July 6, 1974

Dear Bill,

The enclosed notes are incoherent and incomplete. However they may be of use to you. They show that all the questions we discussed can be reduced to one, to question 3 of my previous letter, namely, to the problem of finding a formula for the characters of certain representations of groups whose adjoint form is a product of groups obtained from projective linear groups by restriction of scalars.

The notes contain, in addition to the statement and proof of a technical lemma, reductions of the problems on intertwining operators, reducibility, and orbital integrals to the groups described above.

I hope you can solve the problem of finding a formula for the characters. I will be surprised however if you solve it quickly.

Yours
Bob

Basic Lemma

[**2**] We shall be interested in the following collection of objects.

(a) A subgroup N of the normalizer of $T^{\widehat{0}}$ in $\widehat{G}$ which contains $T^{\widehat{0}}$ and projects onto $\mathbf{Z}$.

(b) A splitting of

$$1 \longrightarrow T^{\widehat{0}}\backslash N^0 \longrightarrow T^{\widehat{0}}\backslash N \longrightarrow \mathbf{Z}$$

where

$$N^0 = N \cap G^{\widehat{0}}$$

(c) A character χ of $\widehat{L}_{\mathrm{sc}}$

They will be subject to the following conditions:

(i) $T^{\widehat{0}}$ is abelian.

(ii) $\chi\big(n(\lambda)\big) = \chi(\lambda)$. $\lambda \in \widehat{L}_{\mathrm{sc},0} \in N$.

(iii) N has no fixed point in $\widehat{L}_{\mathrm{sc}}$ except 0.

(iv) If $\widehat{\alpha}$ is a root then $\chi(\widehat{\alpha}) \neq 1$.

Observe that (iii) is equivalent to

(v) N is contained a no proper PSG of $G^{\widehat{0}}$.

[**3**] Indeed if $\lambda \neq 0$ and $n(\lambda) = \lambda$ for all $n \in N$ then the PSG containing $T^{\widehat{0}}$ whose roots are those $\widehat{\alpha}$ for which $(\widehat{\alpha}, \lambda) \geqslant 0$ is proper and contains N. Conversely the sum of the roots in the unipotent radical of a proper PSG containing N is not zero and is invariant under N.

Given N_1 on $\widehat{G}_1$ and N_2 in $\widehat{G}_2$ and χ_1, χ_2 then

$$N = \big\{ (x_1, x_2) \times z \bigm| x_1 \times z \in N_1,\ x_2 \times z \in N_2 \big\}$$

is a subgroup of

$$\widehat{G} = (\widehat{G}_1^{\widehat{0}} \times \widehat{G}_2^{\widehat{0}}) \times \mathbf{Z}$$

and

$$\chi : (\lambda_1, \lambda_2) \to \chi_1(\lambda_1)\chi_2(\lambda_2)$$

is a character of

$$\widehat{L}_{\mathrm{sc}} = \widehat{L}_{\mathrm{sc},1} \oplus \widehat{L}_{\mathrm{sc},2}$$

Moreover splittings of $T_1^{\widehat{0}}\backslash N_1$ and $T_2^{\widehat{0}}\backslash N_2$ yield a splitting of $T^{\widehat{0}}\backslash N$. If (N_1, χ_1) and (N_2, χ_2) satisfy our four conditions so does (N, χ). If $G_1^{\widehat{0}}$ and $G_2^{\widehat{0}}$ are both different from 1 then (N, χ) is said to be reducible.[1]

We want to analyze pairs (N, χ) satisfying the conditions. It is clear that it is enough to treat the case that G is simply [**4**] connected and (N, χ) is irreducible.

