

June 11, 1977

Dear Bill,

As I said the Rapoport letter contains gaps. I still hope that my conjectural classification of points mod p is alright, but it lies deeper than I thought. There are two major blunders. I misunderstood the situation in regard to lifting. It is more complicated than I imagined. But a careful study of Fontaine might take care of the difficulties. However I was also too facile and thinking that I had identified the action of the Frobenius. At the moment I don't see any way of dealing with this, even for the standard case of the group of symplectic similitudes. To give you an idea of what I had in mind, let me explain the situation in this case.

Start from an abelian variety A mod p and polarization λ on A . Let

$$T^p(A) = \varprojlim_{(n,p)=1} A_n$$

be the Tate module, and M the Dieudonné module of $\tilde{A}$. Thus M is a covariant functor. Let $V(\mathbf{Q})$ be the standard $2n$ -dimensional space over $\mathbf{Q}$ provided with the standard alternating form, and choose an isomorphism $x \rightarrow \psi(x)$ of $T^p(A) \otimes \mathbf{Q}$ with $V(\mathbf{A}_f^p)$ compatible with the bilinear forms on the two spaces. Of course φ is determined up to composition with an element of $G(\mathbf{A}_f^p)$, G being the group of symplectic similitudes. Let N be $M \otimes \mathbf{Q}$.

Whenever we have a lattice $gV(\mathbf{Z}_f^p)$, $g \in G(\mathbf{A}_f^p)$ in $V(\mathbf{A}_f^p)$, and a Dieudonné submodule M' of N whose dual, with respect to the bilinear form defined by the polarization, is a scalar multiple of itself, then we have a complete set of data for the moduli problem.

The associated abelian variety A' is defined by

$$A' \xrightarrow{\psi} A$$

$$T^p(A') \xrightarrow{\psi} T^p(A) \text{ has image } agV(\mathbf{Z}_f^p) \text{ with } a \in \mathbf{Q}^\times$$

$$M(A') \xrightarrow{\psi} M(A) \text{ has image } cM' \text{ with the same } a.$$

In other words there is an isogeny $A' \xrightarrow{\psi} A$ whose associated image in the Tate module or Dieudonné module has the indicated images. The polarization λ' in A' is that defined by the commutativity of

$$\begin{array}{ccc} A' & \xrightarrow{\psi} & A \\ \lambda' \downarrow & & \downarrow \lambda \\ \tilde{A}' & \xleftarrow{\tilde{\psi}} & \tilde{A} \end{array}$$

The identification of $T^p(A')$ with $V(\mathbf{Z}_f^p)$ is obtained by composition

$$T^p(A') \longrightarrow T^p(A) \longleftarrow V(\mathbf{A}_f^p) \xrightarrow{g^{-1}a^{-1}} V(\mathbf{A}_f^p) \quad .$$

The first question to ask is when g_1, M'_1 and g_2, M'_2 define isomorphic data. If $\mathfrak{A} = \text{End } A \otimes \mathbf{Q}$ then an element of $\text{Hom}(A'_1, A'_2)$ is simply an element $a \in \mathfrak{A}$ such that

$$ag_1V(\mathbf{Z}_f^p) \subseteq g_2V(\mathbf{Z}_f^p)$$

and

$$aM'_1 \subseteq M'_2.$$

It is an isomorphism if both relations are equalities. Note that we regard $\mathfrak{A}$ as acting on $V(\mathbf{A}_f^p)$, by means of our identification, and on N . If $a \rightarrow \tilde{a}$ is the involution on $\mathfrak{A}$ defined by the polarization then the associated isomorphism $A'_1 \rightarrow A'_2$ takes λ'_1 to λ'_2 if and only if $\tilde{a}a \in \mathbf{Q}^\times$. This equation defines a group $I(\mathbf{Q})$.

Consequently if X is the set of all Dieudonné submodules of N which are multiples of their duals then the set of points in our moduli space obtained from A, λ is

$$I(\mathbf{Q}) \backslash G(\mathbf{A}_f^p) \times X,$$

where $I(\mathbf{Q})$ is imbedded in $G(\mathbf{A}_f^p)$ in the way indicated, and acts on X through its action on N . Of course, for the true moduli problem one has to divide by an open compact subgroup of $G(\mathbf{A}_f^p)$, but the discussion is, I hope more transparent if one passes to the limit.

