June 19, 1974

Bill,

I'm going to use the tail end of our recent conversation as an excuse to indulge my proclivity for guessing and try to see what your suspicions and mine taken together suggest about the characters of the unitary unramified principal series.

Notation.

- G connected, reductive, quasi-split over the non-archimedean field F, split over an unramified extension.
- $B \subseteq G$: fixed Borel subgroup over F.
- $T^0 \subseteq B$: fixed CSG over F.
- $\varphi : W_{\overline{F}/F} \to G'$, the image of φ consists of s.s. elements and φ factors through $W_{\overline{F}/F} \to \mathbf{Z}$, also eigenvalues of $\operatorname{ad}\varphi(w)$, $w \in W$, and have absolute value 1.
- π_φ: corresponding principal series
- Π_φ: set of irreducible equivalence classes occurring in π_φ.

It is known that: (i) the set of conjugacy classes of special maximal compacts is a principal homogenous space under C, the quotient of

$$\widehat{L}(T^0_{\mathrm{ad}})^{\mathfrak{G}(\overline{F}/F)}$$

by the image of

$$\widehat{L}(T^0)^{\mathfrak{G}(\overline{F}/F)}$$

(ii) There is a map $\lambda : G_{\mathrm{ad}}(F) \to C$ who is kernel contains the image of $G(F)$ so that $\lambda(g)$ takes the conjugacy class of K to that of gKg^{-1}.

You believe. The action of $G_{\mathrm{ad}}(F)$ on Π_φ given by

$$x \longrightarrow \pi(x) \xrightarrow{\ g\ } x \longrightarrow \pi(g^{-1}xg)$$

[**2**] is transitive end factors through C.

Problem I. *Given φ, describe the kernel of the action of C on Π_φ.*

There is a first and tentative suggestion that can be made in this regard. Let $\widehat{T}$ and $B^{\widehat{0}}$ be the CSG and BSG of $G^{\widehat{0}}$ corresponding to $T^{\widehat{0}}$ and B. We may suppose that $\varphi(w)$, $w \in W_{\overline{F}/F}$, normalizes T and B. Let G^φ be the centralizer of φ in $\widehat{G}$. Let $\widehat{S}$ be the centre of the connected component of the inverse image of $G^\varphi \cap G^{\widehat{0}}$ in $G^{\widehat{0}}_{\mathrm{sc}}$. If $a > 1$ and if $t \in \widehat{T}_{\mathrm{der}} \subseteq G^{\widehat{0}}_{\mathrm{der}}$ satisfies $\lambda(t) > 0$ for all weights λ of $\widehat{T}_{\mathrm{der}}$ and $\alpha(\widehat{t}\,) = a$ for all positive simple roots α then $t \in G^\varphi$. Thus $\widehat{S} \subseteq \widehat{T}_{\mathrm{sc}}$ and we have a map

$$\widehat{L}(T^0_{\mathrm{ad}}) = L(\widehat{T}_{\mathrm{sc}}) \to L(\widehat{S}) = L(\widehat{S})^{\mathfrak{G}(\overline{F}/F)}.$$

Let M be the inverse image of $L(\widehat{S})^{G^\varphi}$ in $L(\widehat{T}_{\mathrm{sc}})^{\mathfrak{G}(\overline{F}/F)}$. The kernel *may be* the image of M in C.

So that you can decide whether or not to lend any credibility to this suggestion I give some examples. However I repeat that it is only a suggestion.

