February/March 1973

Dear Bill,

This letter is an excuse to clarify to myself the notion of L-indistinguishability. You may find some things in it of some slight interest; certainly if you don't, no one else will. I shall work on the whole over $\mathbf{R}$ and $\mathbf{C}$. It is a matter of definitions, some simple lemmas, and a rehash of parts of Harish-Chandra's papers. But even simple observations, provided they suggest more precise formulations of the Washington problems, are of some value. I hope in this and a subsequent letter not only to formulate the local problems at infinity more precisely but also to show that in reality they have already been solved.

I begin by pointing out a consequence of Steinberg's paper (Publ. Math. N° 25). G is, and will remain, a *connected* reductive group over the field F. There is a quasi-split group G' over F and an isomorphism $\varphi : G' \to G$ defined over a finite Galois extension K of F such that $a_\sigma = \varphi^{-\sigma}\varphi$ is inner for all $\sigma \in \mathfrak{G}(K/F)$. $\{P\}$ will denote a conjugacy **[2]** class over F of parabolic F-subgroups of G. $\mathfrak{p}_F(G)$ will denote the finite set of such classes. Recall (Borel-Tits 4.13) that two parabolic F-subgroups are conjugate over F if they are conjugate over an extension field of F. There is an injection $\mathfrak{p}_F(G) \hookrightarrow \mathfrak{p}_F(G')$ determined by φ alone. To define it I shall apply a result in the aforementioned paper of Steinberg. He states it over perfect fields. As far as I can tell this assumption is used only to ensure that a maximal torus exists over F. This one now knows (Grothendieck).

If F is finite, the cocycle a_σ is trivial. Thus $a_\sigma = \psi^{-\sigma}\psi$ where ψ is inner. Set $\varphi_1 = \varphi\psi^{-1}$; it is defined over F. We send $\{P\}$ to $\{\varphi_1^{-1}P\}$. This map is independent of the choice of ψ and is bijective. In general let P be an F-parabolic subgroup and let S be a maximal split torus in $R(P)$. By 4.15a of Borel-Tits we may choose an order on the roots of S so that $P = G_\Phi^+$. Let $T \supseteq S$ be **[3]** a Cartan subgroup over F. Suppose we can find an inner automorphism ψ so that if $\varphi_1 = \varphi\psi^{-1}$ then $\varphi_1^{-1}(T) = T'$ is defined over F as is $\varphi_1|T'$. Then $\varphi_1^{-1}(S) = S'$ is a maximal split torus in T'. Since $\Phi(S', G') \simeq \Phi(S, G)$ we can define $P' = G'_{\Phi^+}$. It is clear that $\varphi_1(P') = P$. We map $\{P\} \to \{P'\}$.

We have only to verify that ψ exists for infinite F. Let $\widetilde{G}_0$ be the simply-connected covering group of the semi-simple part of G. Let $\widetilde{T}_0$ be the inverse image of T under the natural map $\widetilde{G}_0 \xrightarrow{\pi} G$. Choose $\widetilde{t}_0 \in \widetilde{T}_0(F)$ so that $\widetilde{T}_0$ is its centralizer. If $\widetilde{G}'_0$ is defined in a similar matter, we may lift φ to $\widetilde{\varphi} : \widetilde{G}'_0 \to \widetilde{G}_0$. $\widetilde{a}_\sigma = \widetilde{\varphi}^{-\sigma}\widetilde{\varphi}$ is inner. Let $\widetilde{\varphi}(\widetilde{t}'_0) = \widetilde{t}_0$. Then $a_\sigma(\widetilde{t}_0'^{\sigma}) = \widetilde{t}'_0$ so the class of $\widetilde{t}'_0$ is defined over F. By Steinberg's Theorem 1.7 there exists an inner automorphism $\widetilde{\psi}$ of $\widetilde{G}'_0$ so that $\widetilde{\psi}(\widetilde{t}'_0)$ is F-rational in G'. Let ψ be the automorphism of G' defined by $\widehat{\psi}$ and let t' be the **[4]** image of $\widetilde{\psi}(\widetilde{t}'_0)$ in G'. Let t be the image of $\widetilde{t}_0$ in G and let $\varphi_1 = \varphi\psi$ then $\varphi_1(t') = -t$. Since both t and t' are F-rational $\varphi_1^{-\sigma}\varphi_1$ lies in T', the centralizer of t'. φ_1 maps T' to T and its restriction to T' is defined over F.

There is an order defined on $\mathfrak{p}(G')$. $\{P'_1\} \succ \{P'\}$ if and only if some element of the first class contains an element of the second. I claim that if $\{P'\}$ is the image of $\{P\}$ and if $\{P'_1\} \succ \{P'\}$ then there is a class $\{P_1\}$ which maps to $\{P_1\}$. Choose $P \in \{P\}$ and then choose S as before.

Let $\{\alpha_1, \dots, \alpha_r\}$ be the simple roots of S. Suppose $P_1 \supseteq P$ is a parabolic subgroup. It must be defined by $\theta \subseteq \{\alpha_1, \dots, \alpha_r\}$ as follows. $S_1 = \left\{ s \in S \;\middle|\; \alpha(s) = 1 \ \forall \alpha \in \theta \right\}$ and let S_1^0 be its connected component. Choose an order on the roots of S_1^0 so that the restriction of every $\alpha \in \Phi^+(S, G)$ is positive or zero. Then $P_1 = G_{\Phi^+(S_1^0, G)}$. Starting **[5]** from P' and P_1' we obtain in the same manner a set $\mathcal{O}'$. If P corresponds to P' in the manner described above then $\mathcal{O}'$ determines a set $\mathcal{O}$ and the class of the associated P_1 maps to $\{P_1'\}$.

Now consider the associated group $\widehat{G} = \widehat{G}_K$ of G, where K/F is finite and Galois. I say that $\widehat{P} \subseteq \widehat{G}$ is parabolic if $\widehat{P}_0 = \widehat{G}_0 \cap \widehat{P}$ is parabolic (we are dealing here with a connected reductive complex group $\widehat{G}_0$) *and* $\widehat{P} \to \mathfrak{G}(K/F)$ is surjective. This notion is invariant under inner automorphisms by elements of $\widehat{G}_0$ (or even elements of $\widehat{G}$). Thus when considering classes of parabolic subgroups of $\widehat{G}$ we need only consider those for which $\widehat{P}_0$ is standard. Then $\widehat{P} = \widehat{P}_0 \times \mathfrak{G}(K/F)$. $\widehat{G}_0$ is defined with respect to a fixed Cartan subgroup T_0 and a fixed set $\widehat{\Delta}$ of its simple roots. Suppose $\widehat{P}_0$ is defined by $\widehat{\theta} \subseteq \widehat{\Delta}$. Recall that $\widehat{\Delta} \leftrightarrow \Delta'$ (the set of simple roots of T', a Cartan subgroup of G' over F contained in a parabolic F-subgroup.) Let $\widehat{\theta} \leftrightarrow \theta'$. Then **[6]** $\widehat{P}_0 \times \mathfrak{G}(K/F)$ is a group if and only if θ' or θ is invariant under $\mathfrak{G}(K/F)$. But the simple roots of G' with respect to S', a maximal F-split torus of T', correspond to the orbits in Δ' under $\mathfrak{G}(K/F)$. Thus there is a bijection from the classes of parabolic subgroups of $\widehat{G}$ to the classes of parabolic F-subgroups of G'. This, together with the earlier considerations, allows us to regard $\mathfrak{p}_F(G)$, which I now write simply as $\mathfrak{p}(G)$, as a set of classes of parabolic subgroups of $\widehat{G}$.

