

Late June 1974

Dear Bill,

I have been ruminating further along the lines of our discussion and I now believe I can analyze the formal aspects of the situation and reduce everything to three specific representation-theoretic problems. Since we are leaving for Montreal today I don't have time to describe the analysis; that I shall postpone to our return. However let me pose the two problems to you now to spur you into solving them. I pose the first for groups quasi-split and split over an unramified extension. You may prefer, at the moment, to treat it only for Chevalley groups.

1. *Does every irreducible factor of a unitary unramified principal series contain the trivial representation of some special maximal compact?*

Of this problem you are of course already aware. It means that the group

$$C = \widehat{L}(T_{\text{ad}}^0) / \text{Im } \widehat{L}(T^0)$$

acts transitively on each Π_φ . Suppose χ is the character of C trivial on the subgroup C_0 of C acting trivially on Π_φ . Choose a special maximal compact K^0 and hence $\pi^0 \in \Pi_\varphi$. If ζ_1, \dots, ζ_r are the values taken by χ set

$$\pi^i = \sum_{\substack{c \in C_0 \setminus C \\ \chi(c) = \zeta_i}} c\pi^0$$

so that

$$(*) \quad \pi_\varphi = \bigoplus \pi^i.$$

Suppose M is a Levi factor of a PSG of G over F . Let S^0 be T^0 regarded as a CSG of M . We have

$$\begin{array}{ccc} \widehat{L}(T^0) & \longrightarrow & L^1(T_{\text{ad}}^0) \\ \wr & & \downarrow (\text{surjective}) \\ \widehat{L}(S^0) & \longrightarrow & \widehat{L}(S_{\text{ad}}^0) \end{array}$$

If χ a character of $\widehat{L}(T_{\text{ad}}^0)$, can be obtained by pulling back a character of $\widehat{L}(S_{\text{ad}}^0)$ and if τ_φ is the principal series of M corresponding to φ so that π_φ is obtained from τ_φ buy a normalized induction, then the decomposition (*) as a consequence of a corresponding decomposition

$$\tau_\varphi = \bigoplus \tau^i.$$

I shall try to convince you in a later letter that, given χ , one can choose M so that M_{ad} is isomorphic over F to a product of groups of the form

$$\text{Res}_{K/F} \text{PSL}(m)$$

with K/F unramified.

These comments may be a help in solving the first problem. They also form an introduction to the second.

Take n unramified extensions K_i $1 \leq i \leq n$, of F and take unramified extensions E_i/K_i of degrees m_i . Choose the basis of O_{E_i} over O_{K_i} (also one of E_i) and use it to imbed E_i^\times in $\text{GL}(m_i, K_i)$. Let G be a closed subgroup of $\prod_i \text{GL}(m_i, K_i)$ containing

$$\left\{ g \mid \eta(g) \in \prod K_i^{m_i} \right\}$$

Here

$$\eta : g \rightarrow \prod \det g_i \in \prod K_i^{m_i}.$$

Let χ be a character of $\prod K_i^\times$ trivial on $\eta(G)$ and such that the kernel of χ in K_i^\times is $\text{Nm } E_i^\times$

Let π be a unitary unramified principal series representation of G and let Π be the set of irreducible components of its restrictions to G .

$$H = \prod K_i^\times / \eta(G)$$

acts on Π . Let its kernel be H^0 .

2. If $m = 1$ and $G = K^\times \text{SL}(m, K)$ then the inverse image of H^0 in K^\times is $\{ \alpha \mid m | o(\alpha) \}$ if and only if π is a principal series representation corresponding to a character

$$\left(\begin{array}{c} \alpha_1 \\ \dots \\ \alpha_m \end{array} \right) \rightarrow \nu(\alpha_1, \dots, \alpha_m) \zeta^{o(\alpha_2) + 2o(\alpha_3) + \dots + (m-1)o(\alpha_m)}.$$

Here $o(\alpha)$ is the order of α .

Anyhow suppose H^0 is contained in the kernel of χ . Let $\pi^0 \in \Pi$ be the representation containing the trivial representation of G in $\prod_i \text{GL}(m, O_{K_i})$.

Form

$$\Theta = \sum_{H^0 \setminus H} \chi(h) \Theta_{h\pi^0}.$$

Let T be the set of all $g = \prod g_i$ in G with $g_i \in E_i^\times$. It is clear that Θ , which one has to prove is a function (this is known only in characteristic 0), has support in $\bigcup_{g \in \prod \text{GL}(m_i, K_i)} g^{-1} T g$.

It is clear that

$$\Theta(\gamma^w) = \Theta(w^{-1} \gamma w) = \chi(\eta(w))^{-1} \Theta(\gamma)$$

3. Find a formula for $\Theta(\gamma)$ when $\gamma \in T$ is regular.

A suggestion

Let

$$\gamma = (\gamma_1, \dots, \gamma_r)$$

and fix

$$\gamma^0 = (\gamma_1^0, \dots, \gamma_r^0) \quad E_i = K_i[\gamma_i^0].$$

For each i , χ defines a character χ_i of K_i^\times and of $\mathfrak{G}(E_i/K_i)$

$$\delta_j = \frac{\sum_{\tau \in \mathfrak{G}(E_i/K_i)} \chi_i(\tau) \tau(\gamma_i)}{\sum \chi_i(\tau) \tau(\gamma_i^0)} \in K_i^\times$$

if γ is regular. We may introduce

$$\prod \chi_i(\delta_i).$$

Then

$$\Theta(\gamma) = c \prod \chi_i(\delta_i)$$

where c is a constant involving orders of Weyl groups and Gaussian sums for the characters χ_i .

I hope to hear from you soon,
Bob

Compiled on February 14, 2025.