Suppose E/F is a finite unramified extension. Suppose $\overline{G}^{\wedge}$ is an associate group over E and $\widehat{G}$ is obtained from it by "restriction of scalars". If $[E : F] = m$ then

$$G^{\widehat{0}} = \overbrace{\overline{G}^{\widehat{0}} \times \cdots \times \overline{G}^{\widehat{0}}}^{m\text{-times}}$$

and if $z \in \mathbf{Z}$

$$z : (g_0, \ldots, g_{m-1}) \to (g_0', \ldots, g_{m-1}')$$

[1] Actually we had best say (N, χ) is reducible if N is a subgroup of the group built up from (N_1, χ_1), (N_2, χ_2) which projects *onto* both N_1 and N_2.

with

$$g_i' = a(g_j)$$

if

$$i + z = am + j \qquad 0 \leqslant j < m.$$

Let $\overline{N}$ and $\overline{\chi}$, together with a lifting of $\overline{T}^{\widehat{0}} \backslash \overline{N} \to \mathbf{Z}$, be given. Define χ by

$$\chi : (\lambda_0, \ldots, \lambda_{m-1}) \to \prod \overline{\chi}(\lambda_i)$$

If the lifting is $a \to n(a) \times a$ where $n(a)$ is only given modulo $\overline{T}^{\widehat{0}}$ we define N to consist of

$$(nn_0, nn_1, \ldots, nn_{m-1}) \times z$$

where [**5**]

$$n_i = n_i(a)$$

if

$$i + z = am + j \qquad 0 \leqslant j < m,$$

and n is any element of $\overline{N}^0$.

Since we have an obvious splitting, we need only check that N is a group and that $T^{\widehat{0}} \backslash N$ is abelian. The other conditions are obvious. It is clear that N will be a group if the lifting

$$z \to (n_0, \ldots, n_{m-1}) \times z$$

given modulo $T^{\widehat{0}}$ is a group homomorphism. It will also follow that $T^{\widehat{0}} \backslash N$ is abelian. However

$$\big\{(n_0, \ldots, n_{m-1}) \times z\big\}\big\{(n_0', \ldots, n_{m-1}') \times z'\big\} = (n_0'', \ldots, n_{m-1}'') \times (z + z')$$

with

$$n_i'' = n_i a(n_j') = n(a) a\big(n(b)\big) = n(a+b) \pmod{\overline{T}^{\widehat{0}}}$$

if

$$i + z = am + j$$
$$j + z' = bm + k$$

Since

$$i + (z + z') = (a+b)m + k$$

[**6**] we are in the clear.

If a given irreducible N, χ, together with the splitting cannot be obtained in the above way with $m > 1$ we say the pair is absolutely irreducible. There is an obvious way of constructing an absolutely irreducible pair. Start from

$$G^{\widehat{0}} = \mathrm{PGL}(n, \mathbf{C}) \qquad \text{(The projective group)}$$

The diagonal matrices will be $T^{\widehat{0}}$. $\widehat{G}$ will be the *direct* product $G^{\widehat{0}} \times \mathbf{Z}$. Start from an abelian group H of order n. The regular representation imbeds it into the Weyl group of $T^{\widehat{0}}$. Let N^0 be a subgroup of the normalizer of $\widehat{T}^0$ in $\widehat{G}^0$ mapping into H. Let h be any element of H such that the image of N^0 together with h generates H. N will be the group generated by N^0 and $\{\, n^z \times z \mid z \in \mathbf{Z} \,\}$. Here n normalizes $T^{\widehat{0}}$ and maps to H. Conditions (i) and (iii) are satisfied.

$$\widehat{L}_{\mathrm{sc}} = \Big\{ (\ell_1, \ldots, \ell_n) \Bigm| \sum \ell_i = 0 \Big\}.$$

[7] Since we can project $\widehat{G}$ to $G^{\widehat{0}}$ we can map N onto H. The second condition means that χ is a character of $H^{-1}(H, \widehat{L}_{\text{sc}})$. The exact sequence

$$0 \longrightarrow \widehat{L}_{\text{sc}} \longrightarrow \mathbf{Z}[H] \longrightarrow \mathbf{Z} \longrightarrow 0$$

leads to

$$H \simeq H^{-2}(H, \mathbf{Z}) \simeq H^{-1}(H, \widehat{L}_{\text{sc}})$$

Under this isomorphism $h \in H$ maps to the image of a root. Consequently a χ can exist satisfying (iv) if and only if H is cyclic. Then χ is a character of H with trivial kernel.