Now take F to be $\mathbf{R}$ or $\mathbf{C}$ and consider maps $\varphi : W_{\mathbf{C}/F} \to \widehat{G} = \widehat{G}_{\mathbf{C}}$ such that $\varphi(w)$ is semi-simple for all $w \in W_{\mathbf{C}/F}$ and such that

$$\begin{array}{ccc} W_{\mathbf{C}/F} & \xrightarrow{\quad\varphi\quad} & \widehat{G} \\ & \searrow \quad \swarrow & \\ & \mathfrak{G}(\mathbf{C}/F) & \end{array}$$

is commutative. φ_1 and φ_2 are *equivalent* if they differ by an inner automorphism by an element of $\widehat{G}_0$. An equivalence class will be written $\{\varphi\}$. Let $\Phi(G)$ be the set of all classes $\{\varphi\}$ such **[7]** that $\varphi(W_{\mathbf{C}/F}) \subseteq \widehat{P}$ implies $\{\widehat{P}\} \in \mathfrak{p}(G)$. Let $\Pi(G)$ be the set of infinitesimal equivalence classes of irreducible admissible representations of $G(F)$. The purpose of this letter is to define a surjective map $\lambda : \Pi(G) \to \Phi(G)$ such that the inverse image of each class $\{\varphi\}$ is finite.

If π (which denotes either a representation or its class) maps to $\{\varphi\}$ and if ρ is a finite-dimensional complex representation of G then, by definition,

$$L(s, \pi, \rho) = L(s, \rho \circ \varphi).$$

Thus if π_1 and π_2 both map to $\{\varphi\}$

$$L(s, \pi_1, \rho) = L(s, \pi_2, \rho)$$

for all ρ. For this reason π_1 and π_2 are said to be L-indistinguishable.

The definition of λ will inevitably seem arbitrary at this stage. It can only be justified by the global theory. However it is so far as I can see the only one compatible with the global form of the Washington **[8]** problems.

In order to be able to consider only the case $F = \mathbf{R}$, I explain how the associated group behaves under restriction of scalars. This I might as well do in general. Take E to be a finite separable extension of the field F. Let $\{\omega_1, \dots, \omega_n\}$ be a basis of E/F and let $\{\eta_1, \dots, \eta_m\}$ be the different imbeddings of E in $\overline{F}$. We regard E as defining an algebra over F in the sense of algebraic geometry. Let

$$\psi : \bigoplus e(\eta) \to \sum \lambda_i \omega_i$$

where

$$e(\eta) = \sum \lambda_i \omega_i^\eta$$

be the standard isomorphism of E with the direct sum of n copies of the one-dimensional algebra defined by F. In the above sum η runs over $\{\eta_1, \dots, \eta_m\}$. (ω_i^η) will stand for the matrix with rows indexed by i and columns indexed by η. Since

$$\big(e(\eta)\big) = (\lambda_i)(\omega_i^\eta)$$

[**9**] we have

$$(\lambda_i) = \big(e(\eta)\big)(\omega_i^\eta)^{-1}.$$

Let $\sigma \in \mathfrak{G}(\overline{F}/F)$. Since

$$\psi^\sigma\big(e(\eta)^\sigma\big) = (\lambda_i^\sigma)$$

and

$$\big(e(\eta)^\sigma\big) = (\lambda_i^\sigma)(\omega_1^{\eta\sigma})$$

the map $\psi^{-\sigma}\psi$ is defined by

$$\big(e(\eta)\big) \to \big(e(\eta_1)\big)(\omega_i^{\eta_1})(\omega_i^{\eta\sigma})^{-1} = \big(e(\eta\sigma)\big)$$

which is just a permutation of the coordinates.

Now let H be a group over E and G the group over F obtained by the restriction of scalars. For each η let $H(\eta)$ be the group over E^η defined by transport of structure. Let H' be a split group, which we may suppose is defined over F, so that H and H' are isomorphic over a finite Galois extension of E. Fix an imbedding $E \to F$ and regard—take $\overline{E}$ to be $\overline{F}$ so that the given isomorphism $\varphi : H' \to H$ may be taken to be defined over a finite separable extension [**10**] of F in $\overline{F}$. Moreover $\{\eta_1, \dots, \eta_n\}$ may be identified with the coset space $\mathfrak{G}(\overline{F}/E)\backslash\mathfrak{G}(\overline{F}/F)$.

The map ψ introduced earlier also defines an isomorphism $\prod H(\eta) \to G$. Since $H'(\eta) = H'$ we may consider the compositum φ_1 of

$$\prod H' \xrightarrow{\prod \varphi^\eta} \prod H(\eta) \xrightarrow{\ \psi\ } G \quad .$$

To be more precise we have to take a set $\mathcal{F}$ of coset representatives for $\mathfrak{G}(\overline{F}/E)\backslash\mathfrak{G}(\overline{F}/F)$ and define φ_1 by

$$\prod H' \xrightarrow{\prod \varphi^\tau} \prod H(\eta) \xrightarrow{\ \psi\ } G \quad .$$

The associated group is defined by a homomorphism $\sigma \to d(\sigma)$ of $\mathfrak{G}(\overline{F}/F)$ into the group $\Gamma\big(\prod H'\big)$ of automorphisms of the group of rational characters of a Cartan subgroup of $\prod H'$. $d(\sigma)$ is the compositum of $\sigma \to \varphi_1^{-\sigma}\varphi_1$ and the standard map of the group of rational automorphisms of G into $\Gamma(G)$. Let $c(\sigma)$, $\sigma \in \mathfrak{G}(\overline{F}/E)$ be the corresponding map for E. [**11**]

$$\varphi_1^{-1}\varphi_1^{\sigma} = \left(\prod_{\tau} \varphi^{-\tau}\right)\psi^{-1}\psi^{\sigma}\Big(\prod \varphi^{\tau\sigma}\Big)$$

Let $\tau\sigma = \delta_\tau(\sigma)\tau'$ with $\delta_\tau(\sigma) \in \mathfrak{G}(\overline{F}/E)$ and $\tau' \in \mathcal{F}$. Then

$$\varphi^{\tau\sigma} = \varphi^{\tau'}(\varphi^{-1}\varphi^{\delta_\tau(\sigma)})^{\tau'}.$$

One sees easily that

$$(\psi^{-1}\psi^{\sigma})\Big(\prod \varphi^{\tau'}\Big) = \Big(\prod \varphi^{\tau}\Big)(\psi^{-1}\psi^{\sigma}).$$

Since H' is split over F, $(\varphi^{-1}\varphi^{\delta_\tau(\sigma)})^{\tau'}$ has the same image in $\Gamma(H')$ as $\varphi^{-1}\varphi^{\delta_\tau(\sigma)}$. Thus

$$d(\sigma) = \left(\prod_{\tau} c\big(\delta_\tau(\sigma)\big)\right)B(\sigma)$$

where $B(\sigma)$ simply permutes the coordinates, replacing the coordinate at η by the coordinate at $\eta\sigma$.

$\widehat{H}$ is a semi-direct product $\widehat{H}_0 \times \mathfrak{G}(K/E)$ where K is a sufficiently large Galois extension of E, which we may suppose Galois over F. The action $c(\sigma)$ of $\mathfrak{G}(K/E)$ on H_0 is determined by $\sigma \to c(\sigma)$. It follows from the preceding discussion and the definition of the associated group that $\widehat{G}$ is a semi-direct product $\widehat{G}_0 \times \mathfrak{G}(K/F)$ with $\widehat{G}_0 = \prod_{\tau\in\mathcal{F}} \widehat{H}_0$. The action of $\mathfrak{G}(K/F)$ is [**12**] given by $\prod \widehat{c}\big(\delta_\tau(\sigma)\big)B(\sigma)$.