Notice that a high power of the Frobenius lies in $I(\mathbf{Q})$. Call it γ . Via the imbedding $I(\mathbf{Q}) \hookrightarrow G(\mathbf{A}_f^p)$, γ defines an element of $G(\mathbf{A}_f^p)$. Since $T^p(A) \otimes \mathbf{Q} \rightarrow V(\mathbf{A}_f^p)$ is not uniquely fixed, it is only the conjugacy class of γ which is uniquely determined.

Suppose we start from the same abelian variety A but another polarization λ' . Then there is a symmetric, positive element $\eta \in \mathfrak{A}$ such that $\lambda' = \lambda \circ \eta$. We want to examine the conjugacy class associated to (A, λ') , and discover its relation to $\{\gamma\}$.

In $\mathfrak{A}(\mathbf{Q})$

$$\eta = \tilde{c}c.$$

Since, for any element ρ of $\text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$,

$$\eta = \rho(\eta) = \rho(\tilde{c})\rho(c)$$

the cocycle $D_\rho = \rho(c)c^{-1}$ satisfies $\tilde{D}_\rho D_\rho = 1$ and therefore lies in $I(\mathbf{Q})$. I claim that it is trivial over $\mathbf{A}_f^p$ if and only if the conjugacy class $\{\gamma'\}$ associated to A, λ' is the same as $\{\gamma\}$.

We have fixed φ so that

$$\{x, \lambda y\} = \alpha \langle \varphi(x), \varphi(y) \rangle$$

for some $\alpha \in I_f^p$. Thus

$$\{x, \lambda' y\} = \langle x, \lambda \eta y \rangle = \alpha \langle \varphi(x), \psi(\eta y) \rangle = \alpha \langle \psi(x), \eta \psi(y) \rangle.$$

For the last equality, we have identified η in $\mathfrak{A}$ with its image in $\text{End}(V(\mathbf{A}_f^p))$. There is a B in $\text{End}(V(\mathbf{A}_f^p))$ such that

$$\langle X, \eta Y \rangle = \langle Bx, By \rangle.$$

If λ is replaced by λ' then φ can be replaced by $\varphi' = B\varphi$, and γ by $\gamma' = B\gamma B^{-1}$. This is conjugate to $\gamma \iff$ there is a g in $G(\mathbf{A}_f)$ with gB centralising γ , i.e., by Tate's theorem, in $\mathfrak{A}(\mathbf{A}_f^p)$. Then

$$\langle X, \eta Y \rangle = \alpha' \langle B_1 X, B_1 Y \rangle \quad \alpha' \in I_f^p.$$

In conclusion, γ and γ' are conjugate $\iff \eta = \alpha' \tilde{B}_1 B_1$ with $\tilde{B}_1$ in $\mathfrak{A}(\mathbf{A}_f^p)$.

However if

$$\eta = \alpha \tilde{c} c = \alpha_1 \tilde{c}_1 c_1$$

with $c, c_1 \in \mathfrak{A}(\overline{\mathbf{Q}})$ then $cc_1^{-1} \in I(\overline{\mathbf{Q}})$ and

$$\rho(c_1 c^{-1}) \rho(c) c^{-1} (cc_1^{-1}) = \rho(c_1) c_1^{-1}.$$

This shows that the cocycle is well-defined, and that it is trivial over $\mathbf{A}_f^p$ if and only if the B_1 above exists.

We are going to put the set corresponding to (A, λ) and (A, λ') if the cocycle defined by η is trivial at every finite place. Since η has to be positive and symmetric, it is automatically trivial at infinity.

Since $H^1(\mathfrak{A}^*)$ is trivial, every cocycle trivial at infinity is defined by an η . Thus the number of sets $I(\mathbf{Q}) \setminus G(\mathbf{A}_f^p) \times X$ that we lump together is equal to the number of elements in $H^1(I)$ that are locally trivial everywhere.

