

June 11, 1977

Dear Bill,

As I said the Rapoport letter contains gaps. I still hope that my conjectural classification of points mod  $p$  is alright, but it lies deeper than I thought. There are two major blunders. I misunderstood the situation in regard to lifting. It is more complicated than I imagined. But a careful study of Fontaine might take care of the difficulties. However I was also too facile and thinking that I had identified the action of the Frobenius. At the moment I don't see any way of dealing with this, even for the standard case of the group of symplectic similitudes. To give you an idea of what I had in mind, let me explain the situation in this case.

Start from an abelian variety  $A$  mod  $p$  and polarization  $\lambda$  on  $A$ . Let

$$T^p(A) = \varprojlim_{(n,p)=1} A_n$$

be the Tate module, and  $M$  the Dieudonné module of  $\tilde{A}$ . Thus  $M$  is a covariant functor. Let  $V(\mathbf{Q})$  be the standard  $2n$ -dimensional space over  $\mathbf{Q}$  provided with the standard alternating form, and choose an isomorphism  $x \rightarrow \psi(x)$  of  $T^p(A) \otimes \mathbf{Q}$  with  $V(\mathbf{A}_f^p)$  compatible with the bilinear forms on the two spaces. Of course  $\varphi$  is determined up to composition with an element of  $G(\mathbf{A}_f^p)$ ,  $G$  being the group of symplectic similitudes. Let  $N$  be  $M \otimes \mathbf{Q}$ .

Whenever we have a lattice  $gV(\mathbf{Z}_f^p)$ ,  $g \in G(\mathbf{A}_f^p)$  in  $V(\mathbf{A}_f^p)$ , and a Dieudonné submodule  $M'$  of  $N$  whose dual, with respect to the bilinear form defined by the polarization, is a scalar multiple of itself, then we have a complete set of data for the moduli problem.

The associated abelian variety  $A'$  is defined by

$$A' \xrightarrow{\psi} A$$

$$T^p(A') \xrightarrow{\psi} T^p(A) \text{ has image } agV(\mathbf{Z}_f^p) \text{ with } a \in \mathbf{Q}^\times$$

$$M(A') \xrightarrow{\psi} M(A) \text{ has image } cM' \text{ with the same } a.$$

In other words there is an isogeny  $A' \xrightarrow{\psi} A$  whose associated image in the Tate module or Dieudonné module has the indicated images. The polarization  $\lambda'$  in  $A'$  is that defined by the commutativity of

$$\begin{array}{ccc} A' & \xrightarrow{\psi} & A \\ \lambda' \downarrow & & \downarrow \lambda \\ \tilde{A}' & \xleftarrow{\tilde{\psi}} & \tilde{A} \end{array}$$

The identification of  $T^p(A')$  with  $V(\mathbf{Z}_f^p)$  is obtained by composition

$$T^p(A') \longrightarrow T^p(A) \longleftarrow V(\mathbf{A}_f^p) \xrightarrow{g^{-1}a^{-1}} V(\mathbf{A}_f^p) \ .$$

The first question to ask is when  $g_1, M'_1$  and  $g_2, M'_2$  define isomorphic data. If  $\mathfrak{A} = \text{End } A \otimes \mathbf{Q}$  then an element of  $\text{Hom}(A'_1, A'_2)$  is simply an element  $a \in \mathfrak{A}$  such that

$$ag_1V(\mathbf{Z}_f^p) \subseteq g_2V(\mathbf{Z}_f^p)$$

and

$$aM'_1 \subseteq M'_2.$$

It is an isomorphism if both relations are equalities. Note that we regard  $\mathfrak{A}$  as acting on  $V(\mathbf{A}_f^p)$ , by means of our identification, and on  $N$ . If  $a \rightarrow \tilde{a}$  is the involution on  $\mathfrak{A}$  defined by the polarization then the associated isomorphism  $A'_1 \rightarrow A'_2$  takes  $\lambda'_1$  to  $\lambda'_2$  if and only if  $\tilde{a}a \in \mathbf{Q}^\times$ . This equation defines a group  $I(\mathbf{Q})$ .

Consequently if  $X$  is the set of all Dieudonné submodules of  $N$  which are multiples of their duals then the set of points in our moduli space obtained from  $A, \lambda$  is

$$I(\mathbf{Q}) \backslash G(\mathbf{A}_f^p) \times X,$$

where  $I(\mathbf{Q})$  is imbedded in  $G(\mathbf{A}_f^p)$  in the way indicated, and acts on  $X$  through its action on  $N$ . Of course, for the true moduli problem one has to divide by an open compact subgroup of  $G(\mathbf{A}_f^p)$ , but the discussion is, I hope more transparent if one passes to the limit.