There is of course an injection $W_{K/E} \hookrightarrow W_{K/F}$ of Weil groups. A variant of Shapiro's lemma is going to give us a one:one correspondence between classes of homomorphisms $W_{K/E} \to \widehat{H}$ and $W_{K/F} \to \widehat{G}$. Since $G(F) = H(E)$ all reasonable conjectures and assertions are invariant under restriction of scalars. This is in particular true of the considerations needed later in this letter so I will be able to take $F = \mathbf{R}$. I continue for a moment with the general situation. Let $W_{K/F}$ be a disjoint union $\bigcup_{v\in\mathfrak{U}} vW_{K/F}$. I suppose $\mathcal{F}$ is of the projection of $\mathfrak{U}$ in $\mathfrak{G}(K/F)$. Note that we are taking K so large that the previous discussion could be carried out with $\overline{F}$ replaced by K. Set $vw = d_v(w)v'$ with $d_v(w) \in W_{K/E}$, $v' \in \mathfrak{U}$. Then $d_v(w_1w_2) = d_v(w_1)d_{v'}(w_2)$. Let $\varphi : W_{K/E} \to \widehat{H}$ and write $\varphi(w) = a(w) \times \sigma(w)$. $\sigma(w)$ is the [**13**] image of w in $\mathfrak{G}(K/E)$. Then $a(w_1w_2) = a(w_1)\sigma(w_1)\big(a(w_2)\big)$. Define $\psi : W_{K/F} \to \widehat{G}$ by

$$\psi(w) = \prod_v a\big(d_v(w)\big) \times \sigma(w)$$

Then

$$\psi(w_1)\psi(w_2) = \left(\prod_v a\big(d_v(w_1)\big)\right)\left(\sigma(w_1)\left(\prod_v a\big(d_v(w_2)\big)\right)\right) \times \sigma(w_1w_2).$$

Since

$$\sigma(w_1)\left(\prod_v a\big(d_v(w_2)\big)\right) = \prod_v \sigma\big(d_v(w_1)\big)\Big(a\big(d_{v'}(w_2)\big)\Big)$$

we have

$$\psi(w_1)\psi(w_2) = \psi(w_1w_2).$$

Replacing φ by $h^{-1}\varphi h$, $h \in \widehat{H}_0$, means we replace $a(w)$ by $h^{-1}a(w)\sigma(h)$, if $\sigma = \sigma(w)$. Then $\psi(w)$ is replaced by

$$\prod_v h^{-1}a\big(d_v(w)\big)\sigma\big(d_v(w)\big)(h) \times \sigma(w)$$

which is

$$\Big(\prod h\Big)^{-1}\psi(w)\Big(\prod h\Big)$$

which is equivalent to the original ψ. If

$$\Big(\prod h_0\Big)^{-1}\psi(w)\Big(\prod h_v\Big) = \psi'(w)$$

[**14**] where ψ' is associated to φ' then

$$h_v^{-1}a\big(d_v(w)\big)\sigma\big(d_v(w)\big)(h_{v'}) = a'\big(d_v(w)\big)$$

for all w and v. Take $w = v^{-1}w_1v$ with $w_1 \in \mathfrak{G}(K/E)$ to see that

$$h_v^{-1}a(w_1)\sigma(w_1)h_v = a'(w_1)$$

Thus φ and φ' are also equivalent. To finish up, I have to show that every ψ is equivalent to an induced homomorphism. Let

$$\psi(w) = \left(\prod_v b_v(w)\right) \times \sigma(w).$$

Then

$$b_v(w_1w_2) = b_v(w_1)\sigma\big(d_v(w_1)\big)b_{v'}(w_2).$$

Fix av_0 and consider $(h_v) = \big(b_v(v^{-1}v_0)\big)$. Then

$$\left(\prod_v h_v^{-1}\right)\psi(w)\left(\prod_v h(v)\right) = \prod_v b_v^{-1}(v^{-1}v_0)b_v(w)\sigma\big(d_v(w)\big)\big(b_{v'}(v^{-1}v_0)\big) \times \sigma(w)$$

Define $\varphi(w) = a(w) \times \sigma(w)$, $w \in W_{K/E}$, by $a(w) = b_{v_0}(v_0^{-1}wv_0)$. I have only to check that

$$b_v^{-1}(v^{-1}v_0)b_v(w)\sigma\big(d_v(w)\big)b_{v'}(v'^{-1}v_0) = a\big(d_v(w)\big)$$

or that [**15**]

$$b_v(w)\sigma\big(d_v(w)\big)\Big(b_{v'}(v'^{-1}v_0)\Big) = b_v(v^{-1}v_0)b_{v_0}\big(v_0^{-1}d_v(w)v_0\big).$$

Both sides are equal to $b_v\big(v^{-1}d_v(w)v_0\big) = b_v(wv'^{-1}v_0)$.

While on this theme let me remark a formal property of L-functions. Suppose μ is a representation of $\widehat{H}$ on V. Define λ, a representation of $\widehat{G}$ on $\bigoplus_\tau V$ by

$$\lambda\left(\prod_\tau g_\tau \times \sigma\right)\Big(\bigoplus v_\tau\Big) = \bigoplus \mu(g_z)\mu\big(\delta_\tau(\sigma)\big)v_{\tau'}.$$

If we apply $\lambda\big(\prod_\tau \overline{g}_\tau \times \overline{\sigma}\big)$ to the left side we obtain, if $\tau'\sigma = \delta_{\tau'}(\sigma)\tau''$.

$$\bigoplus \mu(\overline{g}_\tau)\mu\big(\delta_\tau(\overline{\sigma})\big)\mu\big(g_{\tau'}(\sigma)\big)\mu\big(\delta_{\tau'}(\sigma)\big)v_{\tau''}.$$

Since

$$\left(\prod_\tau \overline{g}_\tau \times \overline{\sigma}\right)\left(\prod_\tau g_\tau \times \sigma\right) = \prod_\tau \overline{g}_\tau\delta_\tau(\overline{\sigma})(g_{\tau'}) \times \overline{\sigma}\sigma$$

and

$$\delta_\tau(\overline{\sigma})\delta_{\tau'}(\sigma) = \delta_\tau(\overline{\sigma}\sigma).$$

λ is in fact a representation. Moreover

$$\lambda \circ \psi = \mathrm{Ind}(W_{K/F}, W_{K/E}, \mu \circ \varphi)$$

because [**16**]

$$\begin{aligned}
\lambda \circ \psi(w)\Big(\bigoplus v_\tau\Big) &= \lambda\left(\prod_v a\big(d_v(w)\big) \times \sigma(w)\right)\Big(\bigoplus v_\tau\Big) \\
&= \bigoplus \mu\Big(a\big(d_v(w)\big)\Big)\mu\Big(\sigma\big(d_v(w)\big)\Big)v_{\tau'} \\
&= \bigoplus \mu\Big(a\big(d_v(w)\big)\Big) \times \sigma\big(d_v(w)\big)v_{\tau'} \\
&= \bigoplus \mu\Big(\big(d_v(w)\big)v_{\tau'}\Big)
\end{aligned}$$

This means that the L-functions do not change upon restriction of scalars, provided μ is replaced by λ.

I return now the question of defining $\Pi(G) \to \Phi(G)$. I may suppose $F = \mathbf{R}$. I shall eventually proceed by associating to each $\{\varphi\} \in \Phi(G)$ a non-empty, finite subset of $\Pi(G)$. Afterwards, I will show that these sets are mutually disjoint with union $\Pi(G)$.

Consider the Zariski-closure H of $\varphi(W_{\mathbf{C}/F})$ in $\widehat{G}$. It is algebraic and supersolvable. Moreover every element of H is semi-simple. To see this imbed $\widehat{G}$ in some $\mathrm{GL}(n, \mathbf{C})$. The elements of $\varphi(\mathbf{C}^\times)$ simultaneously diagonalized. Thus every element in the Zariski-closure H_0 of $\varphi(\mathbf{C}^\times)$ is semi-simple. But $H^2 \subseteq H_0$. By Theorem 5.16 of Springer-Steinberg's article in Lecture Notes #131 [**17**] H normalizes a torus in $\widehat{G}_0$ which we may take to be $\widehat{T}$ (the standard maximal torus). The action of $\varphi(\mathbf{C}^\times)$ on $\widehat{T}$ must be trivial so $\varphi(\mathbf{C}^\times) \subseteq \widehat{T}$. Let s be an element of $W_{\mathbf{C}/\mathbf{R}}$ not in $\mathbf{C}^\times$ such that $s^2 = -1$. Let ω be the image of $\varphi(s)$ in $N(\widehat{T})/\widehat{T}$, where $N(\widehat{T})$ is the normalizer of $\widehat{T}$ in $\widehat{G}$. ω is an involution.