The pair constructed above is called a standard pair.

Lemma. *An absolutely irreducible pair is (isomorphic to) a standard pair.*

It is clear that this lemma yields a complete classification. Let $\widehat{\alpha}$ be a root of $T^{\widehat{0}}$. Suppose that $n\widehat{\alpha} = -\widehat{\alpha}$ for some $n \in N$. Then $\chi(\widehat{\alpha}) = -1$. Thus $\chi\big(n(\widehat{\alpha})\big) = -1$ for all $n \in N$ so that $n_1\widehat{\alpha} \pm n_2\widehat{\alpha}$ is never a root. Thus **[8]**

$$\langle n_1\widehat{\alpha}, n_2\widehat{\alpha}\rangle = 0$$

if $n_1\widehat{\alpha} \neq \pm n_2\widehat{\alpha}$.

For the moment we suppose that $\chi(\widehat{\alpha}) \neq -1$. Since

$$\chi\big(n(\widehat{\alpha})\big) = \chi(\widehat{\alpha})$$

$n_1\widehat{\alpha} - n_2\widehat{\alpha}$ is never a root. Therefore

$$\langle n_1\widehat{\alpha}, n_2\widehat{\alpha}\rangle \leqslant 0$$

if $n_1\widehat{\alpha} \neq n_2\widehat{\alpha}$. Let $\widehat{\alpha}_1, \dots, \widehat{\alpha}_r$ be an enumeration of the elements in $\{\, n\widehat{\alpha} \mid n \in N \,\}$. Since all these roots have the same length

$$\langle \widehat{\alpha}_i, \widehat{\alpha}_j\rangle = \begin{cases} -\frac{1}{2}\langle \widehat{\alpha}, \widehat{\alpha}\rangle \\ \text{or} & \qquad i \neq j \\ 0 \end{cases}$$

Suppose we have found a sequence

$$\widehat{\beta}_k, \widehat{\beta}_{k+1}, \dots, \widehat{\beta}_\ell \qquad k \leqslant \ell,\ k, \ell \in \mathbf{Z}$$

of distinct elements of $\{\widehat{\alpha}_1, \dots, \widehat{\alpha}_r\}$ so that

$$\langle \widehat{\beta}_i, \widehat{\beta}_{i+1}\rangle = -\frac{1}{2}\langle \widehat{\alpha}, \widehat{\alpha}\rangle \qquad k \leqslant i < \ell$$

and so that **[9]**

$$\langle \widehat{\beta}_i, \widehat{\beta}_j\rangle = 0$$

if $i \neq j \pm 1, j$ except perhaps for $i = k$, $j = \ell$, or $i = \ell$, $j = k$ and so that

$$\widehat{\gamma}_{k',\ell'} = \sum_{i=k'}^{\ell'} \widehat{\beta}_i \qquad k \leqslant k' \leqslant \ell' \leqslant \ell$$

is a root except perhaps for $k' = k$, $\ell' = \ell$. However if $\gamma_{k,\ell}$ is not a root then it must be 0.

If $k' \neq k$ or $\ell' \neq \ell$ then

$$\langle \widehat{\gamma}_{k',\ell'}, \widehat{\gamma}_{k',\ell'}\rangle = (\ell' - k' + 1)\langle \widehat{\alpha}, \widehat{\alpha}\rangle - (\ell' - k')\langle \widehat{\alpha}, \widehat{\alpha}\rangle = \langle \widehat{\alpha}, \widehat{\alpha}\rangle.$$

Also

$$\langle\widehat{\gamma}_{k,\ell},\widehat{\gamma}_{k,\ell}\rangle = (\ell-k+1)\langle\widehat{\alpha},\widehat{\alpha}\rangle - (\ell-k)\langle\widehat{\alpha},\widehat{\alpha}\rangle + 2\langle\widehat{\beta}_k,\widehat{\beta}_\ell\rangle.$$

Thus if $\widehat{\gamma}_{k',\ell'}$ is a root its length is the same as that of $\widehat{\alpha}$. Moreover $\widehat{\gamma}_{k',\ell'}$ is a root if and only if $\langle\widehat{\beta}_k,\widehat{\beta}_\ell\rangle \neq 0$.