Notice that a high power of the Frobenius lies in  $I(\mathbf{Q})$ . Call it  $\gamma$ . Via the imbedding  $I(\mathbf{Q}) \hookrightarrow G(\mathbf{A}_f^p)$ ,  $\gamma$  defines an element of  $G(\mathbf{A}_f^p)$ . Since  $T^p(A) \otimes \mathbf{Q} \rightarrow V(\mathbf{A}_f^p)$  is not uniquely fixed, it is only the conjugacy class of  $\gamma$  which is uniquely determined.

Suppose we start from the same abelian variety  $A$  but another polarization  $\lambda'$ . Then there is a symmetric, positive element  $\eta \in \mathfrak{A}$  such that  $\lambda' = \lambda \circ \eta$ . We want to examine the conjugacy class associated to  $(A, \lambda')$ , and discover its relation to  $\{\gamma\}$ .

In  $\mathfrak{A}(\overline{\mathbf{Q}})$

$$\eta = \tilde{c}c.$$

Since, for any element  $\rho$  of  $\text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ ,

$$\eta = \rho(\eta) = \rho(\tilde{c})\rho(c)$$

the cocycle  $D_\rho = \rho(c)c^{-1}$  satisfies  $\tilde{D}_\rho D_\rho = 1$  and therefore lies in  $I(\mathbf{Q})$ . I claim that it is trivial over  $\mathbf{A}_f^p$  if and only if the conjugacy class  $\{\gamma'\}$  associated to  $A, \lambda'$  is the same as  $\{\gamma\}$ .

We have fixed  $\varphi$  so that

$$\{x, \lambda y\} = \alpha \langle \varphi(x), \varphi(y) \rangle$$

for some  $\alpha \in I_f^p$ . Thus

$$\{x, \lambda' y\} = \langle x, \lambda \eta y \rangle = \alpha \langle \varphi(x), \psi(\eta y) \rangle = \alpha \langle \psi(x), \eta \psi(y) \rangle.$$

For the last equality, we have identified  $\eta$  in  $\mathfrak{A}$  with its image in  $\text{End}(V(\mathbf{A}_f^p))$ . There is a  $B$  in  $\text{End}(V(\mathbf{A}_f^p))$  such that

$$\langle X, \eta Y \rangle = \langle Bx, By \rangle.$$

If  $\lambda$  is replaced by  $\lambda'$  then  $\varphi$  can be replaced by  $\varphi' = B\varphi$ , and  $\gamma$  by  $\gamma' = B\gamma B^{-1}$ . This is conjugate to  $\gamma \iff$  there is a  $g$  in  $G(\mathbf{A}_f)$  with  $gB$  centralising  $\gamma$ , i.e. by Tate's theorem, in  $\mathfrak{A}(\mathbf{A}_f^p)$ . Then

$$\langle X, \eta Y \rangle = \alpha' \langle B_1 X, B_1 Y \rangle \quad \alpha' \in I_f^p.$$

In conclusion,  $\gamma$  and  $\gamma'$  are conjugate  $\iff \eta = \alpha' \tilde{B}_1 B_1$  with  $\tilde{B}_1$  in  $\mathfrak{A}(\mathbf{A}_f^p)$ .

However if

$$\eta = \alpha \tilde{c} c = \alpha_1 \tilde{c}_1 c_1$$

with  $c, c_1 \in \mathfrak{A}(\overline{\mathbf{Q}})$  then  $cc_1^{-1} \in I(\overline{\mathbf{Q}})$  and

$$\rho(c_1 c^{-1}) \rho(c) c^{-1} (c c_1^{-1}) = \rho(c_1) c_1^{-1}.$$

This shows that the cocycle is well-defined, and that it is trivial over  $\mathbf{A}_f^p$  if and only if the  $B_1$  above exists.

We are going to put the set corresponding to  $(A, \lambda)$  and  $(A, \lambda')$  if the cocycle defined by  $\eta$  is trivial at every finite place. Since  $\eta$  has to be positive and symmetric, it is automatically trivial at infinity.

Since  $H^1(\mathfrak{A}^*)$  is trivial, every cocycle trivial at infinity is defined by an  $\eta$ . Thus the number of sets  $I(\mathbf{Q}) \setminus G(\mathbf{A}_f^p) \times X$  that we lump together is equal to the number of elements in  $H^1(I)$  that are locally trivial everywhere.