There is a pointed to be noticed. Namely replacing λ by λ' replaces $I(\overline{\mathbf{Q}})$ by

$$I'(\overline{\mathbf{Q}}) = \left\{ c^{-1} h c \mid h \in I(\overline{\mathbf{Q}}) \right\}.$$

and $c^{-1} h c$ is rational if and only if

$$\rho(c^{-1}) \rho(h) \rho(c) = c^{-1} h c$$

or

$$D_\rho^{-1} \rho(h) D_\rho = h.$$

Thus I' is obtained from I by twisting by the cocycle D_ρ . By Hasse's theorem, if it is trivial locally then $\text{Ad } D_\rho$ is trivial. Thus the groups $I(\mathbf{Q})$ are in fact the same for all the (A, λ') lumped together with (A, λ) . However the imbeddings of $I(\mathbf{Q})$ in $G(\mathbf{A}_f^D)$ vary. This I did not stress in the DeKalb talk. In fact, I was not explicitly aware of it, but it is perhaps important.

So far, I have said nothing that was not formal. Now, I want to begin, and immediately difficulties arise. First of all, I want to say that given (A, λ) I can find an isogenous A' , so that if λ' is defined by

$$\begin{array}{ccc} A' & \xrightarrow{\psi} & A \\ \lambda' \downarrow & & \downarrow \lambda \\ \tilde{A}' & \xleftarrow[\tilde{\psi}]{} & \tilde{A} \end{array}$$

then A', λ' can be lifted to $\tilde{A}, \tilde{\lambda}$ over the ring of integers in $\overline{\mathbf{Q}}_p$, the algebraic closure of $\mathbf{Q}_p$. Then, if I have fixed $\overline{\mathbf{Q}}_p \subseteq \mathbf{C}$, this gives me a variety over $\mathbf{C}$. Moreover, I want $\tilde{A}$ to be of CM type, i.e., to contain a commutative endomorphism algebra, and I want that endomorphism algebra to be stable under the involution defined by $\tilde{\lambda}$.

I gave in the Rapoport letter an argument for this. But I used a result on deformations which simply does not exist, and is even false, although an approximation may be true. None the less suppose $\tilde{A}, \tilde{\lambda}$ exist. Since the set associated to A', λ' is the same as that associated to A, λ , I may as well suppose that $A = A'$ and $\lambda = \lambda'$. Thus

$$T^p(\tilde{A}) \leftrightarrow T^p(A)$$

and the identification $T^p(A) \otimes \mathbf{Q} \leftrightarrow V(\mathbf{A}_f^p)$ may be taken to be that defined by $T^p(\tilde{A}) \otimes \mathbf{Q} \leftrightarrow V(\mathbf{A}_f^p)$, which in its turn may be taken to be that provided by an identification of $H_1(\tilde{A}(\mathbf{C})) \otimes \mathbf{Q}$ with $V(\mathbf{Q})$ preserving the alternating form. Now γ becomes an endomorphism of $\tilde{A}$ and hence defines an element of $G(\mathbf{Q})$. Moreover there is an associated h , that defining the Hodge structure of H_1 (or its inverse). This is the pair (γ, h) associated to (A, λ) , or to the associated $I(\mathbf{Q}) \backslash G(\mathbf{A}_f^p) \times X$.

It is clear that the conjugacy class of this γ in $G(\mathbf{A}_f^p)$ is well-defined. If (A, λ) and (A', λ') determine a pair γ, γ' which are conjugate in $G(\mathbf{A}_f^p)$ then by Tate's theorem A' and A are isogenous. Thus we have

$$\begin{array}{ccc} A & \xrightarrow{\psi} & A' \\ \lambda'' \downarrow & & \downarrow \lambda' \\ \tilde{A} & \xleftarrow[\tilde{\psi}]{} & \tilde{A}' \end{array}$$

Replacing (A', λ') by (A, λ'') , we might as well suppose that $A = A'$. Then the above discussion shows that $\lambda' = \lambda \circ \eta$ and that the cocycle determined by η is trivial at all primes different from p .

We are beginning to see how the conditions of my DeKalb lecture arose, but now we come to the most serious point which, alas!, I passed over too glibly in my letter to Rapoport.