$\widehat{G}_0 = \widehat{G}_0^1 \cdot \widehat{Z}_0^1$ where $\widehat{G}_0^1$ is semi-simple and $\widehat{Z}_0^1$ is a torus and $\widehat{G}_0^1 \cap \widehat{Z}_0^1$ is finite. $\widehat{L}^1$ is the group of rational characters of $\widehat{T}^1 = \widehat{T} \cap \widehat{G}_0^1$ and $\widehat{L}^0$ is the group of rational characters of $\widehat{Z}_0$. $\widehat{L}$, the group of rational characters of $\widehat{T}$ is contained in $\widehat{L}^1 \oplus \widehat{L}^0$ and $\widehat{L} \otimes \mathbf{R} = \widehat{L}^1 \otimes \mathbf{R} \oplus \widehat{L}^0 \otimes \mathbf{R}$. ω acts on $\widehat{L} \otimes \mathbf{R}$. Suppose it has a fixed point $\widehat{\lambda}$. After conjugation we may suppose that $\widehat{\lambda}$ lies in the closure of the positive Weyl chamber, ie. $\langle \alpha, \widehat{\lambda} \rangle \geqslant 0$ if $\alpha > 0$. Then $\left\{ \alpha \in \Phi(\widehat{T}, \widehat{G}_0) \;\middle|\; \langle \alpha, \widehat{\lambda} \rangle \geqslant 0 \right\}$ defines a parabolic subgroup fixed by $\varphi(W_{\mathbf{C}/\mathbf{R}})$.

We begin with the case that $\varphi(W_{\mathbf{C}/\mathbf{R}})$ is contained in no proper [**18**] parabolic subgroup of $\widehat{G}$. Then $\widehat{\lambda}$ must lie in $\widehat{L}^0 \otimes \mathbf{R}$ and ω must act on $\widehat{T}^1$ as $t \to t^{-1}$. Let G_1 be the semi-simple part of G. I claim that if ω acts on $\widehat{T}^1$ as $t \to t^{-1}$ then $G_1(\mathbf{R})$ has a compact Cartan subgroup. If σ is the non-trivial element in $\mathfrak{G}(\mathbf{C}/\mathbf{R})$ then G and hence G_1 is determined by an automorphism a_σ of a split group G' (or G_1'). a_σ determines an element ω_σ of $\Gamma(G')$, then the lattice of rational characters of a Cartan subgroup of G', ω_σ actually leaves the set of simple roots, with respect to a predetermined order, invariant and acts on the Dynkin diagram. As explained in the Washington notes ω_σ determines an automorphism of $\widehat{G}_0$ and by definition $(1 \times \sigma)(\widehat{g} \times 1)(1 \times \sigma^{-1}) = \omega_\sigma(\widehat{g}) \times 1$ in $\widehat{G}$. Since $\varphi(s)$ is of the form $\widehat{h} \times \sigma$, we conclude that the automorphism of $\widehat{G}^1$ fixing $\widehat{T}^1$ and taking each root to its negative differs from ω_σ by an inner automorphism. ω_σ also determines an automorphism of G' (the "straight extension" in Freudenthal-de Vries). It differs from a_σ by an inner automorphism. Thus **[19]** there is an automorphism of G'_1 which differs from a_σ, which we can assume takes a given Cartan subgroup T' to itself, by an inner automorphism and takes t in T'_1 to t^{-1}.

a_σ is determined by $\psi : G' \to G$, $a_\sigma = \psi^{-\sigma}\psi$. We may suppose that if $T_1 = \varphi(T'_1)$ then $T_1(\mathbf{R})$ contains a maximal torus (in the sense of Lie groups) of $G_1(\mathbf{R})$. We may also regard G as arising from G' by a twist with respect to a_σ. Let b_σ be the twist which gives the compact form. We may suppose b_σ leaves T' fixed and sends $t \to t^{-1}$. Then, if we restrict to G'_1, $a_\sigma b_\sigma^{-1}$ is inner. On the other hand we may actually assume (Freudenthal-de Vries, Theorem 51.9) that $a_\sigma b_\sigma^{-1}$ leaves a chamber invariant. Thus the action of $a_\sigma b_\sigma^{-1}$ on G'_1 must be given by an element of T'_1. In particular a_σ and b_σ define the same twisting of T'_1 and $T_1(\mathbf{R})$ is compact. As you can imagine the discrete series will soon come into play. Unfortunately, because of the appearance of one-half the sum of the positive roots, some preliminaries are required. It is at this point, trivial as it appears, there is something to be understood. (I don't yet understand however.)

[20] Let

$$\widehat{M} = \left\{ \widehat{\lambda} \in \widehat{L} \;\middle|\; \omega(\widehat{\lambda}) = -\widehat{\lambda} \right\}$$

and let $\widehat{N} = \widehat{L}/\widehat{M}$. We have the exact sequence

$$0 \longrightarrow \widehat{H} \longrightarrow \widehat{L} \longrightarrow \widehat{N} \longrightarrow 0$$

and its dual

$$0 \longrightarrow N \longrightarrow L \longrightarrow M \longrightarrow 0 \quad .$$

Choosing any splitting $M \to L$ of the second sequence, we obtain ω as a matrix

$$\begin{pmatrix} I & A \\ 0 & -I \end{pmatrix}$$

For this we regard the elements of L as column vectors and those of $\widehat{L}$ as row vectors. A maybe supposed of the form

$$\begin{pmatrix} \alpha_1 & 0 \cdots\cdots & 0 \cdots & 0 \\ 0 & \alpha_2 \cdots\cdots & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ 0 & & \alpha_* & 0 \cdots 0 \\ 0 & \cdots\cdots & \cdots & 0 \\ 0 & \cdots\cdots & \cdots & 0 \end{pmatrix}$$

Let the elements of $T'(\mathbf{C})$ be represented by $(u, \cdot) = (u_1, \ldots, u_p, v_1, \ldots, v_q)$. Then [text cut off] acts by

$$(u, v) \to (u, u^{+A}v^{-1}).$$

[21] Thus $T(\mathbf{R})$ is defined by

$$u = \overline{u}$$

$$v\overline{v} = u^{+A}$$

Thus each u_i is real and u_i is positive if $i \leqslant r$ and α_i is odd. Let C be defined by $C = \{ t \in T \mid \lambda(t) = 1 \text{ if } \omega(\lambda) = -\lambda \}$. Then C is central and $C(\mathbf{R})$ is defined by $v_i = 1$ if $i > r$, $u_i^{\alpha_i/2} = v_i$ if α_i even. $u_i^{\alpha_i} = v_i^2$ if α_i odd.

Let $G^0(\mathbf{R})$, $G_1^0(\mathbf{R})$, and $T^0(\mathbf{R})$ be the topological connected components of $G(\mathbf{R})$, $G_1(\mathbf{R})$, and $T(\mathbf{R})$ respectively. I point out that T is now $\psi(T')$. $T^0(\mathbf{R})$ corresponds to those (u,v) for which $u_i > 0$ for all i. $G^0(\mathbf{R}) = G_1^0(\mathbf{R})T^0(\mathbf{R})$ and $T(\mathbf{R}) = T^0(\mathbf{R})C(\mathbf{R})$. Set $\widetilde{G}(\mathbf{R}) = G^0(\mathbf{R})T(\mathbf{R}) = G^0(\mathbf{R})C(\mathbf{R})$. If $g \in C(\mathbf{R})$ it is congruent modulo $G_1^0(\mathbf{R})$ to an element in the normalizer of $T_1(\mathbf{R})$, a connected torus. Thus g determines a unique element of $\Omega_1\backslash\Omega$. Ω is the Weyl group of $\{G(\mathbf{C}), T(\mathbf{C})\}$ and Ω_1 is the Weyl group of $\{G_1^0(\mathbf{R}), T_1(\mathbf{R})\}$. We obtain a map $G_1^0\backslash G(\mathbf{R}) \to \Omega_1\backslash\Omega$. Suppose g maps to the trivial coset. Multiplying on the left **[22]** by an element of $G_1^0(\mathbf{R})$ we may suppose it centralizes $T_1(\mathbf{R})$ and hence $T^0(\mathbf{R})$. Since $T^0(\mathbf{R})$ is Zariski-dense in T, it lies in $T(\mathbf{R})$. Thus the inverse image of the trivial coset is $\widetilde{G}(\mathbf{R})$.

