Princeton, NJ
September 15, 1975

Dear Bill,

I was delighted to read of your walks in Northumbrian. Perhaps there will be soon a conference with congenial mathematics and mathematicians, as well as a pleasant countryside, and then you and I will be able to take a tour together.

Our summer, France with the Laurentians as a chaser, and no serious obligations, was pleasantly and profitably spent. I hope to repeat it, with variations, frequently in the future. Ann Arbor was brief and pleasant—lots of cronies and an opportunity to become better acquainted with some Japanese, especially Ihara and Shintani, as well as Shimura.

The idea I mentioned to you is working out well. There are a few lemmas left to prove, and some more writing to do, but I anticipate no difficulty. If it were not that a residue of superstition remains from my Catholic childhood, I would be willing to call the results theorems. Prudence is perhaps not called for, but certainly does no harm.

Before describing the results, let me state a corollary.

If ρ is a two-dimensional representation of $\mathfrak{G}(\overline{\mathbf{Q}}/\mathbf{Q})$ of tetrahedral or octahedral type for which the Frobenius substitution at ∞ has eigenvalues $\{1,-1\}$, then $L(s,\rho)$ is entire, that is, the Artin conjecture is valid for ρ.

However the results are potentially much stronger than this corollary. In order to formulate them easily I recall some conventions. If $\pi = \bigotimes \pi_v$ is an automorphic representation of $\mathrm{GL}(2,\mathbf{A}_F)$ then for almost all v there is associated to π_v a conjugacy class

$$\left\{ t(\pi_v) = \begin{pmatrix} a(\pi_v) & 0 \\ 0 & b(\pi_v) \end{pmatrix} \right\}$$

[**2**] in $\mathrm{GL}(2,\mathbf{C})$. If ρ_v is the representation of the Weil group given by

$$W_{F_v} \longrightarrow \mathbf{Z} \longrightarrow \mathrm{GL}(2,\mathbf{C})$$

$$n \longrightarrow t(\pi_v)^n$$

then $\pi_v = \pi(\rho_v)$. I forgot to mention that F is an arbitrary number field.

Let E being a cyclic extension of prime degree ℓ. We say that an automorphic representation Π of $\mathrm{GL}(2,\mathbf{A}_E)$ is a lifting of π if for almost all places w of E the representation ρ_w is the restriction to $W_{E_w} \hookrightarrow W_{\rho_v}$ of ρ_v. Here $w|v$, $\Pi_w = \pi(\rho_w)$, and $\pi_v = \pi(\rho_v)$.

Here are the global results.

1) Every π has a unique lifting.
2) Define ${}^\sigma\Pi$ for $\sigma \in \mathfrak{G}(E/F)$ by ${}^\sigma\Pi(g) = \Pi(g^\sigma)$. Then Π is a lifting of some π if and only if ${}^\sigma\Pi \simeq \Pi$ for all σ.
3) Suppose π lifts to Π. If $\pi = \pi(\mu,\nu)$ with two quasi-characters μ, ν of the idèle class group then the only other representations lifting to Π are $\pi(\mu_1\nu, \nu_1\nu)$ where μ_1, ν_1 are

characters of $F^\times N_{E/F} I_E \backslash I_F$. Otherwise π' lifts to Π if and only if $\pi' = \omega \otimes \pi$ where ω is again a character of $F^\times N_{E/F} I_E \backslash I_F$. The number of such π' is ℓ, unless $\ell = 2$ and $\pi = \pi(\rho)$ where ρ is a two-dimensional dihedral representation of W_F defined by a quasi-character of $E^\times \backslash I_E$, when it is 1, for $\pi \simeq \omega \otimes \pi$ in this case.

4) Suppose $k \subset F \subset E$ and $k \backslash F$, $k \backslash E$ are Galois. If $\tau \in \mathfrak{G}(E/k)$ has image $\overline{\tau}$ in $\mathfrak{G}(F/k)$ and Π is a lifting of π then ${}^\tau\Pi$ is a lifting of ${}^{\overline{\tau}}\pi$.

Notice that identical statements are valid for two-dimensional representations of W_F, W_E if lifting is replaced by restriction. If σ is a two-dimensional representation of W_F by semi-simple matrices we say that **[3]** $\pi = \pi(\sigma)$ if $\pi_v = \pi(\sigma_v)$ for almost all v. Here σ_v is the restriction of σ to W_{F_v}. If $\pi(\sigma)$ exists, the Artin conjecture is valid for σ.

σ is either reducible, dihedral, or of tetrahedral, octahedral, or icosahedral type. The first two are understood, and about the last I have nothing to say. Consider the other two.

A) **Tetrahedral type.** There is a cyclic extension of degree 3 so that Σ, the restriction of σ to W_E, is dihedral. Therefore $\Pi = \pi(\Sigma)$ exists. Since $\Sigma = {}^\tau\Sigma$ and $\pi({}^\tau\Sigma) = {}^\tau\pi(\Sigma)$, we have ${}^\tau\Pi \simeq \Pi$ for all $\tau \in \mathfrak{G}(E/F)$. Thus Π is a lifting of some π. π is determined up to tensoring with ω, a character of $F^\times N_{E/F} I_E \backslash I_F$. Tensoring σ with such an ω yields the same set of π. Thus, modulo such tensoring we have $\sigma \to \pi = \pi_{\text{pseudo}}(\sigma)$ defined for tetrahedral σ. I have not yet tried to prove that one of the three $\pi_{\text{pseudo}}(\sigma)$ is $\pi(\sigma)$. This may be difficult. Note that

$$\pi_{\text{pseudo}}(\sigma) = \pi_{\text{pseudo}}(\sigma') \implies \pi(\Sigma) = \pi(\Sigma') \implies \Sigma = \Sigma' \implies \sigma' \simeq \omega \otimes \sigma$$

with ω a character of $N_{E/F} C_E \backslash C_F$. Thus our map is an injection.