There is a pointed to be noticed. Namely replacing  $\lambda$  by  $\lambda'$  replaces  $I(\overline{\mathbf{Q}})$  by

$$I'(\overline{\mathbf{Q}}) = \left\{ c^{-1} h c \mid h \in I(\overline{\mathbf{Q}}) \right\}.$$

and  $c^{-1} h c$  is rational if and only if

$$\rho(c^{-1}) \rho(h) \rho(c) = c^{-1} h c$$

or

$$D_\rho^{-1} \rho(h) D_\rho = h.$$

Thus  $I'$  is obtained from  $I$  by twisting by the cocycle  $D_\rho$ . By Hasse's theorem, if it is trivial locally then  $\text{Ad } D_\rho$  is trivial. Thus the groups  $I(\mathbf{Q})$  are in fact the same for all the  $(A, \lambda')$  lumped together with  $(A, \lambda)$ . However the imbeddings of  $I(\mathbf{Q})$  in  $G(\mathbf{A}_f^p)$  vary. This I did not stress in the DeKalb talk. In fact, I was not explicitly aware of it, but it is perhaps important.

So far, I have said nothing that was not formal. Now, I want to begin, and immediately difficulties arise. First of all, I want to say that given  $(A, \lambda)$  I can find an isogenous  $A'$ , so that if  $\lambda'$  is defined by

$$\begin{array}{ccc} A' & \xrightarrow{\psi} & A \\ \lambda' \downarrow & & \downarrow \lambda \\ \tilde{A}' & \xleftarrow{\tilde{\psi}} & \tilde{A} \end{array}$$

then  $A', \lambda'$  can be lifted to  $\tilde{A}, \tilde{\lambda}$  over the ring of integers in  $\overline{\mathbf{Q}}_p$ , the algebraic closure of  $\mathbf{Q}_p$ . Then, if I have fixed  $\overline{\mathbf{Q}}_p \subseteq \mathbf{C}$ , this gives me a variety over  $\mathbf{C}$ . Moreover, I want  $\tilde{A}$  to be of CM type, i.e. to contain a commutative endomorphism algebra, and I want that endomorphism algebra to be stable under the involution defined by  $\tilde{\lambda}$ .

I gave in the Rapoport letter an argument for this. But I used a result on deformations which simply does not exist, and is even false, although an approximation may be true. None the less suppose  $\tilde{A}, \tilde{\lambda}$  exist. Since the set associated to  $A', \lambda'$  is the same as that associated to  $A, \lambda$ , I may as well suppose that  $A = A'$  and  $\lambda = \lambda'$ . Thus

$$T^p(\tilde{A}) \leftrightarrow T^p(A)$$

and the identification  $T^p(A) \otimes \mathbf{Q} \leftrightarrow V(\mathbf{A}_f^p)$  may be taken to be that defined by  $T^p(\tilde{A}) \otimes \mathbf{Q} \leftrightarrow V(\mathbf{A}_f^p)$ , which in its turn may be taken to be that provided by an identification of  $H_1(\tilde{A}(\mathbf{C})) \otimes \mathbf{Q}$  with  $V(\mathbf{Q})$  preserving the alternating form. Now  $\gamma$  becomes an endomorphism of  $\tilde{A}$  and hence defines an element of  $G(\mathbf{Q})$ . Moreover there is an associated  $h$ , that defining the Hodge structure of  $H_1$  (or its inverse). This is the pair  $(\gamma, h)$  associated to  $(A, \lambda)$ , or to the associated  $I(\mathbf{Q}) \backslash G(\mathbf{A}_f^p) \times X$ .

It is clear that the conjugacy class of this  $\gamma$  in  $G(\mathbf{A}_f^p)$  is well-defined. If  $(A, \lambda)$  and  $(A', \lambda')$  determine a pair  $\gamma, \gamma'$  which are conjugate in  $G(\mathbf{A}_f^p)$  then by Tate's theorem  $A'$  and  $A$  are isogenous. Thus we have

$$\begin{array}{ccc} A & \xrightarrow{\psi} & A' \\ \lambda'' \downarrow & & \downarrow \lambda' \\ \tilde{A} & \xleftarrow{\tilde{\psi}} & \tilde{A}' \end{array}$$

Replacing  $(A', \lambda')$  by  $(A, \lambda'')$ , we might as well suppose that  $A = A'$ . Then the above discussion shows that  $\lambda' = \lambda \circ \eta$  and that the cocycle determined by  $\eta$  is trivial at all primes different from  $p$ .

We are beginning to see how the conditions of my DeKalb lecture arose, but now we come to the most serious point which, alas!, I passed over too glibly in my letter to Rapoport.