An irreducible admissible representation π^0 of $G^0(\mathbf{R})$ together with a quasi-character χ of $C(\mathbf{R})$ such that $\chi(t)I = \pi(t)$, $t \in C(\mathbf{R}) \cap G^0(\mathbf{R})$ determine a representation $\widetilde{\pi}$ of $\widetilde{G}(\mathbf{R})$. If π^0 or rather its restriction to $G_1^0(\mathbf{R})$ lies in the discrete series then the formula for the character of π on $T_1(\mathbf{R})$ shows that when h_1 and h_2 are in $G(\mathbf{R})$ the representations $g \to \widetilde{\pi}(h_1^{-1}gh_1)$ and $g \to \widetilde{\pi}(h_2^{-1}gh_2)$ are equivalent only if $h_1^{-1}h_2 \in \widetilde{G}(\mathbf{R})$. Thus

$$\pi = \operatorname{Ind}\Big(G(\mathbf{R}), \widetilde{G}(\mathbf{R}), \widetilde{\pi}\Big)$$

is irreducible.

What we must do now is return to $\varphi : W_{\mathbf{C}/\mathbf{R}} \to \widehat{G}$ and to show how it determines a representation π^0 in the discrete series and a compatible quasi-character χ. Recall that we have arranged matters in such a way that $\varphi(\mathbf{C}^\times) \subseteq \widehat{T}$. If $\widehat{\lambda} \in \widehat{L}$ let

$$\widehat{\lambda}\big(\varphi(z)\big) = z^{\langle\mu,\widehat{\lambda}\rangle}\overline{z}^{\langle\mu,\widehat{\lambda}\rangle}$$

[23] with μ, ν in $L \otimes \mathbf{C}$ and $\mu - \nu$ in L. Then

$$\overline{z}^{\langle\mu,\widehat{\lambda}\rangle} z^{\langle\nu,\widehat{\lambda}\rangle} = \widehat{\lambda}\big(\varphi(\overline{z})\big) = \omega^{-1}\widehat{\lambda}\big(\varphi(z)\big) = z^{\langle\omega\mu,\widehat{\lambda}\rangle}\overline{z}^{\langle\omega\nu,\widehat{\lambda}\rangle}$$

so $\nu = \omega\mu$ and $\mu = \omega\nu$. Also

$$\widehat{\lambda}\big(\varphi(s)^2\big) = \widehat{\lambda}\big(\varphi(-1)\big) = (-1)^{\langle\mu-\nu,\widehat{\lambda}\rangle}.$$

Replacing φ by $\operatorname{Ad}\widehat{h} \circ \varphi$ if necessary we may suppose that $\operatorname{Re}\mu$ lies in the closure of a positive Weyl chamber (with respect to some order).

Let δ be one-half the sum of the positive roots with respect to this order. I shall show in a moment that

$$\widehat{\lambda}\big(\varphi(s)^2\big) = (-1)^{\langle 2\delta,\widehat{\lambda}\rangle} = (-1)^{\langle\delta-\omega\delta,\widehat{\lambda}\rangle} \qquad \widehat{\lambda} \in \widehat{M}. \tag{1}$$

Thus $\Big\langle \mu - \delta - \omega(\mu-\delta), \widehat{\lambda} \Big\rangle$ is even for all $\widehat{\lambda} \in \widehat{M}$ and

$$\frac{(\mu-\delta) - \omega(\mu-\delta)}{2} \in L \oplus N \otimes \mathbf{C}$$

and

$$\mu - \delta = \frac{(\mu-\delta)-\omega(\mu-\delta)}{2} + \frac{(\mu-\delta)+\omega(\mu-\delta)}{2} \in L + (N \otimes \mathbf{C}).$$

It follows readily that

$$\exp\Big(H + \omega(\overline{H})\Big) \to e^{(\mu-\delta)\big(H+\omega(\overline{H})\big)}$$

[24] is a well-defined quasi-character ϵ of $T^0(\mathbf{R})$. Indeed if $\exp\Big(H + \omega(\overline{H})\Big) = 1$ then $\lambda\Big(H + \omega(\overline{H})\Big) \in 2\pi i\mathbf{Z}$ for all $\lambda \in L$. In particular $H + \omega(\overline{H})$ is purely imaginary; so $\omega\Big(H + \omega(\overline{H})\Big) = -\omega\Big(\overline{H} + \omega(H)\Big) = -\Big(H + \omega(\overline{H})\Big)$. Thus if $\mu - \delta = \mu_1 + \mu_2$ with $\mu_1 \in L$, $\mu_2 \in N \otimes \mathbf{C}$ then

$$(\mu - \delta)\Big(H + \omega(\overline{H})\Big) = \mu_1\Big(H + \omega(\overline{H})\Big) \in 2\pi i\mathbf{Z}.$$

I observe next that $\langle \mu, \widehat{\alpha}\rangle \neq 0$ for all roots $\widehat{\alpha}$. Suppose the contrary. Since the projection of μ on $L_1 \otimes \mathbf{R}$, where L_1 is the lattice of rational characters of T_1, is real and $\operatorname{Re}\mu$ lies in the closure of a positive Weyl chamber we may suppose $\widehat{\alpha}$ is simple. Let $X_{\widehat{\alpha}}$ be the corresponding root vector. Let $U_{\widehat{\alpha}} = X_{\widehat{\alpha}} + \varphi(s)(X_{\widehat{\alpha}})$ and let $\mathcal{W}_{\widehat{\alpha}}$ be the set of U in the Lie algebra of $\widehat{T}$ for which $\widehat{\alpha}(U) = 0$. Then $\mathcal{W}_{\widehat{\alpha}} + \mathbf{C}U_{\widehat{\alpha}}$ is the Lie algebra of a Cartan subgroup containing $\varphi(\mathbf{C}^\times)$. $\mathcal{W}_{\widehat{\alpha}}$ is invariant under $\varphi(s)$ and $\varphi(s)(U_{\widehat{\alpha}}) = \varphi(s)X_{\widehat{\alpha}} + \widehat{\alpha}\big(\varphi(s)^2\big)X_{\widehat{\alpha}} = U_{\widehat{\alpha}}$. It is clear that $\varphi(s)$ takes this Cartan subgroup to itself but does not take every positive root to its negative. We know that this is **[25]** incompatible with our assumption that $\varphi(W_{\mathbf{C}/\mathbf{R}})$ is contained in no proper parabolic subgroup.

If Z_1 is the centre of G then $G^0(\mathbf{R}) = G_1^0(\mathbf{R})Z_1^0(\mathbf{R})$. Since $\mu - \delta \in L$ the projection μ_1 of μ on L_1 defines not merely one but several discrete series $\pi_{\pi\mu_1}$, $\tau \in \Omega$, the character of $\pi_{\tau\mu_1}$ on $T_1(\mathbf{R})$ is

$$\chi_{\tau\mu_1}(\exp H) = (-1)^m\epsilon(\tau\mu_1)\sum_{\sigma\in\Omega_1}\frac{\operatorname{sgn}\sigma\exp\sigma\tau\mu_1(H)}{\prod_{\alpha>0}\Big(\exp\frac{\alpha(H)}{2} - \exp\Big(-\frac{\alpha(H)}{2}\Big)\Big)}.$$

which equals

$$(-1)^m\epsilon(\tau\mu_1)\sum_{\sigma\in\Omega_1}\frac{\operatorname{sgn}\sigma\exp\sigma\tau(\mu_1-\delta)(H)\exp(\sigma\tau\delta-\delta)(H)}{\prod_{\alpha>0}\Big(1-\exp\big(-\alpha(H)\big)\Big)}$$

$\pi_{\tau\mu_1}$ extends to a representation π^0 of $G^0(\mathbf{R})$ with character

$$(-1)^m\epsilon(\tau\mu_1)\sum_{\sigma\in\Omega_1}\frac{\operatorname{sgn}\sigma\exp\sigma\tau(\mu-\delta)(H)\exp(\sigma\tau\delta-\delta)(H)}{\prod_{\alpha>0}\Big(1-\exp\big(-a(H)\big)\Big)}$$

for H in the Lie algebra of $T^0(\mathbf{R})$, ie. $H = H_1 + \omega(\overline{H}_1)$ with H_1 in the Lie algebra of $T'(\mathbf{C})$.