I'm still supposing that $\tilde{A}, \tilde{\lambda}$ exist. We therefore have two objects on which γ acts.

$$Q = H^1(\tilde{A}(\mathbf{Q})) \otimes k = T_p(\tilde{A}) \otimes_{\mathbf{Z}_p} k \leftrightarrow V(k) = V(\mathbf{Q}_p) \otimes k$$

and N . They are both provided with a bilinear form, and the associated involutions have the same effect on γ , namely replace it by $p\gamma^{-1}$. Thus, there is an isomorphism

$$\psi : N \rightarrow Q = V(\mathbf{Q}_p) \otimes k$$

which preserves the form, and commutes with the action of γ . If σ is the Frobenius on k , then σ acts on N in a semi-linear fashion because N has been obtained from a Dieudonné module. It also acts on $\mathbf{Q}$ through its action on k . There is therefore a b so that

$$\psi(\sigma(x)) = b\sigma(\psi(x)).$$

If we modify ψ to $B\psi$ with B in $I^0(k)$, the centralizer of γ in $G(k)$, and this is the only way we are allowed to modify it, then b is replaced by $Bb\sigma(B^{-1})$.

It is this b , up to the indicated ambiguity, that I thought I had found (cf. especially the construction in the app[endix] to the paper on Shimura varieties submitted to the Can. Jour.) However, my argument is not complete, and I am beginning to believe that the problem of identifying it is much deeper than I had originally thought, and perhaps tied up with the questions in my Corvallis talk. Sometimes it can be verified, but I have not yet tried to see exactly when this is easy.

Finally I would like to add a remark which may make the condition at p clearer. Suppose that A, λ is the reduction of $\tilde{A}, \tilde{\lambda}$ and E is a commutative endomorphism algebra of $\tilde{A}$ stable under the involution defined by $\tilde{\lambda}$, or rather the algebra tensored with $\mathbf{Q}$. E acts on N and on Q and I may choose ψ so that it respects the action of E .

Suppose that $\eta \in E$ and is positive and symmetric. Then $\tilde{\lambda} \circ \eta$ is a polarization reducing to $\lambda \circ \eta$.

- (a) Suppose $\langle X, \eta Y \rangle = \langle BX, BY \rangle$ with B in $\mathfrak{A}(\mathbf{Q}_p)$. In other words the cohomology class defined by η is to be trivial at p . The map $\psi : N \rightarrow V(\mathbf{Q}_p) \otimes k$ is to [be] replaced by $B\psi$. Thus $\gamma \rightarrow B\gamma B^{-1} = \gamma$ and $b \rightarrow Bb\sigma(B^{-1})$. However $\mathfrak{A}(\mathbf{Q}_p)$ consists precisely of those elements of $\text{End } N$ which commute with γ and σ , i.e., in terms of its realization B on $\mathbf{Q}$

$$b\sigma(B)b^{-1} = B$$

or

$$Bb\sigma(B^{-1}) = b.$$

Thus γ and b are left unchanged.

- (b) At all events, η comes from an endomorphism of $\tilde{A}$ and thus its action on V is defined over $\mathbf{Q}$. Hence

$$\langle X, \eta Y \rangle = \langle BX, BY \rangle$$

with B an endomorphism of $V(\mathbf{Q})$. When λ is replaced by λ' then ψ must be replaced by $B\psi$. Thus $\gamma \rightarrow \gamma' = B\gamma B^{-1}$ and $b \rightarrow b' = Bb\sigma(B^{-1})$. However there exists a $g \in G(k)$ so that $g\gamma g^{-1} = \gamma'$. Thus $g^{-1}B$ commutes with γ . Suppose, and this is the condition for equivalence at p introduced in the DeKalb lecture,

$$b' = agb\sigma(g^{-1})\sigma(c^{-1})$$

with c in the centralizer of γ in $G(k)$. Then

$$B^{-1}cg\gamma g^{-1}c^{-1}B = \gamma$$

and

$$b = B^{-1}cgb\sigma(g^{-1}c^{-1}B).$$

Thus $g^{-1}c^{-1}B$ lies in $\mathfrak{A}(\mathbf{Q}_p)$ and

$$\langle X, \eta Y \rangle = \langle Bx, By \rangle = \alpha \langle g^{-1}c^{-1}BX, g^{-1}c^{-1}BY \rangle.$$

It follows at the cocycle defined by η is trivial in $I(\mathbf{Q}_p)$.

Since the other demands of the DeKalb lecture are fairly simple, these remarks may give you some idea of what I had in mind. However I did not have in mind that the Rapoport letter was incomplete.

None the less I am not unhappy, only embarrassed, for the whole complex of problems associated with Shimura varieties is beginning to assume a coherence, and thus an attraction, that it lacked before. But can they be solved? Perhaps, but not before Rennes!

Yours
Bob.

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