Suppose $\widehat{\gamma}_{k,\ell}$ is a root. Let I be the set of i, $1 \leqslant i \leqslant r$ such that $\widehat{\alpha}_i \notin \{\widehat{\beta}_k,\dots,\widehat{\beta}_\ell\}$. Then

$$0 = \left\langle \widehat{\gamma}_{k,\ell}, \sum_{i=1}^{r}\widehat{\alpha}_i \right\rangle = \langle\widehat{\gamma}_{k,\ell},\widehat{\gamma}_{k,\ell}\rangle + \left\langle \widehat{\gamma}_{k,\ell}, \sum_{i\in I}\widehat{\alpha}_i \right\rangle.$$

[10] We conclude that for some $i \in I$,

$$\langle\widehat{\gamma}_{k,\ell},\widehat{\alpha}_i\rangle = -\frac{1}{2}\langle\widehat{\alpha},\widehat{\alpha}\rangle.$$

Thus there is exactly one j, $k \leqslant j \leqslant \ell$ so that

$$\langle\widehat{\beta}_j,\widehat{\alpha}_i\rangle = -\frac{1}{2}\langle\widehat{\alpha},\widehat{\alpha}\rangle.$$

(Observe: we use again and again that if two roots have the same length as $\widehat{\alpha}$ then their inner product is either 0 or $-1/2$ times $\langle\widehat{\alpha},\widehat{\alpha}\rangle$. If $j' \neq j$ then

$$\langle\widehat{\beta}_{j'},\widehat{\alpha}_i\rangle = 0.$$

Note however that, for a given s,

$$0 = \left\langle \widehat{\alpha}_s, \sum_{t=1}^{r}\widehat{\alpha}_t \right\rangle = \langle\widehat{\alpha},\widehat{\alpha}\rangle = \sum_{t\neq s}\langle\widehat{\alpha}_s,\widehat{\alpha}_t\rangle.$$

Thus $\langle\widehat{\alpha}_s,\widehat{\alpha}_t\rangle \neq 0$ for exactly two t. We conclude that $j = k$ or $j = \ell$. If $j = k$ we set $\widehat{\beta}_{k-1} = \widehat{\alpha}_i$. If $j = \ell$ we set $\widehat{\beta}_{\ell+1} = \widehat{\alpha}_i$. If $j = k$

$$\langle\widehat{\gamma}_{k,\ell'},\widehat{\beta}_{k-1}\rangle = -\frac{1}{2}\langle\widehat{\alpha},\widehat{\alpha}\rangle \qquad \ell' \neq \ell$$

so that $\widehat{\gamma}_{k-1,\ell'}$ is a root. A similar statement applies to $\gamma_{k',\ell+1}$. **[11]** Also $\widehat{\gamma}_{k-1,\ell'}$ is a root unless $\langle\widehat{\beta}_{k-1},\widehat{\beta}_\ell\rangle = -\frac{1}{2}\langle\widehat{\alpha},\widehat{\alpha}\rangle$. Otherwise it is 0.

Thus by induction we repeat the process until we arrive at the stage at which $\widehat{\gamma}_{k,\ell} = 0$. Then we define $\widehat{\beta}_j$ for all $j \in \mathbf{Z}$ by setting

$$\widehat{\beta}_{j+\ell-k+1} = \widehat{\beta}_j$$

It is clear that the Lie group generated by the one parameter subgroups corresponding to $X_{\pm\widehat{\beta}_j}$ is of type $A_{\ell-k}$ with $\beta_1,\dots,\beta_{\ell-k}$ as a fundamental system of roots. We may suppose $k = 0$.