[26] If e is the number of cosets in $\Omega_1\backslash\Omega$ lying in the image of $G(\mathbf{R})$ we now just about know how to associate to each class $\{\varphi\}$ such that $\varphi(W_{\mathbf{C}/\mathbf{R}})$ is contained in no proper parabolic subgroup of $\widehat{G}$ several, to be exact $[\Omega : \Omega_1]/e$ irreducible admissible representations of $G(\mathbf{R})$, the restriction of each of which to $G_1^0(\mathbf{R})$, is the direct sum of e distinct discrete series representations.

To completely verify this we have to prove (1), to define the quasi-character χ of $C(\mathbf{R})$, and to prove that the choices made during the construction have no effect on the result.

To construct χ is easy. If $M' = \{\, \lambda \in M \mid \omega\lambda = -\lambda \,\}$ and $N' = L/M'$ then N' is the group of rational characters of C. We have an exact sequence

$$0 \longrightarrow \widehat{N'} \longrightarrow \widehat{L} \longrightarrow \widehat{M'} \longrightarrow 0$$

which yields

$$\widehat{T} \to \widehat{C}_0$$

if $\widehat{\widehat{C}}_0$ is the connected component of $\widehat{C}_0$, the associated group of C. Since

$$\widehat{N} = \left\{ \widehat{\lambda} \in \widehat{L} \;\middle|\; \omega\widehat{\lambda} = \widehat{\lambda} \right\}$$

and $\widehat{\lambda}\big(\varphi(s)^2\big) = (-1)^{\langle \mu - \omega\mu, \widehat{\lambda}\rangle}$, the image of **[27]** $\varphi(s)^2$ in $\widehat{C}_0$ is 1. Thus φ determines $\psi : W_{\mathbf{C}/F} \to \widehat{C}$. The construction of "Representations of abelian algebraic groups" determines χ from ψ. I recapitulate the construction for the ground field $\mathbf{R}$, which is much simpler than the general local field and is, besides, the field under discussion. Take an element $x : w \to x(w)$ in $H_1(W_{K/F}, \widehat{N}')$. Note in particular that $x(w) = 0$ for all but a finite number of w. $H_1(W_{K/F}, \widehat{N}')$ is isomorphic to $C(\mathbf{R})$. The isomorphism, constructed in the notes just mentioned, is such that if x corresponds to t then

$$\lambda(t) = \left\{ \prod_{w = a\times 1} a^{\langle \lambda, x(w)\rangle} \overline{a}^{\langle \lambda, \omega x(w)\rangle} \right\} \left\{ \prod_{w = a\times s} a^{\langle \lambda, x(w)\rangle} (-\overline{a})^{\langle \lambda, \omega x(w)\rangle} \right\} \qquad \lambda \in N'.$$

The set $\Phi(C)$ is $H^1_{\mathrm{cont}}(W_{K/F}, \widehat{C}_0)$. The natural pairing

$$H^1(W_{K/F}, \widehat{C}_0) \times H_1(W_{K/F}, \widehat{N}') \to \mathbf{C}^\times$$

associates to each $\{\psi\}$ in $\Phi(C)$ a quasi-character χ of $C(\mathbf{R})$. Take in particular $x(w) = 0$ unless w is of the form $a \times 1$. Then the pairing gives

$$\prod_{w = (a\times 1)} x(w)\big(\psi(w)\big)$$

which, since $x(w)\big(a(w)\big)$ equals, in the notation used above, $a^{\langle \mu, x(w)\rangle} \overline{a}^{\langle \omega\mu, x(w)\rangle}$ **[28]** is equal to

$$\left(\prod_{w = a\times 1} a \right)^{\langle \mu, x(w)\rangle} \left(\prod_{w = a\times 1} \overline{a}^{\langle \mu, \omega x(w)\rangle} \right)$$

which, if we define $H(w)$ by

$$a^{\langle \lambda, x(w)\rangle} = e^{\lambda'(H(w))}$$

if $\lambda' \in L$ maps to λ in N', is equal to, because $\delta \in M' \otimes \mathbf{R}$.

$$e^{\mu\left(H + \omega(\overline{H})\right)} = e^{(\mu - \delta)\left(H + \omega(\overline{H})\right)}$$

for

$$H = \sum_{w = a\times 1} H(w).$$

This shows that $\pi(t) = \chi(t)I$ for t in the connected component $C^0(\mathbf{R})$ of $C(\mathbf{R})$. But the centralizer of $T_1(\mathbf{R})$ in $G_1^0(\mathbf{R})$ is connected and is therefore equal to $T_1(\mathbf{R})Z_1^0(\mathbf{R}) = T^0(\mathbf{R})$. Thus $C(\mathbf{R}) \cap G^0(\mathbf{R}) = C(\mathbf{R}) \cap T^0(\mathbf{R}) = C^0(\mathbf{R})$.

Now I come to (1). It is a delicate point and it shows that we have been lucky in our definition of the associated group. Let r be any element in the normalizer of $\widehat{T}$ projecting to the non-trivial element σ of $\mathfrak{G}(\mathbf{C}/\mathbf{R})$ and such that $rtr^{-1} = \varphi(s)t\varphi(s)^{-1} = \omega(t)$, $t \in \widehat{T}$. If $t \in \widehat{T}$

$$(tr)^2 = t\omega(t)r^2$$

[**29**] and

$$\widehat{\lambda}\big(t\omega(t)\big) = \widehat{\lambda}(t)\omega\widehat{\lambda}(t) = 1 \qquad \widehat{\lambda} \in \widehat{M}.$$

Since $\varphi(s)$ is of the form tr we see that $\widehat{\lambda}\big(\varphi(s)\big)^2$ depends only on ω. We could take $r = n \times \sigma$ where n lies in the normalizer of $\widehat{T}$ in $\widehat{G}_0^1$ and it takes positive roots to negative roots. Then $r^2 = n\sigma(n) \times \sigma^2 = n\sigma(n) \times 1$. In other words for the proof we may as well work in $\widehat{G}_0^1 \times \mathfrak{G}(\mathbf{C}/\mathbf{R})$ which is itself an associated group. Moreover we may as well suppose that $\widehat{G}_0^1$ is simply-connected and, since it will be a product, simple.

Let $\widehat{\beta}$ be the top root as defined in Freudenthal-de Vries. Then $\sigma(e_{\widehat{\beta}}) = \eta e_{\widehat{\beta}}$ with $\eta = \pm 1$. $e_{\widehat{\beta}}$ is a root vector. σ is the non-trivial element of $\mathfrak{G}(\mathbf{C}/\mathbf{R})$. It acts on $\widehat{G}_0^1$ as described in the Washington lectures, ie. by taking the straight extension (in the sense of Freudenthal-de Vries) of the automorphism of the Dynkin diagram defined by σ. For groups admitting no outer automorphisms σ of course acts trivially and $\eta = 1$. In general I claim that if $\widehat{\beta} = \sum n_i\widehat{\alpha}_i$ is the expression of $\widehat{\beta}$ as a sum of simple roots and ℓ is one-half the sum of those [**30**] n_i for which $\alpha_i \neq \sigma\alpha_i$, and $(\alpha_i, \sigma\alpha_i) \neq 0$ then $\eta = (-1)^\ell$. This statement is not true for $\widehat{\beta}$ alone but for any positive root $\widehat{\gamma} = \sum m_i\widehat{\alpha}_i$ invariant under σ. η of course must be replaced by $\eta(\widehat{\gamma})$ where $\sigma(e_{\widehat{\gamma}}) = \eta(\widehat{\gamma})e_{\widehat{\gamma}}$. I prove it by induction on $\sum m_i$. Choose $\widehat{\alpha}_j$ so that $(\widehat{\gamma}, \widehat{\alpha}_j) > 0$. If $\widehat{\alpha}_k = \sigma(\widehat{\alpha}_j)$ then $(\widehat{\gamma}, \widehat{\alpha}_k) = (\gamma, \widehat{\alpha}_j)$. If $\widehat{\alpha}_k = \widehat{\alpha}_j$ then $\widehat{\gamma}' = \widehat{\gamma} - \widehat{\alpha}_j$ is also a root and

