June 15, 1978

Dear Bill,

Reciting poetry lulled Robert Frost to sleep, but composing a letter to you had the opposite effect on me. At all counts let me describe two typical cases, one in which it can easily be shown that b has the required form, and another in which this is not so clear.

I start from a commutative semi-simple algebra E of dimension $2n$, an involution $E \to \widetilde{E}$, and a skew-symmetric element α. Then

$$\operatorname{tr}_{E/\mathbf{Q}} x\alpha\widetilde{y}$$

is an alternating form on E. I fix an isomorphism of E with the standard symplectic space $V(\mathbf{Q})$, compatible with the alternating forms. I suppose as well that I have h, defining a complex structure on $V(\mathbf{C})$, compatible with the action of E and the symplectic form. Then any lattice L in E satisfying

$$\widetilde{L} = \left\{ x \mid \langle x, y\rangle \in \mathbf{Z} \ \forall y \in L \right\} = aL$$

for some a in $\mathbf{Q}$ defines an abelian variety $\widetilde{A}$ with principal polarization $\widetilde{\lambda}$, and I can reduce it modulo p to obtain A and λ.

As before, $N = M(A) \otimes k$ and

$$Q = V(\mathbf{Q}) \times k$$

and we fix an isomorphism $\psi : N \to Q$ which is compatible with the actions of E and the bilinear forms. As in the previous letter, this defines b.

I begin with a remark. Suppose η in E is positive and symmetric, and we replace α by α_η. If

$$\langle x, \eta y\rangle = \langle Bx, By\rangle$$

for some B in $\operatorname{End} V(\mathbf{Q})$ then we have to replace ψ by $B\psi$. This replaces γ by $B\gamma B^{-1}$ and b by BbB^{-1}, and of course the CSG T of G is replaced by BTB^{-1}, and h by BhB^{-1}. It is therefore clear that the conjectural form of b is correct for $\widetilde{A}$, $\widetilde{\lambda}$ it is also correct for $\widetilde{A}$, $\widetilde{\lambda} \circ \eta$.

Thus I can assume the following: If O_E is the ring of integers in E then

$$\left\{ x \mid \langle x, y\rangle \in \mathbf{Z}_p \ \forall y \in O_E \right\} = O_E \otimes \mathbf{Z}_p.$$

In other words, I can choose L so that $L_p = O_E \otimes \mathbf{Z}_p$. This gives me considerable control over the Dieudonné module $M(A)$ of A. It will be invariant under $O_E \otimes \mathbf{Z}_p = O_p$ and under $O_p \otimes W$, if W is the ring of Witt vectors.

a) Suppose E is unramified at p. Then

$$O_p \otimes W = \overbrace{W \oplus \cdots \oplus W}^{2n \text{ times}}$$

and

$$M(A) = M'_1 \oplus \cdots \oplus M'_{2n}.$$

More precisely, if
$$O_p = O_1 \oplus \cdots \oplus O_r$$
we first write
$$M(A) = \bigoplus M_i \qquad M_i = O_i M(A).$$
Then
$$\alpha \otimes \beta \to \alpha\beta, \alpha\sigma(\beta), \ldots, \alpha\sigma^{n-1}(\beta) \qquad n = [O_p : \mathbf{Z}_p]$$
identifies
$$O_i \otimes W$$
with
$$W \oplus \cdots \oplus W.$$
and
$$M_i \leftrightarrow W \oplus \cdots \oplus W$$
Thus $\mathbf{F}$ acts on the Dieudonné module M_i as
$$(x_1, \ldots, x_n) \to \big(a_1x_2, a_2x_3, \ldots, a_{n-1}x_n, a_n\sigma^n(x_2)\big)$$
A little reflection shows that the collections of a_j, one for each i, define an element of $T(k)$, add to that the element is the b defined as m in the previous letter.

If my calculations are correct, and one takes the Dieudonneé module as I indicated, namely so that it is the usual Dieudonné module of the dual variety, then the tangent space to A is $M(A)/FM(A)$. Because $\langle Fx, Fy\rangle = p\langle x, y\rangle$, the a_i are units or units times p.

Recall that E has $2n$ distinct homomorphisms to $\overline{\mathbf{Q}}$ and that the coweight μ can be regarded as a function which assigns the value 0 or 1 to each of these homomorphisms according as it does or does not occur in the action on the tangent space to $\widetilde{A}$. (There is room here for confusion of signs, and of interchanges of 0's and 1's!). However because there is no ramification, I can determine the action on the tangent space to $\widetilde{A}$ by examining the action of $O \otimes \mathbf{F}_p$ on the tangent space to A. (Notice that our choice of L allows $O_E \otimes \mathbf{F}_p$ to act). Namely there is a 1-1 correspondence between $E \to \overline{\mathbf{Q}}$ and $O_E \otimes \mathbf{F}_p \to \overline{\mathbf{F}}_p$.

For a given i, the jth component of M^i contributes nothing to the tangent space of A if a_j is a unit. Otherwise it contributes one copy of $\overline{\mathbf{F}}_p$, ie a one-dimensional subspace. The corresponding $O_E \otimes \mathbf{F}_p \to \overline{\mathbf{F}}_p$ is that obtained from
$$\alpha \to \sigma^{-j}(\alpha)$$
taking

$O_i \to W$. Notice that to describe M_i we have fixed $O_i \hookrightarrow W$.

Another way of saying the above is that if λ is a weight of T then
$$\big|\lambda(b)\big| = |p|^{\langle \mu, \lambda\rangle}$$
(I apologize for the allusive style of this letter, but without developing an elaborate notation, I see no other way of expressing myself.)

On the other hand, compute according to the recipe of the appendix of the paper on Shimura varieties submitted to the Canadian Journal. Choose the k_p of that letter to be unramified, as we may because E is unramified at p, and take $w = (1, \sigma)$ where σ is the Frobenius. We may suppose
$$a_{\sigma^i, \sigma} = \begin{cases} 1 & 0 \leqslant i < [k_p : \mathbf{Q}_p] - 1 \\ p & i = [k_p : \mathbf{Q}_p] - 1 \end{cases}.$$

This gives

$$b_w = p^\mu \qquad (\mu = \mu^\nu).$$

Since there is no cohomology in the maximal compact subgroup of the torus, and thus b_w differs from the true b by an element in the maximal compact, the construction leads to the correct result.

b) Now suppose E is completely ramified at p, and, in particular, that p stays prime in E. Here I think there is no difficulty in showing that the true b differs from the conjectured b by an element of the maximal compact of $T(k)$, but now the maximal compact has cohomology and I don't see what to do at the moment.

I will think about it, but for now I just want to send you this brief supplement to the other letter.

Yours,
Bob

Compiled on July 3, 2024.