I'm still supposing that  $\tilde{A}, \tilde{\lambda}$  exist. We therefore have two objects on which  $\gamma$  acts.

$$Q = H^1(\tilde{A}(\mathbf{Q})) \otimes k = T_p(\tilde{A}) \otimes_{\mathbf{Z}_p} k \leftrightarrow V(k) = V(\mathbf{Q}_p) \otimes k$$

and  $N$ . They are both provided with a bilinear form, and the associated involutions have the same effect on  $\gamma$ , namely replace it by  $p\gamma^{-1}$ . Thus, there is an isomorphism

$$\psi : N \rightarrow Q = V(\mathbf{Q}_p) \otimes k$$

which preserves the form, and commutes with the action of  $\gamma$ . If  $\sigma$  is the Frobenius on  $k$ , then  $\sigma$  acts on  $N$  in a semi-linear fashion because  $N$  has been obtained from a Dieudonné module. It also acts on  $\mathbf{Q}$  through its action on  $k$ . There is therefore a  $b$  so that

$$\psi(\sigma(x)) = b\sigma(\psi(x)).$$

If we modify  $\psi$  to  $B\psi$  with  $B$  in  $I^0(k)$ , the centralizer of  $\gamma$  in  $G(k)$ , and this is the only way we are allowed to modify it, then  $b$  is replaced by  $Bb\sigma(B^{-1})$ .

It is this  $b$ , up to the indicated ambiguity, that I thought I had found (cf. especially the construction in the app[endix] to the paper on Shimura varieties submitted to the Can. Jour.) However, my argument is not complete, and I am beginning to believe that the problem of identifying it is much deeper than I had originally thought, and perhaps tied up with the questions in my Corvallis talk. Sometimes it can be verified, but I have not yet tried to see exactly when this is easy.

Finally I would like to add a remark which may make the condition at  $p$  clearer. Suppose that  $A, \lambda$  is the reduction of  $\tilde{A}, \tilde{\lambda}$  and  $E$  is a commutative endomorphism algebra of  $\tilde{A}$  stable under the involution defined by  $\tilde{\lambda}$ , or rather the algebra tensored with  $\mathbf{Q}$ .  $E$  acts on  $N$  and on  $Q$  and I may choose  $\psi$  so that it respects the action of  $E$ .

Suppose that  $\eta \in E$  and is positive and symmetric. Then  $\tilde{\lambda} \circ \eta$  is a polarization reducing to  $\lambda \circ \eta$ .

- (a) Suppose  $\langle X, \eta Y \rangle = \langle BX, BY \rangle$  with  $B$  in  $\mathfrak{A}(\mathbf{Q}_p)$ . In other words the cohomology class defined by  $\eta$  is to be trivial at  $p$ . The map  $\psi : N \rightarrow V(\mathbf{Q}_p) \otimes k$  is to [be] replaced by  $B\psi$ . Thus  $\gamma \rightarrow B\gamma B^{-1} = \gamma$  and  $b \rightarrow Bb\sigma(B^{-1})$ . However  $\mathfrak{A}(\mathbf{Q}_p)$  consists precisely of those elements of  $\text{End } N$  which commute with  $\gamma$  and  $\sigma$ , i.e. in terms of its realization  $B$  on  $\mathbf{Q}$

$$b\sigma(B)b^{-1} = B$$

or

$$Bb\sigma(B^{-1}) = b.$$

Thus  $\gamma$  and  $b$  are left unchanged.

- (b) At all events,  $\eta$  comes from an endomorphism of  $\tilde{A}$  and thus its action on  $V$  is defined over  $\mathbf{Q}$ . Hence

$$\langle X, \eta Y \rangle = \langle BX, BY \rangle$$

with  $B$  an endomorphism of  $V(\mathbf{Q})$ . When  $\lambda$  is replaced by  $\lambda'$  then  $\psi$  must be replaced by  $B\psi$ . Thus  $\gamma \rightarrow \gamma' = B\gamma B^{-1}$  and  $b \rightarrow b' = Bb\sigma(B^{-1})$ . However there exists a  $g \in G(k)$  so that  $g\gamma g^{-1} = \gamma'$ . Thus  $g^{-1}B$  commutes with  $\gamma$ . Suppose, and this is the condition for equivalence at  $p$  introduced in the DeKalb lecture,

$$b' = agb\sigma(g^{-1})\sigma(c^{-1})$$

with  $c$  in the centralizer of  $\gamma$  in  $G(k)$ . Then

$$B^{-1}cg\gamma g^{-1}c^{-1}B = \gamma$$

and

$$b = B^{-1}cgb\sigma(g^{-1}c^{-1}B).$$

Thus  $g^{-1}c^{-1}B$  lies in  $\mathfrak{A}(\mathbf{Q}_p)$  and

$$\langle X, \eta Y \rangle = \langle Bx, By \rangle = \alpha \langle g^{-1}c^{-1}BX, g^{-1}c^{-1}BY \rangle.$$

It follows that the cocycle defined by  $\eta$  is trivial in  $I(\mathbf{Q}_p)$ .

Since the other demands of the DeKalb lecture are fairly simple, these remarks may give you some idea of what I had in mind. However I did not have in mind that the Rapoport letter was incomplete.

None the less I am not unhappy, only embarrassed, for the whole complex of problems associated with Shimura varieties is beginning to assume a coherence, and thus an attraction, that it lacked before. But can they be solved? Perhaps, but not before Rennes!

Yours  
Bob.

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