$$e_{\widehat{\gamma}} = [e_{\widehat{\alpha}_j}, \alpha_{\widehat{\gamma}'}]$$

so $\eta(\widehat{\gamma}) = \eta(\widehat{\gamma}')$. (N.B. by construction $\eta(\widehat{\alpha}_j) = 1$). If $\widehat{\alpha}_k \neq \widehat{\alpha}_j$ and $(\widehat{\alpha}_j, \widehat{\alpha}_k) = 0$ then $[e_{\widehat{\alpha}_j}, e_{\widehat{\alpha}_k}] = 0$. Take $\widehat{\gamma}' = \widehat{\gamma} - \widehat{\alpha}_j - \widehat{\alpha}_k$. It is a root and

$$e_{\widehat{\gamma}} = \big[e_{\widehat{\alpha}_j}[e_{\widehat{\alpha}_k}, e_{\widehat{\gamma}'}]\big] = \big[e_{\widehat{\alpha}_k}[e_{\widehat{\alpha}_j}, e_{\widehat{\gamma}'}]\big]$$

so $\eta(\widehat{\gamma}) = \eta(\widehat{\gamma}')$. If $(\widehat{\alpha}_j, \widehat{\alpha}_k) \neq 0$ then $\widehat{\alpha} = \widehat{\alpha}_j + \widehat{\alpha}_k$ is a root and $e_{\widehat{\alpha}} = [e_{\widehat{\alpha}_j}, \widehat{e}_{\alpha_k}]$. Thus $\eta(e_{\widehat{\alpha}}) = -1$. Take $\widehat{\gamma}' = \widehat{\gamma} - \widehat{\alpha}$; then $e_{\widehat{\gamma}} = [e_{\widehat{\alpha}}, e_{\widehat{\gamma}'}]$ so $\eta(\widehat{\gamma}) = -\eta(\widehat{\gamma}')$.

Since $(\widehat{\beta}, \widehat{\alpha}) \geqslant 0$ for all positive roots, every route perpendicular to $\widehat{\beta}$ is a linear combination of simple roots perpendicular to it. Let $\widehat{H}_0$ be the group [**31**] corresponding to the Lie algebra generated by $\left\{ e_{\widehat{\alpha}} \;\middle|\; (\widehat{\alpha}, \widehat{\beta}) = 0 \right\}$ and let $\widehat{J}_0$ be the group corresponding to the algebra spanned by $e_{\widehat{\beta}}$, $e_{-\widehat{\beta}}$ and $[e_{\widehat{\beta}}, e_{-\widehat{\beta}}]$. Both groups are invariant under $\mathfrak{G}(\mathbf{C}/\mathbf{R})$ and they commute with each other. Let n_1 be an element of $\widehat{H}_0$ normalizing $\widehat{T}$ which takes positive roots of $\widehat{H}_0$ to negative roots and let n_2 be an element of $\widehat{J}_0$ normalizing $\widehat{T}$ and taking $\widehat{\beta}$ to $-\widehat{\beta}$. If $(\widehat{\alpha}, \widehat{\beta}) \neq 0$ and $\widehat{\alpha} \neq \widehat{\beta}$ then $n_2(\widehat{\alpha}) < 0$. If not $n_2(\widehat{\alpha}) = \widehat{\alpha} - \frac{2(\widehat{\alpha},\widehat{\beta})}{(\widehat{\beta},\widehat{\beta})}\widehat{\beta}$ would be positive and $\widehat{\beta} - \widehat{\alpha} \leqslant \frac{2(\widehat{\alpha},\widehat{\beta})}{(\widehat{\beta},\widehat{\beta})}\widehat{\beta} - \widehat{\alpha}$ would be negative. This is impossible since $\widehat{\beta}$ is a top root. Take $n = n_1n_2 = n_2n$. Then n normalizes $\widehat{T}$ and takes every positive root to a negative root. Note that if $(\widehat{\alpha}, \widehat{\beta}) > 0$ then $(n_2\widehat{\alpha}, \widehat{\beta}) > 0$. Thus $\operatorname{Ad} n_1 \circ \sigma$ takes every positive root of $\widehat{H}_0$ to its own negative and $\widehat{H} = \widehat{H}_0 \times \mathfrak{G}(\mathbf{C}/\mathbf{R}) \subseteq \widehat{G}^1$ is a group to which an induction assumption may be applied.

$$\widehat{\lambda}\big(n\sigma(n)\big) = \widehat{\lambda}\big(n, \sigma(n_1)\big)\widehat{\lambda}\big(n_2\sigma(n_2)\big)$$

and, by induction

$$\widehat{\lambda}\big(n,\sigma(n_1)\big)=(-1)^{\sum_{\substack{\widehat{\alpha}>0\\(\widehat{\alpha},\widehat{\beta})=0}}\langle\alpha,\widehat{\lambda}\rangle}.$$

[32] Note that $\langle\alpha,\widehat{\lambda}\rangle=\widehat{\lambda}(H_{\widehat{\alpha}})$ so the left side has the same meaning for $\widehat{H}$ as for $\widehat{G}^1$. Now $\widehat{J}_0$ is covered by $\mathrm{SL}(2,\mathbf{C})$ and we may suppose $n_1\leftarrow\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$. Then

$$n_1\sigma(n_1)\leftrightarrow\eta\begin{pmatrix}-1&0\\0&-1\end{pmatrix}=\begin{pmatrix}(-1)^{\ell+1}&\\&(-1)^{\ell+1}\end{pmatrix}$$

if ℓ has its previous meaning. Thus

$$\widehat{\lambda}\big(n_2\sigma(n_2)\big)=(-1)^{(\ell+1)\langle\beta,\widehat{\lambda}\rangle}.$$

It remains to show that

$$\ell\langle\beta,\widehat{\lambda}\rangle\equiv\sum_{\substack{\widehat{\alpha}>0\\\widehat{\alpha}\neq\widehat{\beta}\\(\widehat{\alpha},\widehat{\beta})\neq0}}(\alpha,\widehat{\lambda})\pmod 2.$$

Let ν be the permutation of the set of positive roots $\widehat{\alpha}$ given by $\operatorname{Ad}n_1\circ\sigma$. ν may also be regarded as a permutation of the α and

$$\nu(\alpha)=\text{“}-\operatorname{Ad}n_2(\alpha)\text{”}=\frac{2(\alpha,\beta)}{(\beta,\beta)}\beta-\alpha.$$

If $\widehat{\alpha}$ is in the set over which the last sum is taken then $\alpha\neq\nu(\alpha)$ and $\alpha+\nu(\alpha)=\frac{2(\alpha,\beta)}{(\beta,\beta)}\beta$. Set $\ell'=\sum_{\substack{\widehat{\alpha}>0\\\widehat{\alpha}\neq\widehat{\beta}\\(\widehat{\alpha},\widehat{\beta})\neq0}}\frac{(\alpha,\beta)}{(\beta,\beta)}$. We have to show $\ell\equiv\ell'\pmod 2$. Clearly **[33]**

$$\ell'+1=\sum_{\alpha>0}\frac{(\alpha,\beta)}{(\beta,\beta)}=\langle\delta,\widehat{\beta}\rangle=\sum m_i\qquad\left(\delta=\frac{1}{2}\sum_{\alpha>0}\alpha\right)$$

if $\widehat{\beta}=\sum m_i\widehat{\alpha}_i$ is the expression of $\widehat{\beta}$ as a linear combination of simple roots.

At this point I must, unfortunately, but I see no way out at the moment, use classification. $\sum m_i$ is the altitude of $\widehat{\beta}$. Freudenthal and de Vries give a list of the altitudes of the top roots (p. 534). This altitude is odd except in the case of A_m, m even. Thus ℓ' is even except in this case when it is odd. On the other hand the structure of the Dynkin diagrams of all other groups is such that no automorphism of them even take a root into another root distinct from and yet not orthogonal to itself. Thus for these ℓ is also even. For A_m, m even the automorphism of the Dynkin diagram coming into question is the flip. It interchanges one pair of *nm*-perpendicular roots. Since these enter into the top root with coefficient 1, ℓ is odd in this case and we are done.

Since $\langle\mu,\widehat{\alpha}\rangle\neq0$ for all roots $\widehat{\alpha}$, the Cartan subgroup containing $\varphi(\mathbf{C}^\times)$ is unique and μ is determined up to the action of the Weyl group. This shows that the set of representations associated to the class $\{\varphi\}$ is well-defined.

