June 11, 1977

Dear Bill,

As I said the Rapoport letter contains gaps. I still hope that my conjectural classification of points mod p is alright, but it lies deeper than I thought. There are two major blunders. I misunderstood the situation in regard to lifting. It is more complicated than I imagined. But a careful study of Fontaine might take care of the difficulties. However I was also too facile and thinking that I had identified the action of the Frobenius. At the moment I don't see any way of dealing with this, even for the standard case of the group of symplectic similitudes. To give you an idea of what I had in mind, let me explain the situation in this case.

Start from an abelian variety A mod p and polarization λ on A. Let

$$T^p(A) = \varprojlim_{(n,p)=1} A_n$$

be the Tate module, and M the Dieudonné module of $\widetilde{A}$. Thus M is a covariant functor. Let $V(\mathbf{Q})$ be the standard $2n$-dimensional space over $\mathbf{Q}$ provided with the standard alternating form, and choose an isomorphism $x \to \psi(x)$ of $T^p(A) \otimes \mathbf{Q}$ with $V(\mathbf{A}_f^p)$ compatible with the bilinear forms on the two spaces. Of course φ is determined up to composition with an element of $G(\mathbf{A}_f^p)$, G being the group of symplectic similitudes. Let N be $M \otimes \mathbf{Q}$.

Whenever we have a lattice $gV(\mathbf{Z}_f^p)$, $g \in G(\mathbf{A}_f^p)$ in $V(\mathbf{A}_f^p)$, and a Dieudonné submodule M' of N whose dual, with respect to the bilinear form defined by the polarization, is a scalar multiple of itself, then we have a complete set of data for the moduli problem.

The associated abelian variety A' is defined by

$$A' \xrightarrow{\psi} A$$

$$T^p(A') \xrightarrow{\psi} T^p(A) \text{ has image } agV(\mathbf{Z}_f^p) \text{ with } a \in \mathbf{Q}^\times$$

$$M(A') \xrightarrow{\psi} M(A) \text{ has image } cM' \text{ with the same } a.$$

In other words there is an isogeny $A' \xrightarrow{\psi} A$ whose associated image in the Tate module or Dieudonné module has the indicated images. The polarization λ' in A' is that defined by the commutativity of

$$\begin{array}{ccc} A' & \xrightarrow{\psi} & A \\ \lambda' \downarrow & & \downarrow \lambda \\ \widetilde{A}' & \xleftarrow[\widetilde{\psi}]{} & \widetilde{A} \end{array}$$

The identification of $T^p(A')$ with $V(\mathbf{Z}_f^p)$ is obtained by composition

$$T^p(A') \longrightarrow T^p(A) \longleftrightarrow V(\mathbf{A}_f^p) \xrightarrow{g^{-1}a^{-1}} V(\mathbf{A}_f^p) \quad .$$

The first question to ask is when g_1, M_1' and g_2, M_2' define isomorphic data. If $\mathfrak{A} = \operatorname{End} A \otimes \mathbf{Q}$ then an element of $\operatorname{Hom}(A_1', A_2')$ is simply an element $a \in \mathfrak{A}$ such that

$$a g_1 V(\mathbf{Z}_f^p) \subseteq g_2 V(\mathbf{Z}_f^p)$$

and

$$a M_1' \subseteq M_2'.$$

It is an isomorphism if both relations are equalities. Note that we regard $\mathfrak{A}$ as acting on $V(\mathbf{A}_f^p)$, by means of our identification, and on N. If $a \to \widetilde{a}$ is the involution on $\mathfrak{A}$ defined by the polarization then the associated isomorphism $A_1' \to A_2'$ takes λ_1' to λ_2' if and only if $\widetilde{a}a \in \mathbf{Q}^\times$. This equation defines a group $I(\mathbf{Q})$.

Consequently if X is the set of all Dieudonné submodules of N which are multiples of their duals then the set of points in our moduli space obtained from A, λ is

$$I(\mathbf{Q})\backslash G(\mathbf{A}_f^p) \times X,$$

where $I(\mathbf{Q})$ is imbedded in $G(\mathbf{A}_f^p)$ in the way indicated, and acts on X through its action on N. Of course, for the true moduli problem one has to divide by an open compact subgroup of $G(\mathbf{A}_f^p)$, but the discussion is, I hope more transparent if one passes to the limit.

