in my Antwerp notes that  $p \neq 2$ , because I couldn't prove the following facts, which is of course a consequence of "The restriction..."

*The support of the character of an absolutely cuspidal representation of  $GL(2, F)$ ,  $F$  a non-archimedean local field, does not contain  $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$  if  $|\alpha| \neq |\beta|$ .*

As an aside, what does the word  $p$ -adic mean to you? Can a  $p$ -adic field have positive characteristic? I knew of course that this assertion should be a consequence of a Shalika-Mautner type theorem. But my attempts at finding a statement like your Theorem 3 failed.

Just to make sure that I am using your results correctly let me sketch the verification of the above corollary.  $\sigma$  : the representation of  $U = Z(F)G(O_F)$  which you denote  $\epsilon P$ .  $\Pi = \text{Ind}(\sigma, G(F, u))$ . Commuting algebra of  $\Pi$  formed by functions  $\lambda$  such that

$$(*) \quad \lambda(k_1 h k_2) = \sigma(k_2^{-1})\lambda(h)\sigma(k_1^{-1}).$$

If  $\varphi$  such that  $\varphi(ug) = \sigma(u)\varphi(g)$ ,  $u \in U$ , then  $\lambda : \varphi \rightarrow \psi$  with

$$\psi(h) = \int_{Z(F)\backslash G(F)} \lambda(g)\varphi(gh)$$

Consider a  $\lambda$  satisfying (\*) and take

$$h = \begin{pmatrix} a & y \\ 0 & b \end{pmatrix} \quad |a| \leq |b|$$

$\varpi$  is a generator of the maximal ideal of  $O_F$ . From (\*)

$$\sigma(k^{-1})\lambda(h) = \lambda(h)\sigma(hkh^{-1})$$

if

$$k \in U \cap h^{-1}Uh.$$

In particular if  $x \in O_F$

$$\sigma \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \lambda(h) = \lambda(h)\sigma \begin{pmatrix} 1 & \frac{ax}{b} \\ 0 & 1 \end{pmatrix}$$

It follows from your analysis that if your  $c$  is even then  $\lambda(h)$  is 0 unless  $|a| = |b|$  and that if  $c$  is odd  $\lambda(h)$  is 0 unless  $|a| = |b|$  or  $|\frac{a}{b}| = |\varpi|$ . Moreover in the second case, trace  $\lambda(h) = 0$  when  $|\frac{a}{b}| = |\varpi|$ . More generally if  $h$  has eigenvalues in  $F$ , there is a  $k$  in  $G(O_F)$  so that

$$k^{-1}hk = \begin{pmatrix} a & y \\ 0 & b \end{pmatrix} = g \quad \left| \frac{a}{b} \right| \leq 1.$$

If  $|a| < |b|$

$$\text{trace } \lambda(h) = \text{trace } \sigma(k^{-1})\lambda(h)\sigma(k) = \text{trace } \lambda(g) = 0.$$

Let  $\{g_\alpha\}$  be a set of representations for  $U \backslash G(F)$ . Let  $\sigma$  act on  $V$  and  $\Pi$  on  $W$ . If  $\{v_\beta\}$  is a basis of  $V$ , we may take as a basis for  $W$  the functions  $\varphi_{\alpha\beta}$ , where  $\varphi_{\alpha\beta}$  is 0 outside of  $Ug_\alpha$  and  $\varphi_{\alpha\beta}(ug_\alpha) = \sigma(u)v_\beta$ . We take  $g_\alpha$  of the form  $t_m k$  where

$$t_m = \begin{pmatrix} \varpi^m & 0 \\ 0 & 1 \end{pmatrix} \quad m \geq 0$$

and where  $k$  runs over a set of representatives for  $K \cap t_m^{-1} K t_m \backslash K$ . The  $\varphi_{\alpha\beta}$  can be used to calculate the trace.

What we have to do to prove the corollary is to show that if

$$h_0 = \begin{pmatrix} a_0 & 0 \\ 0 & b_0 \end{pmatrix} \quad \left| \frac{a_0}{b_0} \right| \neq 1$$

and if  $t$  is a function with support in a small neighbourhood of  $h_0$  then

$$\text{trace } \lambda \Pi(f) = 0$$

for any  $\lambda$  in the commuting algebra.

We may suppose that if  $f(h) \neq 0$  then  $h$  has eigenvalues in  $F$  of different absolute values. Let  $W_\alpha$  be the functions with support in  $Kg_\alpha$ . Let  $\lambda \Pi(f) = A$  and let  $A = (A_{\alpha\beta})$  where  $A_{\alpha\beta} : W_\alpha \rightarrow W_\beta$ . We have to show that  $\text{trace } A_{\alpha\alpha} = 0$ . Now  $A : \varphi \rightarrow \psi$  with

$$\psi(x) = \int_{G(F)} \lambda(g) \varphi(gxh) dg$$

$A_{\alpha\alpha} : \varphi \rightarrow \psi$  with

$$\begin{aligned} \psi(g_\alpha) &= \int_{\{g \mid gg_\alpha h = ug_\alpha, u \in U\}} \lambda(g) \varphi(gg_\alpha h) dg \\ &= \int_{\{g \mid gg_\alpha h = ug_\alpha\}} \lambda(g) \sigma(u) \varphi(g_\alpha) dg \end{aligned}$$

Thus

$$A_{\alpha\alpha} = \int_{\{g \mid gg_\alpha h = ug_\alpha\}} \lambda(g) \sigma(u) dg.$$

Since

$$\lambda(g) \sigma(u) = \lambda(u^{-1}g) = \lambda(g_\alpha h^{-1} g_\alpha^{-1})$$

we have

$$\text{trace } A_{\alpha\alpha} = \int \text{trace } \lambda(g_\alpha h^{-1} g_\alpha^{-1}) = 0.$$

Thanks once again for the preprints,  
Bob

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