(i) The centralizer of φ in $\widehat{G} = G^{\widehat{0}} \times W_{\overline{F}/F}$ is contained in $T^{\widehat{0}} \times W_{\overline{F}/F}$. Then

$$M = L(\widehat{T}_{\mathrm{sc}})^{\mathfrak{G}(\overline{F}/F)} = \widehat{L}(T^0_{\mathrm{ad}})^{\mathfrak{G}(\overline{F}/F)}.$$

On the other hand, the Bruhat theory shows presumably that in this case π_φ is irreducible so that Π_φ consists of one element. [**3**]

(ii) $\varphi : w \to 1 \times w \in \widehat{G} = G^{\widehat{0}} \times W_{\overline{F}/F}$. This map factors through $\mathbf{Z}$ (if we regard it as mapping to $G^{\widehat{0}} \times \mathbf{Z}$). Again

$$M = \widehat{L}(T^0_{\mathrm{ad}})^{\mathfrak{G}(\overline{F}/F)}.$$

Thus, π_φ should be irreducible in this case also. Is this a known fact?

(iii) $G = \mathrm{SL}(2)$. $\widehat{C}$ is the direct product of $\mathrm{PGL}(2, \mathbf{C})$ with $W_{\overline{F}/F}$. The only case not covered by (i) and (ii) is

$$\varphi(\mathrm{Frob}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \times \mathrm{Frob}$$

where the matrix is taken modulo ± 1, Then

$$L(\widehat{S}) = L(\widehat{T}_{\mathrm{sc}}) = L(\widehat{T}_{\mathrm{sc}})^{\mathfrak{G}(\overline{F}/F)} \simeq \mathbf{Z}.$$

However

$$G^\varphi = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} 0 & \alpha \\ -\alpha^{-1} & 0 \end{pmatrix} \right\} \quad \mathrm{mod}\ \pm 1.$$

Thus $M = 0$ and the image of M in $C \simeq \mathbf{Z}/2\mathbf{Z}$ is 0. Thus π_φ should split into two parts as it does.

These are the only cases in which I know how π_φ decomposes. My suggestion was manufactured to account for them.

In the conversation we wanted to find pairs T, χ yielding $\widehat{H}$ (or $H^{\widehat{0}}$) so that T is (isomorphic to) a CSG in a BSG of the [**4**] corresponding quasi-split H. Now I work backwards. I start from $\widehat{H}$.

$H^{\widehat{0}} \supseteq \widehat{T}$ and is defined by giving those roots $\widehat{\alpha}$ which are roots of $H^{\widehat{0}}$. $\widehat{H}$ is defined by giving ω in the normalizer of $\widehat{T}$ which takes positive roots of $\widehat{T}$ in $H^{\widehat{0}}$ to positive roots and then letting $\omega \times \mathrm{Frob}$ be the action of the Frobenius on $H^{\widehat{0}}$. $\widehat{H}$ is the group generated by $H^{\widehat{0}}$ and $\omega \times \mathrm{Frob}$ in $\widehat{G} = G^{\widehat{0}} \times \mathbf{Z}$ (with $1 = \mathrm{Frob}$). The sequence

$$0 \longrightarrow H^{\widehat{0}} \longrightarrow \widehat{H} \longrightarrow \mathbf{Z} \longrightarrow 0$$

is exact. Taking ω so that

$$\omega X_{\widehat{\alpha}} = X_{\omega\widehat{\alpha}}$$

for $\widehat{\alpha}$ a simple root of $\widehat{H}$ we get a splitting of the above exact sequence which turns $\widehat{H}$ into an associate group.

Notation. This is splitting is not uniquely determined for we can multiply ω by any element of the centre of $H^{\widehat{0}}$. In other words if we regard $\widehat{H}$ has the associate group of H, then, although H is uniquely determined, the imbedding $\widehat{H} \hookrightarrow \widehat{G}$ is not.

