

Late June 1974

Dear Bill,

I have been ruminating further along the lines of our discussion and I now believe I can analyze the formal aspects of the situation and reduce everything to three specific representation-theoretic problems. Since we are leaving for Montreal today I don't have time to describe the analysis; that I shall postpone to our return. However let me pose the two problems to you now to spur you into solving them. I pose the first for groups quasi-split and split over an unramified extension. You may prefer, at the moment, to treat it only for Chevalley groups.

1. *Does every irreducible factor of a unitary unramified principal series contain the trivial representation of some special maximal compact?*

Of this problem you are of course already aware. It means that the group

$$C = \widehat{L}(T_{\text{ad}}^0) / \text{Im } \widehat{L}(T^0)$$

acts transitively on each  $\Pi_\varphi$ . Suppose  $\chi$  is the character of  $C$  trivial on the subgroup  $C_0$  of  $C$  acting trivially on  $\Pi_\varphi$ . Choose a special maximal compact  $K^0$  and hence  $\pi^0 \in \Pi_\varphi$ . If  $\zeta_1, \dots, \zeta_r$  are the values taken by  $\chi$  set

$$\pi^i = \sum_{\substack{c \in C_0 \setminus C \\ \chi(c) = \zeta_i}} c\pi^0$$

so that

$$(*) \quad \pi_\varphi = \bigoplus \pi^i.$$

Suppose  $M$  is a Levi factor of a PSG of  $G$  over  $F$ . Let  $S^0$  be  $T^0$  regarded as a CSG of  $M$ . We have

$$\begin{array}{ccc} \widehat{L}(T^0) & \longrightarrow & L^1(T_{\text{ad}}^0) \\ \wr & & \downarrow (\text{surjective}) \\ \widehat{L}(S^0) & \longrightarrow & \widehat{L}(S_{\text{ad}}^0) \end{array}$$

If  $\chi$  a character of  $\widehat{L}(T_{\text{ad}}^0)$ , can be obtained by pulling back a character of  $\widehat{L}(S_{\text{ad}}^0)$  and if  $\tau_\varphi$  is the principal series of  $M$  corresponding to  $\varphi$  so that  $\pi_\varphi$  is obtained from  $\tau_\varphi$  buy a normalized induction, then the decomposition (\*) as a consequence of a corresponding decomposition

$$\tau_\varphi = \bigoplus \tau^i.$$

I shall try to convince you in a later letter that, given  $\chi$ , one can choose  $M$  so that  $M_{\text{ad}}$  is isomorphic over  $F$  to a product of groups of the form

$$\text{Res}_{K/F} \text{PSL}(m)$$

with  $K/F$  unramified.

These comments may be a help in solving the first problem. They also form an introduction to the second.

Take  $n$  unramified extensions  $K_i$   $1 \leq i \leq n$ , of  $F$  and take unramified extensions  $E_i/K_i$  of degrees  $m_i$ . Choose the basis of  $O_{E_i}$  over  $O_{K_i}$  (also one of  $E_i$ ) and use it to imbed  $E_i^\times$  in  $\text{GL}(m_i, K_i)$ . Let  $G$  be a closed subgroup of  $\prod_i \text{GL}(m_i, K_i)$  containing

$$\left\{ g \mid \eta(g) \in \prod K_i^{m_i} \right\}$$

Here

$$\eta : g \rightarrow \prod \det g_i \in \prod K_i^{m_i}.$$

Let  $\chi$  be a character of  $\prod K_i^\times$  trivial on  $\eta(G)$  and such that the kernel of  $\chi$  in  $K_i^\times$  is  $\text{Nm } E_i^\times$

Let  $\pi$  be a unitary unramified principal series representation of  $G$  and let  $\Pi$  be the set of irreducible components of its restrictions to  $G$ .

$$H = \prod K_i^\times / \eta(G)$$

acts on  $\Pi$ . Let its kernel be  $H^0$ .

2. If  $m = 1$  and  $G = K^\times \text{SL}(m, K)$  then the inverse image of  $H^0$  in  $K^\times$  is  $\{ \alpha \mid m | o(\alpha) \}$  if and only if  $\pi$  is a principal series representation corresponding to a character

$$\left( \begin{array}{c} \alpha_1 \\ \dots \\ \alpha_m \end{array} \right) \rightarrow \nu(\alpha_1, \dots, \alpha_m) \zeta^{o(\alpha_2) + 2o(\alpha_3) + \dots + (m-1)o(\alpha_m)}.$$

Here  $o(\alpha)$  is the order of  $\alpha$ .

Anyhow suppose  $H^0$  is contained in the kernel of  $\chi$ . Let  $\pi^0 \in \Pi$  be the representation containing the trivial representation of  $G$  in  $\prod_i \text{GL}(m, O_{K_i})$ .

Form

$$\Theta = \sum_{H^0 \setminus H} \chi(h) \Theta_{h\pi^0}.$$

Let  $T$  be the set of all  $g = \prod g_i$  in  $G$  with  $g_i \in E_i^\times$ . It is clear that  $\Theta$ , which one has to prove is a function (this is known only in characteristic 0), has support in  $\bigcup_{g \in \prod \text{GL}(m_i, K_i)} g^{-1} T g$ .

It is clear that

$$\Theta(\gamma^w) = \Theta(w^{-1} \gamma w) = \chi(\eta(w))^{-1} \Theta(\gamma)$$

3. Find a formula for  $\Theta(\gamma)$  when  $\gamma \in T$  is regular.

A suggestion

Let

$$\gamma = (\gamma_1, \dots, \gamma_r)$$

and fix

$$\gamma^0 = (\gamma_1^0, \dots, \gamma_r^0) \quad E_i = K_i[\gamma_i^0].$$

For each  $i$ ,  $\chi$  defines a character  $\chi_i$  of  $K_i^\times$  and of  $\mathfrak{G}(E_i/K_i)$

$$\delta_j = \frac{\sum_{\tau \in \mathfrak{G}(E_i/K_i)} \chi_i(\tau) \tau(\gamma_i)}{\sum \chi_i(\tau) \tau(\gamma_i^0)} \in K_i^\times$$

if  $\gamma$  is regular. We may introduce

$$\prod \chi_i(\delta_i).$$

Then

$$\Theta(\gamma) = c \prod \chi_i(\delta_i)$$

where  $c$  is a constant involving orders of Weyl groups and Gaussian sums for the characters  $\chi_i$ .

I hope to hear from you soon,  
Bob

Compiled on July 3, 2024.