We can form a graph with vertices $\{\widehat{\alpha}_1,\dots,\widehat{\alpha}_r\}$. We join $\widehat{\alpha}_i$ and $\widehat{\alpha}_j$ if and only if $\langle\widehat{\alpha}_i,\widehat{\alpha}_j\rangle \neq 0$. N acts on this graph and permutes the connected components fixed. One connected component is $\{\widehat{\beta}_0,\dots,\widehat{\beta}_\ell\}$, for if $\widehat{\alpha}_1 \notin \{\widehat{\beta}_0,\dots,\widehat{\beta}_k\}$ then **[12]**

$$\left\langle \sum_{j=0}^{\ell}\widehat{\beta}_j,\widehat{\alpha}_i \right\rangle = 0$$

and

$$\langle\widehat{\beta}_j,\widehat{\alpha}_i\rangle \leqslant 0.$$

Let N^1 be the stabilizer of the connected component $\{\widehat{\beta}_0, \ldots, \widehat{\beta}_\ell\}$. Take $n \in N^1$ and let $n\widehat{\beta}_0 = \widehat{\beta}_s$. Then $n\widehat{\beta}_1 = \widehat{\beta}_{s+\delta}$ with $\delta = \pm 1$. By induction, for $\ell \geqslant 2$, since we are assuming $n\widehat{\alpha} \neq -\widehat{\alpha}$ for all n,

$$n\widehat{\beta}_i = \widehat{\beta}_{s+\delta_i}.$$

If $\delta = -1$ then

$$n^2\widehat{\beta}_0 = \widehat{\beta}_0.$$

which implies that $\chi(\widehat{\alpha}) = -1$. This is the excluded case. Thus $\delta = 1$ and the elements of N^1 act as translations. This shows that N^1 modulo the stabilizer of $\widehat{\alpha}$ is cyclic of order $\ell + 1$.

Since N acts transitively on $\{\widehat{\alpha}_i\}$ we can break this set up into the components

$$\{\widehat{\beta}_{0,i}, \widehat{\beta}_{1,i}, \ldots, \widehat{\beta}_{\ell,i} \qquad 1 \leqslant i \leqslant t\}$$

where **[13]**

$$\widehat{\beta}_{0,i}, \ldots, \widehat{\beta}_{\ell,i}$$

is obtained from

$$\widehat{\beta}_0, \ldots, \widehat{\beta}_\ell$$

by applying an element of N.

The algebra generated by the root vectors $X_{\pm\widehat{\alpha}_i}$ is a direct sum of algebras of type A_ℓ. (Note that what we are now saying also applies to the case $\chi(\widehat{\alpha}) = -1$, which was formerly excluded. In this case $\ell = 1$.) If $\widehat{M}$ is the lattice generated by $\{\widehat{\alpha}_i\}$ we may write an element of $\widehat{M} \otimes \mathbf{Q}$ as $(b_1, \ldots, b_t)$ with

$$b_i = \{b_{0,i}, \ldots, b_{\ell,i}\}$$

and

$$\sum_j b_{j,i} = 0.$$

For example

$$\widehat{\beta}_{j,i} = (0, \ldots, 0, b_i, 0, \ldots, 0)$$

with

$$b_i = (0, \ldots, 0, 1, -1, 0, \ldots, 0) \qquad 1 \leqslant j \leqslant \ell.$$

[14] Every root of $\widehat{G}^0$ defines an element of $M \otimes \mathbf{Q}$ and there is a constant c so that if $\widehat{\beta} \sim (b_1, \ldots, b_t)$, $\overline{\beta}^\wedge \sim (\bar{b}_1, \ldots, \bar{b}_t)$ then

$$\langle \widehat{\beta}, \overline{\beta}^\wedge \rangle = c \sum_{i,j} b_{ij} \bar{b}_{ij}$$

provided $\overline{\beta}^\wedge$ is a linear combination of $\{\widehat{\alpha}_i\}$.

If for some i

$$b_{01} \leqslant b_{11} \leqslant \cdots \leqslant b_{\ell i} \leqslant b_{0i}$$

the $b_i = 0$. If $\widehat{\beta}$ is not orthogonal to all $\widehat{\alpha}_i$ there is an i so that $b_i \neq 0$. Choose j so that $b_{j0} < b_{j-1,i}$. (Let $b_{j,i}$ be periodic of period $\ell + 1$ in j.) Then

$$\langle \widehat{\beta}, \widehat{\beta}_{ij} \rangle > 0$$

and $\widehat{\beta} - \widehat{\beta}_{ij}$ is a root or 0.