Notice that a high power of the Frobenius lies in $I(\mathbf{Q})$. Call it γ. Via the imbedding $I(\mathbf{Q}) \hookrightarrow G(\mathbf{A}_f^p)$, γ defines an element of $G(\mathbf{A}_f^p)$. Since $T^p(A) \otimes \mathbf{Q} \to V(\mathbf{A}_f^p)$ is not uniquely fixed, it is only the conjugacy class of γ which is uniquely determined.

Suppose we start from the same abelian variety A but another polarization λ'. Then there is a symmetric, positive element $\eta \in \mathfrak{A}$ such that $\lambda' = \lambda \circ \eta$. We want to examine the conjugacy class associated to (A, λ'), and discover its relation to $\{\gamma\}$.

In $\mathfrak{A}(\overline{\mathbf{Q}})$

$$\eta = \widetilde{c}c.$$

Since, for any element ρ of $\operatorname{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$,

$$\eta = \rho(\eta) = \rho(\widetilde{c})\rho(c)$$

the cocycle $D_\rho = \rho(c)c^{-1}$ satisfies $\widetilde{D}_\rho D_\rho = 1$ and therefore lies in $I(\mathbf{Q})$. I claim that it is trivial over $\mathbf{A}_f^p$ if and only if the conjugacy class $\{\gamma'\}$ associated to A, λ' is the same as $\{\gamma\}$.

We have fixed φ so that

$$\{x, \lambda y\} = \alpha\big\langle \varphi(x), \varphi(y) \big\rangle$$

for some $\alpha \in I_f^p$. Thus

$$\{x, \lambda' y\} = \langle x, \lambda\eta y\rangle = \alpha\big\langle \varphi(x), \psi(\eta y)\big\rangle = \alpha\big\langle \psi(x), \eta\psi(y)\big\rangle.$$

For the last equality, we have identified η in $\mathfrak{A}$ with its image in $\operatorname{End}\Big(V(\mathbf{A}_f^p)\Big)$. There is a B in $\operatorname{End}\Big(V(\mathbf{A}_f^p)\Big)$ such that

$$\langle X, \eta Y\rangle = \langle Bx, By\rangle.$$

If λ is replaced by λ' then φ can be replaced by $\varphi' = B\varphi$, and γ by $\gamma' = B\gamma B^{-1}$. This is conjugate to $\gamma \iff$ there is a g in $G(\mathbf{A}_f)$ with gB centralising γ, i.e. by Tate's theorem, in $\mathfrak{A}(\mathbf{A}_f^p)$. Then

$$\langle X, \eta Y\rangle = \alpha'\langle B_1 X, B_1 Y\rangle \qquad \alpha' \in I_f^p.$$

In conclusion, γ and γ' are conjugate $\iff \eta = \alpha' \widetilde{B}_1 B_1$ with $\widetilde{B}_1$ in $\mathfrak{A}(\mathbf{A}_f^p)$.

However if

$$\eta = \alpha\tilde{c}c = \alpha_1\tilde{c}_1 c_1$$

with c, $c_1 \in \mathfrak{A}(\overline{\mathbf{Q}})$ then $cc_1^{-1} \in I(\overline{\mathbf{Q}})$ and

$$\rho(c_1c^{-1})\rho(c)c^{-1}(cc_1^{-1}) = \rho(c_1)c_1^{-1}.$$

This shows that the cocycle is well-defined, and that it is trivial over $\mathbf{A}_f^p$ if and only if the B_1 above exists.

We are going to put the set corresponding to (A,λ) and (A,λ') if the cocycle defined by η is trivial at every finite place. Since η has to be positive and symmetric, it is automatically trivial at infinity.

Since $H^1(\mathfrak{A}^*)$ is trivial, every cocycle trivial at infinity is defined by an η. Thus the number of sets $I(\mathbf{Q})\backslash G(\mathbf{A}_f^p) \times X$ that we lump together is equal to the number of elements in $H^1(I)$ that are locally trivial everywhere.