[34] Suppose we have an irreducible admissible representation π of $G(\mathbf{R})$. It is more a less clear, on the basis of standard theorems, that $\pi|G_1^0(\mathbf{R})$ is the direct sum of finitely many irreducible admissible representations, which are permuted amongst themselves by the action of $G(\mathbf{R})$ on $G_1^0(\mathbf{R})$. If one belongs to the discrete series, they all do.

I claim that if this is the case then the class of π' corresponds to a $\{\varphi\}$ such that $\varphi(W_{\mathbf{C}/\mathbf{R}})$ is contained in no proper parabolic subgroup. The restriction of π to $\widetilde{G}(\mathbf{R})$ is also the sum of finitely many irreducible representations. Let one of them be $\widetilde{\pi}$. On $T^0(\mathbf{R})C(\mathbf{R})$ the character of $\widetilde{\pi}$ has the form

$$\pm \sum_{\tau\in\Omega_1} \frac{\operatorname{sgn}\tau\epsilon\big(\tau(t)\big)-\delta_\tau(t)}{\prod_{\alpha>0}\big(1-\alpha^{-1}(t)\big)}$$

with $\delta_\tau = \tau\delta-\delta$. We have to construct φ from ϵ.

Write

$$\epsilon(\exp H) = \exp\nu(H)$$

[**35**] or better

$$\epsilon\Big(\exp H+\omega(\overline{H})\Big) = \exp\nu\Big(H+\omega(\overline{H})\Big).$$

If $\lambda\Big(H+\omega(\overline{H})\Big) \in 2\pi i\mathbf{Z}$ for all $\lambda \in L$ then $\nu\Big(H+\omega(\overline{H})\Big) \in 2\pi i\mathbf{Z}$. Consider such an $H_1 = H+\omega(\overline{H})$. $\omega(H_1) = \overline{H}_1$ because $H_1 = H+\omega(\overline{H})$ and $\overline{H}_1 = -H_1$ because $\lambda(H_1)$ is purely imaginary for all λ. Thus H_1 is purely imaginary, and lies in $2\pi i(\widehat{M}\otimes\mathbf{R})$. Otherwise it is arbitrary. We conclude that $\nu \in L+N\oplus\mathbf{C}$. Set $\mu = \nu+\delta$. Then

$$\mu-\omega\mu = \nu-\omega\nu+2\delta \in L.$$

We define φ in $\mathbf{C}^\times$ by

$$\widehat{\lambda}\big(\varphi(z)\big) = z^{\langle\mu,\widehat{\lambda}\rangle}\overline{z}^{\langle\omega\mu,\widehat{\lambda}\rangle}$$

In particular

$$\widehat{\lambda}\big(\varphi(-1)\big) = (-1)^{\langle\mu-\omega\mu,\widehat{\lambda}\rangle} = (-1)^{\langle 2\delta,\widehat{\lambda}\rangle}(-1)^{\langle\nu-\omega\nu,\widehat{\lambda}\rangle}.$$

If $\widehat{\lambda}\in\widehat{M}$ then $(-1)^{\langle\nu-\omega\nu,\widehat{\lambda}\rangle} = (-1)^{\langle 2\nu,\widehat{\lambda}\rangle} = 1$.

To define φ completely we have to define $\varphi(s)$. It will be of the form $tn\times\sigma$. Then [**36**]

$$\varphi(s)^2 = t\omega(t)n\sigma(n)\times 1.$$

There are two conditions to be satisfied.

$$(-1)^{\langle\mu-\omega\mu,\widehat{\lambda}\rangle} = \widehat{\lambda}\big(\varphi(s)^2\big) = \widehat{\lambda}\big(t\omega(t)\big)\widehat{\lambda}\big(n\sigma(n)\big). \tag{2}$$

Moreover the image of t in $\widehat{C}_0$ is specified by the theory in "Representations of abelian algebraic groups." Choose t_0 so that the second condition is satisfied and set $t = t_0t_1$ where $\widehat{\lambda}(t_1) = 1$ of $\widehat{\lambda}\in\widehat{N}'$. The condition (2) is then automatically satisfied for $\widehat{\lambda}$ in $\widehat{N}'$.

If $t_2\in\widehat{T}$ and $\lambda(t_2) = 1$ for $\widehat{\lambda}\in\widehat{N}'$ then $\widehat{\lambda}\big(t_2\omega(t_2)\big) = (\omega\widehat{\lambda}+\widehat{\lambda})(t_2) = 1$ for all $\widehat{\lambda}$ and $\omega(t_2) = t_2^{-1}$. We can always choose t_2 so that $t_2^2 = t_1$. Then

$$t_2^{-1}(t_1t_0n\times\sigma)t_2 = t_0n\times\sigma.$$

Thus the conjugacy class of $t_1t_0n\times\sigma$ is independent of t_1. Since $\widehat{\lambda}\big(n\sigma(n)\big)$ depends only on the restriction of $\widehat{\lambda}$ to $\widehat{T}_1$ it equals $(-1)^{\langle 2\rho,\widehat{\lambda}\rangle}$. We need to show that

$$(-1)^{\langle\nu-\omega\nu,\widehat{\lambda}\rangle} = \widehat{\lambda}\big(t\omega(t)\big).$$

The right side is $(\widehat{\lambda}+\omega\widehat{\lambda})(t)$ and, since $\widehat{\lambda}+\omega\widehat{\lambda}\in\widehat{N}'$, it is by definition the value of ϵ at the element of $C(\mathbf{R})$ defined by [**37**]

$$\lambda(c) = (-1)^{\langle\lambda,\widehat{\lambda}+\omega\widehat{\lambda}\rangle}$$

Thus if $H = \pi \cdot \widehat{\lambda}$ ($\widehat{L} \otimes \mathbf{C}$ may be identified with the Lie algebra of $T(\mathbf{C})$)

$$e = \exp\Big(H - \omega(\overline{H})\Big) = \exp\Big(H + \omega(\overline{H})\Big)$$

and

$$\epsilon(c) = \exp \nu\Big(H + \omega(\overline{H})\Big) = \exp \pi i \langle \nu, \widehat{\lambda} - \omega\widehat{\lambda}\rangle = (-1)^{\langle \nu, \widehat{\lambda} - \omega\widehat{\lambda}\rangle}.$$

With this we are done. I will save the rest of the definition for another letter. Let me remark however two problems which are of some importance for the trace formula. A graduate student at Yale, Diana Shelstad, is supposed to be working on these problems. I haven't heard from her for some time so I don't know how far along she has come. Suppose G_1 and G_2 are two groups over $\mathbf{R}$ and $\widehat{G}_1 = \widehat{G}_2$. Suppose $\varphi : W_{\mathbf{C}/\mathbf{R}} \to \widehat{G}_1 = \widehat{G}_2$ and $\varphi(W_{\mathbf{C}/\mathbf{R}})$ is contained in no proper parabolic subgroup. Let $\Pi_\varphi(G_i)$ be the finite set of classes in $\Pi(G_i)$ corresponding to φ. Let χ_i^φ be the sum of the characters of the classes in $\Pi_\varphi(G_i)$. Let $\psi : G_1 \to G_2$ be defined over the finite Galois extension **[38]** K (ie in our context $\mathbf{C}$) and such that $\psi^{-\sigma}\psi$ is inner for all $\sigma \in \mathfrak{G}(K/F)$ (ie $\mathfrak{G}(\mathbf{C}/\mathbf{R})$). Suppose $\varphi : T_1 \to T_2$ where T_i is a Cartan subgroup of G_i over $F(= \mathbf{R})$ and φ restricted to T_1 is defined over F. Is there an equality

$$\chi_1^\varphi(t) = c\chi_2^\varphi\big(\psi(t)\big) \qquad t \in T_1(F)$$

where c is a constant depending, most optimistically, only on G_1 and G_2, but perhaps also on T_1, T_2, and φ. Observe that the question is non-trivial even when $G_1 = G_2$ and ψ is inner. Observe also that $\widehat{G}_1$ and $\widehat{G}_2$ may be identified only after ψ is chosen up to an inner automorphism.

I look forward to seeing you next year.

All the best,
Bob

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