Anyhow given $\widehat{H}$ there is an H and by Steinberg a CSG in a BSG of H is isomorphic to a CSG of T (all over F). The question is whether we can find a character χ of $H^{-1}\Big(\widehat{L}(T_{\mathrm{sc}})\Big)$ trivial on the kernel of

$$H^{-1}\Big(\widehat{L}(T_{\mathrm{sc}})\Big) \to H^{-1}\Big(\widehat{L}(T)\Big)$$

so that the pair (T,χ) yields $H^{\widehat{0}}$. $\widehat{L}(T_{\mathrm{sc}})$ is the lattice spanned by the roots $\widehat{\alpha}$. If $\widehat{M}$ is the lattice **[5]** spanned by the roots $\widehat{\alpha}$ of $H^{\widehat{0}}$, one of our conditions is that χ be trivial on the image of $H^{-1}(\widehat{M})$. The other is that if $\mathrm{Nm}\,\widehat{\alpha} = 0$ and $\chi(\widehat{\alpha}) = 1$ then $\widehat{\alpha}$ is a root of $H^{\widehat{0}}$.

Suppose for a moment that we have T, χ, and H and our disposal. I want to associate to χ a character of C. We may identify $\widehat{L}(T^0_{\mathrm{sc}})$ and $\widehat{L}(T_{\mathrm{sc}})$ as groups. If σ^0 denotes the action of Frobenius on the first and σ its action on the second then

$$\sigma = \omega\sigma^0.$$

(Here the element ω in the normalizer of $\widehat{T}$ is regarded as an element of the Weyl group.) Let

$$C_{\mathrm{sc}} = \widehat{L}(T^0_{\mathrm{ad}})^{\mathfrak{G}(\overline{F}/F)}/\widehat{L}(T^0_{\mathrm{sc}})^{\mathfrak{G}(\overline{F}/F)}.$$

There is a surjection

$$C_{\mathrm{sc}} \to C.$$

There is also an injection

$$C_{\mathrm{sc}} \hookrightarrow H^0\Big(\widehat{L}(T^0_{\mathrm{ad}})/\widehat{L}(T^0_{\mathrm{sc}})\Big) = H^0\Big(\widehat{L}(T_{\mathrm{ad}})/\widehat{L}(T_{\mathrm{sc}})\Big) \simeq H^{-2}\Big(\widehat{L}(T_{\mathrm{ad}})/\widehat{L}(T_{\mathrm{sc}})\Big).$$

These cohomology groups are taken with respect to the Galois group of a large unramified extension of F. The last isomorphism is periodicity. Because of

$$H^{-2}\Big(\widehat{L}(T_{\mathrm{ad}})/\widehat{L}(T_{\mathrm{sc}})\Big) \longrightarrow H^{-1}\Big(\widehat{L}(T_{\mathrm{sc}})\Big) \longrightarrow H^{-1}\Big(\widehat{L}(T_{\mathrm{ad}})\Big)$$

χ can be pulled back to a character of C_{sc}. If x in C_{sc} is represented by **[6]** $\overline{y} \in \widehat{L}(T^0_{\mathrm{ad}})$ which is the image of y in $\widehat{L}(T_{\mathrm{ad}})$, then x maps to $\sigma\overline{y} - \overline{y}$ in $H^{-1}\Big(\widehat{L}(T_{\mathrm{sc}})\Big)$. However we have

$$\begin{array}{ccc} \widehat{L}(T_{\mathrm{sc}}) & \hookrightarrow & \widehat{L}(T_{\mathrm{ad}}) \\ \downarrow & \nearrow & \\ \widehat{L}(T) & & \end{array}$$

Since

$$\sigma\overline{y} - \overline{y} = \sigma y - y$$

in $\widehat{L}(T)$, $\sigma\overline{y} - \overline{y}$ is in $H^{-1}\Big(\widehat{L}(T)\Big)$. Thus χ takes the value 1 on it, so the pullback of χ takes the value 1 on x. Thus the pullback induces a character η of C.

What do we want?

If

$$\begin{array}{ccc} \varphi & & \\ \downarrow & \searrow & \\ \widehat{H} & \longrightarrow & \widehat{G} \end{array}$$

is unramified and if we fix a special maximal compact and hence $\pi^0 \in \Pi_\varphi$ and Θ_π is the character of π then

$$\sum_C \eta(c)\Theta_{c\pi} = \Theta$$

should be non-zero and should have support on T and the Cartan subgroups stably conjugate to F.