There is a pointed to be noticed. Namely replacing λ by λ' replaces $I(\overline{\mathbf{Q}})$ by

$$I'(\overline{\mathbf{Q}}) = \left\{\, c^{-1}hc \;\middle|\; h \in I(\overline{\mathbf{Q}}) \,\right\}.$$

and $c^{-1}hc$ is rational if and only if

$$\rho(c^{-1})\rho(h)\rho(c) = c^{-1}hc$$

or

$$D_\rho^{-1}\rho(h)D_\rho = h.$$

Thus I' is obtained from I by twisting by the cocycle D_ρ. By Hasse's theorem, if it is trivial locally then $\operatorname{Ad} D_\rho$ is trivial. Thus the groups $I(\mathbf{Q})$ are in fact the same for all the (A,λ') lumped together with (A,λ). However the imbeddings of $I(\mathbf{Q})$ in $G(\mathbf{A}_f^D)$ vary. This I did not stress in the DeKalb talk. In fact, I was not explicitly aware of it, but it is perhaps important.

So far, I have said nothing that was not formal. Now, I want to begin, and immediately difficulties arise. First of all, I want to say that given (A,λ) I can find an isogenous A', so that if λ' is defined by

$$\begin{array}{ccc} A' & \xrightarrow{\ \psi\ } & A \\ {\scriptstyle\lambda'}\big\downarrow & & \big\downarrow{\scriptstyle\lambda} \\ \widetilde{A}' & \xleftarrow[\ \widetilde{\psi}\ ]{} & \widetilde{A} \end{array}$$

then A', λ' can be lifted to $\widetilde{A}$, $\widetilde{\lambda}$ over the ring of integers in $\overline{\mathbf{Q}}_p$, the algebraic closure of $\mathbf{Q}_p$. Then, if I have fixed $\overline{\mathbf{Q}}_p \subseteq \mathbf{C}$, this gives me a variety over $\mathbf{C}$. Moreover, I want $\widetilde{A}$ to be of CM type, i.e. to contain a commutative endomorphism algebra, and I want that endomorphism algebra to be stable under the involution defined by $\widetilde{\lambda}$.

I gave in the Rapoport letter an argument for this. But I used a result on deformations which simply does not exist, and is even false, although an approximation may be true. None the less suppose $\widetilde{A}$, $\widetilde{\lambda}$ exist. Since the set associated to A', λ' is the same as that associated to A, λ, I may as well suppose that $A = A'$ and $\lambda = \lambda'$. Thus

$$T^p(\widetilde{A}) \leftrightarrow T^p(A)$$

and the identification $T^p(A)\otimes\mathbf{Q}\leftrightarrow V(\mathbf{A}_f^p)$ may be taken to be that defined by $T^p(\widetilde{A})\otimes\mathbf{Q}\leftrightarrow V(\mathbf{A}_f^p)$, which in its turn may be taken to be that provided by an identification of $H_1\big(\widetilde{A}(\mathbf{C})\big)\otimes\mathbf{Q}$ with $V(\mathbf{Q})$ preserving the alternating form. Now γ becomes an endomorphism of $\widetilde{A}$ and hence defines an element of $G(\mathbf{Q})$. Moreover there is an associated h, that defining the Hodge structure of H_1 (or its inverse). This is the pair (γ,h) associated to (A,λ), or to the associated $I(\mathbf{Q})\backslash G(\mathbf{A}_f^p)\times X$.

It is clear that the conjugacy class of this γ in $G(\mathbf{A}_f^p)$ is well-defined. If (A,λ) and (A',λ') determine a pair γ, γ' which are conjugate in $G(\mathbf{A}_f^p)$ then by Tate's theorem A' and A are isogenous. Thus we have

$$\begin{array}{ccc} A & \xrightarrow{\ \psi\ } & A' \\ {\scriptstyle\lambda''}\big\downarrow & & \big\downarrow{\scriptstyle\lambda'} \\ \widetilde{A} & \xleftarrow[\ \widetilde{\psi}\ ]{} & \widetilde{A}' \end{array}$$

Replacing (A',λ') by (A,λ''), we might as well suppose that $A=A'$. Then the above discussion shows that $\lambda'=\lambda\circ\eta$ and that the cocycle determined by η is trivial at all primes different from p.

We are beginning to see how the conditions of my DeKalb lecture arose, but now we come to the most serious point which, alas!, I passed over two glibly in my letter to Rapoport.