Suppose the image of χ consists of the nth roots of unity. Fix a primitive nth root of unity. We choose $\widehat{\alpha}$ so that $\chi(\widehat{\alpha}) = \zeta^m$, $0 < m < n$ with m as small as possible. Suppose $\widehat{\beta}$ is a **[15]** root which is not a root in the algebra generated by $\{X_{\pm\widehat{\alpha}_i}\}$ but which is also not orthogonal to all $\widehat{\alpha}_i$. Let $\chi(\widehat{\beta}) = \zeta^f$. We choose β so that f is as small as possible. As we have just seen there is an $\widehat{\alpha}_i$ so that $\widehat{\beta} - \widehat{\alpha}_i$ is a root. It is certainly not 0. But

$$\chi(\widehat{\beta} - \widehat{\alpha}_i) = \zeta^{f-m}$$

and $0 < f - m < f$. This is a contradiction.

We conclude that the algebra generated by $\{X_{\pm\widehat{\alpha}_i}\}$ is a direct summand of the Lie algebra of $G^{\widehat{0}}$. By the assumption of irreducibility it is all of the larger Lie algebra.

Let $n(1) \pmod{T^{\widehat{0}}}$ be one of the liftings of $1 \in \mathbf{Z}$. We may take

$$\widehat{\beta}_{j,i} = n(1)^{i-1}(\widehat{\beta}_j) \qquad 1 \leqslant i < t.$$

Also if $n \in N$ projects to 0 in $\mathbf{Z}$ then since it must commute with $n(1)$ module $T^{\widehat{0}}$ and preserve each summand of $G^{\widehat{0}}$ which is now a product of copies of $\mathrm{PGL}(\ell + 1)$ it must be of the form $(n', \ldots, n') \pmod{T^{\widehat{0}}}$. **[16]** This shows that if (N, χ) is irreducible. We are dealing with a standard pair.

The lemma is proved.

[Second set of notes (Incomplete)]

[17] The formula on the preceding page can be put into a more elegant form.

We have H, G and $\widehat{H} \hookrightarrow \widehat{G}$. Moreover, the imbedding is such that if $\widehat{T}_G$, $\widehat{T}_H$, $\widehat{B}_G$, $\widehat{B}_H$ are the CSG's and BSG's of $\widehat{G}$ and $\widehat{H}$ then

$$\widehat{T}_H = \widehat{T}_G = \widehat{T}$$

and

$$\widehat{B}_H = \widehat{B}_G \cap \widehat{H}.$$

Let

$$\widehat{L} = L(\widehat{T}) \qquad L = \widehat{L}(\widehat{T})$$

If T_H and T_G are CSG's over F of H and G respectively we have families $\Xi(T_G)$ and $\Xi(T_H)$ of isomorphisms

$$L(T_G) \simeq L$$
$$L(T_H) \simeq L.$$

They are principal homogeneous under the Weyl groups $\Omega(\widehat{T}, G^{\widehat{0}})$ and $\Omega(\widehat{T}, H^{\widehat{0}})$. Let $\mathfrak{W}(T_H, T_G)$ be the set of isomorphisms

$$\mu : T_H \to T_G$$

defined over F for which there exists ξ, ζ in $\Xi(T_G)$, $\Xi(T_H)$ **[18]** making

$$\begin{array}{ccc} L(T_G) & & \\ {\scriptstyle\mu^*}\downarrow & \overset{\xi}{\searrow} & \\ & & L \\ & \underset{\zeta}{\nearrow} & \\ L(T_H) & & \end{array}$$

commutative.