B) **Octahedral type.** There is a sequence $F \subseteq E \subseteq E_1$ with E_1/F, E/F Galois so that $\Sigma_1 = \sigma | W_{E_1}$ is dihedral and $\Sigma = \sigma | W_E$ is tetrahedral. E/F is cyclic of degree two and E_1/E cyclic of degree 3. We may introduce $\pi_{\text{pseudo}}(\Sigma)$. If $\tau \in \mathfrak{G}(E/F)$ then $\Sigma \simeq {}^\tau\Sigma$. Thus if $\Pi = \pi_{\text{pseudo}}(\Sigma)$ then ${}^\tau\Pi = \pi_{\text{pseudo}}({}^\tau\Sigma)$ is $\omega \otimes \Pi$ where ω is a character of $E^\times N_{E_1/E} I_{E_1} \backslash I_E$. We may then find another ω_1 so that $\omega\, {}^\tau\omega_1 = \omega_1$ for all τ. Replacing Π by $\omega_1 \otimes \Pi$ if necessary, we may assume that ${}^\tau\Pi = \Pi$ for all τ. Then Π is the lifting of some π. π is determined up to tensoring with one of the two characters of $F^\times N_{E/F} I_E \backslash I_F$. Thus we may introduce $\pi_{\text{pseudo}}(\sigma)$ once again.

[4] Observe that if $\pi(\sigma)$ does happen to exist then $\pi(\sigma)$ is a $\pi_{\text{pseudo}}(\sigma)$.

How does one deduce the statement of the first page? Start with such a ρ and let $\pi = \pi_{\text{pseudo}}(\rho)$. It follows easily from the methods used to establish 1)–4) that $\pi_\infty = \pi(\rho_\infty)$. The reason is that the extensions E, E_1 above are now totally real. It follows from Deligne-Serre that a ρ' exists for which $\pi = \pi(\rho')$. We conclude that $\rho = \omega \otimes \rho'$. Hence $\omega \otimes \pi$ is in fact $\pi(\rho)$.

It is also possible to introduce the notion of lifting locally. Let F now be a local field. Let E be either a direct sum of ℓ-copies of F or a cyclic extension of prime degree. In the first case we say that a representation Π of $\mathrm{GL}(2, E) \simeq \mathrm{GL}(2, F) \times \cdots \times \mathrm{GL}(2, F)$ is a lifting of π if $\Pi = \pi \otimes \cdots \otimes \pi$. This being trivial concentrate on the second. To define the notion of lifting one needs to fix a generator σ of $\mathfrak{G}(E/F)$. It is a theorem that the notion is independent of this choice.

Given σ and $g \in \mathrm{GL}(2, E)$ set

$$Ng = g^{\sigma^{\ell-1}} g^{\sigma^{\ell-2}} \cdots g^\sigma g.$$

Only the conjugacy class of Ng is relevant. It can always be realized by an h in $\mathrm{GL}(2, F)$.

Local lifting. Suppose Π is a representation (irreducible admissible) of $\mathrm{GL}(2, E)$ and ${}^{\sigma}\Pi \simeq \Pi$. We say that Π is a lifting of π if one of the following conditions is satisfied.

(a) $\Pi = \pi(\mu, \nu)$, $\pi = \pi(\mu', \nu')$ and $\mu(x) = \mu'(N_{E/F}x)$, $\nu(x) = \nu'(N_{E/F}x)$ [**5**]

(b) If ${}^{\sigma}\Pi \simeq \Pi$ then Π extends in ℓ ways to a representation Π' of the semi-direct product $\mathfrak{G}(E/F) \times \mathrm{GL}(2, E)$. The character of Π' exists as a locally-integrable function. The second way for Π to be a lifting of π is for them both to be finite-dimensional or both infinite-dimensional and for the equality

$$\chi_{\Pi'}(\sigma \times g) = \chi_{\pi}(h)$$

to hold when $h \in G(F)$ is conjugate to Ng and has distinct eigenvalues.

Statements 1)–4) are again valid and if $\Pi = \bigotimes \Pi_v$ is a lifting of $\pi = \bigotimes \pi_v$ then Π_v is a lifting of π_v for all v. It is again impossible to define $\pi_{\text{pseudo}}(\sigma)$. I have not yet tried to prove that every representation of $\mathrm{GL}(2, F)$, F a local field, is either special or a $\pi_{\text{pseudo}}(\sigma)$.

If you glance at Shintani's paper in the proceedings of the Ann Arbor seminar you will see that he has very similar results. However locally he must assume that the residual characteristic is not two. Globally he must assume that the fields are totally real and that the representations lie in the discrete series at infinity. He was aware that his results should be more general; but the trace formula caused him some difficulty. I started from his ideas.

I was surprised when he wrote, in response to my enquiries, that the applications to Artin L-functions had not occurred to him.

All the best
Bob

Compiled on November 12, 2024.