It can be non-zero only if the stabilizer of π^0 in C is contained in the kernel of η. This is something to be checked.

[**7**] Moreover if $\gamma \in T(F)$ is regular and $\delta \in \mathcal{D}(T)$ then $\Theta(\gamma^\delta)$ should equal $\chi(\delta)^{-1}\Theta(\gamma)$ (except perhaps for a factor or depending on the orders of the Weyl groups over F of T and T^δ.)

Notice also that our demand means that T (or at least its stable conjugacy class) is determined by η. This is by no means obvious.

If Θ_{π_φ} is the character of the representation π_φ of H we should have

$(*)\quad \Theta(f) =$

$$\frac{1}{w}\int_{T(F)} \Theta_{\pi_\varphi}(\gamma)\epsilon(\gamma)\left\{\prod_{\substack{\widehat{\alpha}>0 \\ \widehat{\alpha} \text{ root of } G^{\widehat{0}} \\ \text{not a root of } H^{\widehat{0}}}} \left|1-\alpha^{-1}(\gamma)\right|\right\}\left|\rho_{G,H}(\gamma)\right|\Phi_f^\chi(\gamma) \prod_{\widehat{\alpha} \text{ root of } H^{\widehat{0}}} \left|1-\alpha^{-1}(\gamma)\right| d\gamma$$

w is the order of the Weyl group of $T(F)$ in $H(F)$. $\epsilon(\gamma)$ is a factor still to be determined. It depends in a simple way on the choice of the imbedding $\widehat{H} \hookrightarrow \widehat{G}$ (cf. (iii) on p. 39 of my notes on real groups).

Simple manipulations lead from x to the conclusion that as a function

$$\Theta(\gamma) = \frac{1}{w} \sum_{\substack{\mu\in\Omega\left(T'(F),G'(F)\right) \\ \delta\in\mathcal{D}(F,T) \\ \overline{\gamma}\in T(F) \\ \overline{\gamma}^{\delta\mu}=\gamma}} \frac{\Theta_{\pi_\varphi}(\overline{\gamma})\epsilon(\overline{\gamma})\chi(\delta)}{\left\{\prod_{\substack{\widehat{\alpha}>0 \\ \widehat{\alpha} \text{ root of } G^{\widehat{0}} \\ \text{not of } H^{\widehat{0}}}} \left|1-\alpha(\gamma)\right|\right\}\left|\rho_{G,H}(\gamma)\right|^{-1}}$$

Here $\rho_{G,H}$ is

$$\frac{1}{2}\sum_{\substack{\widehat{\alpha}>0 \\ \widehat{\alpha} \text{ root of } G^{\widehat{0}} \\ \text{not of } H^{\widehat{0}}}} \alpha.$$

[**8**] Also $\gamma \in T'$ with T' stably conjugate to T.

Taking account of the standard formula for Θ_{π_φ}, we see that $\Theta(\gamma)$ should be the quotient of

$$\frac{1}{w}\sum_{\substack{\nu\in\Omega\left(T(F),G(F)\right) \\ \delta\in\mathcal{D}(F,T) \\ \overline{\gamma}\in T(F) \\ \overline{\gamma}^{\delta\mu}=\gamma \\ \mu\in\Omega\left(T'(F),G'(F)\right)}} \xi(\overline{\gamma}^\nu)\epsilon(\overline{\gamma})\chi(\delta)$$

by

$$\prod_{\alpha>0} \left|1-\alpha(\gamma)\right| \left|\rho_G(\gamma)\right|^{-1}$$

This formula should by the way suggest something about the form of ϵ. Here ξ is the character of $T(F)$ defined by φ (or rather one of the characters defined by φ.)

Compiled on July 3, 2024.