I'm still supposing that $\widetilde{A}$, $\widetilde{\lambda}$ exist. We therefore have two objects on which γ acts.

$$Q=H^1\big(\widetilde{A}(\mathbf{Q})\big)\otimes k=T_p(\widetilde{A})\otimes_{\mathbf{Z}_p}k\leftrightarrow V(k)=V(\mathbf{Q}_p)\otimes k$$

and N. They are both provided with a bilinear form, and the associated involutions have the same effect on γ, namely replace it by $p\gamma^{-1}$. Thus, there is an isomorphism

$$\psi: N\to Q=V(\mathbf{Q}_p)\otimes k$$

which preserves the form, and commutes with the action of γ. If σ is the Frobenius on k, then σ acts on N in a semi-linear fashion because N has been obtained from a Dieudonné module. It also acts on $\mathbf{Q}$ through its action on k. There is therefore a b so that

$$\psi\big(\sigma(x)\big)=b\sigma\big(\psi(x)\big).$$

If we modify ψ to $B\psi$ with B in $I^0(k)$, the centralizer of γ in $G(k)$, and this is the only way we are allowed to modify it, then b is replaced by $Bb\sigma(B^{-1})$.

It is this b, up to the indicated ambiguity, that I thought I had found (cf. especially the construction in the app[endix] to the paper on Shimura varieties submitted to the Can. Jour.) However, my argument is not complete, and I am beginning to believe that the problem of identifying it is much deeper than I had originally thought, and perhaps tied up with the questions in my Corvallis talk. Sometimes it can be verified, but I have not yet tried to see exactly when this is easy.

Finally I would like to add a remark which may make the condition at p clearer. Suppose that A, λ is the reduction of $\widetilde{A}$, $\widetilde{\lambda}$ and E is a commutative endomorphism algebra of $\widetilde{A}$ stable under the involution defined by $\widetilde{\lambda}$, or rather the algebra tensored with $\mathbf{Q}$. E acts on N and on Q and I may choose ψ so that it respects the action of E.

Suppose that $\eta\in E$ and is positive and symmetric. Then $\widetilde{\lambda}\circ\eta$ is a polarization reducing to $\lambda\circ\eta$.

(a) Suppose $\langle X, \eta Y\rangle = \langle BX, BY\rangle$ with B in $\mathfrak{A}(\mathbf{Q}_p)$. In other words the cohomology class defined by η is to be trivial at p. The map $\psi : N \to V(\mathbf{Q}_p) \otimes k$ is to [be] replaced by $B\psi$. Thus $\gamma \to B\gamma B^{-1} = \gamma$ and $b \to Bb\sigma(B^{-1})$. However $\mathfrak{A}(\mathbf{Q}_p)$ consists precisely of those elements of End N which commute with γ and σ, i.e. in terms of its realization B on $\mathbf{Q}$

$$b\sigma(B)b^{-1} = B$$

or

$$Bb\sigma(B^{-1}) = b.$$

Thus γ and b are left unchanged.

(b) At all events, η comes from an endomorphism of $\widetilde{A}$ and thus its action on V is defined over $\mathbf{Q}$. Hence

$$\langle X, \eta Y\rangle = \langle BX, BY\rangle$$

with B an endomorphism of $V(\mathbf{Q})$. When λ is replaced by λ' then ψ must be replaced by $B\psi$. Thus $\gamma \to \gamma' = B\gamma B^{-1}$ and $b \to b' = Bb\sigma(B^{-1})$. However there exists a $g \in G(k)$ so that $g\gamma g^{-1} = \gamma'$. Thus $g^{-1}B$ commutes with γ. Suppose, and this is the condition for equivalence at p introduced in the DeKalb lecture,

$$b' = agb\sigma(g^{-1})\sigma(c^{-1})$$

with c in the centralizer of γ in $G(k)$. Then

$$B^{-1}cg\gamma g^{-1}c^{-1}B = \gamma$$

and

$$b = B^{-1}cgb\sigma(g^{-1}c^{-1}B).$$

Thus $g^{-1}c^{-1}B$ lies in $\mathfrak{A}(\mathbf{Q}_p)$ and

$$\langle X, \eta Y\rangle = \langle Bx, By\rangle = \alpha\langle g^{-1}c^{-1}BX, g^{-1}c^{-1}BY\rangle.$$

It follows at the cocycle defined by η is trivial in $I(\mathbf{Q}_p)$.

Since the other demands of the DeKalb lecture are fairly simple, these remarks may give you some idea of what I had in mind. However I did not have in mind that the Rapoport letter was incomplete.

None the less I am not unhappy, only embarrassed, for the whole complex of problems associated with Shimura varieties is beginning to assume a coherence, and thus an attraction, that it lacked before. But can they be solved? Perhaps, but not before Rennes!

Yours
Bob.

Compiled on February 14, 2025.