If
$$\mathfrak{A}(T_G, T'_G) = \{\, g \in \mathfrak{A}(T_G) \mid g^{-1} T_G g = T'_G \,\}$$
then
$$\mathfrak{W}(T_G, T'_G) = T_G(\overline{F}) \backslash \mathfrak{A}(T_G, T'_G).$$
If $\mu \in \mathfrak{W}(T_H, T_G)$ then
$$\mathfrak{W}(T_H, T'_G) = \mu \mathfrak{W}(T_G, T'_G)$$
because
$$\Omega(\widehat{T}, \widehat{G^0}) \supseteq \Omega(\widehat{T}, \widehat{H^0})$$

If we assume, as we should, that the ϵ' appearing earlier is invariant under $\Omega\big(T_H(F), H(F)\big)$ (The old T is now T_H; the old T' is T'_G) then we may write the numerator on p. 8[2] as
$$\sum_{\mu \in \mathfrak{W}(T_H, T'_G)} \xi\big(\mu^{-1}(\gamma)\big)\epsilon(\gamma, \mu).$$
Here $\epsilon(\gamma, \mu)$ should satisfy **[19]**

(i)
$$\epsilon(\gamma, \delta_\mu) = \epsilon(\gamma)$$
if
$$\delta \in \mathfrak{W}(T_H, T_H)$$
which is, in the present circumstances (T_H is a CSG of a BSG of H over F),
$$\Omega\big(T_H(F), H(F)\big)$$

(ii) If T_G is the CSG of G fixed above and referred to as T, then
$$\epsilon(\gamma, \mu\delta) = \epsilon(\gamma, \mu)\chi(\delta)$$
if
$$\delta \in \mathfrak{W}(T_G, T'_G)$$
and if we also regard χ as a function on $\mathfrak{A}(T_G)$ by lifting it from $\mathfrak{W}(T_G)$.

The problem then is to determine $\epsilon(\gamma, \mu)$.

[2][Pages 1 up to two pages just before the present page of the original second set of handwritten notes are missing. Page 8 of this second set of handwritten notes is among the missing pages.]

[Third set of notes]

[20]

A further remark. Suppose we construct $\widehat{H^0}$ as before. Consider the set X of roots $\widehat{\alpha}$ of $\widehat{H^0}$ for which $\operatorname{Nm}\widehat{\alpha} = 0$. The norm is taken with respect to the action σ of the Frobenius on $\widehat{L}$. X clearly has the following two properties.

(i) It is invariant under $\mathfrak{G}(\overline{F}/F)$. (The Frobenius acts as σ.)

(ii) Any root which is a linear combination of roots in X is again in X.

The first property implies that the group $\widehat{\overline{H}^0}$ containing $\widehat{T}$ with X as its set of roots itself satisfies our conditions. The second implies that $\widehat{\overline{H}^0}$ is a Levi factor of a PSG of $\widehat{H^0}$ and that $\overline{H}$ is a Levi factor of a PSG of H which is defined over F.

Since it is pretty clear that it is enough to work with $\overline{H}$ we may suppose from the beginning that $\widehat{\alpha}$ is a root of $\widehat{H^0}$ if and only if $\operatorname{Nm}\widehat{\alpha} = 0$ and $\chi(\widehat{\alpha}) = 1$.

With this extra condition, T and χ determine $\widehat{H^0}$. Since T_G is unramified by an action $\sigma = \omega\sigma^0$ of the Frobenius in $\widehat{L}$ (once we have identified $\widehat{L}(T_G)$ and $\widehat{L}$) we are going to get an H satisfying our conditions only if this action preserves the set of roots of $\widehat{H^0}$ positive with respect to some order.

If $\widehat{\alpha}$ is positive with respect to this order, then so are $\sigma\widehat{\alpha}, \sigma^2\widehat{\alpha}, \sigma^3\widehat{\alpha}, \ldots$ and

$$\operatorname{Nm}\widehat{\alpha} \neq 0.$$

This is impossible. Hence $\widehat{H^0} = \widehat{T}$ and H is a torus.

Thus χ must satisfy: **[21]**

If $\operatorname{Nm}\widehat{\alpha} = 0$ *then* $\chi(\widehat{\alpha}) \neq 1$.

Compiled on February 14, 2025.