October 8, 1980

Dear Bill,

First of all congratulations to you and Yuko go on the birth of your daughter.

Part of the notes for my lectures in Paris has been written out and typed, but the technical part remains. I have been working on it since I came to Bonn, but only intermittently and it is advancing rather slowly. I could send you what is ready, but it is probably best to wait until I am finished and then to send everything. There is a possibility that the notes will be typed and bound like the notes from the Gérardin-Labesse seminars and distributed by the University of Paris.

I hadn't intended when after seeing Thompson's book in your office, I took it up to read to spend much time studying statistical mechanics, but because it is related to other questions which I had wanted to understand and because there is a great deal of solid mathematical literature available, for example the notes of Ruelle or of Georgii, I have worked on it a fair amount and I'm beginning to understand some things. But I am far from a mastery of the subject.

As you probably know the central mathematical object in the theory is a convex hypersurface of codimension 1 in a linear space. The linear parameters are βU, $\beta\mu$, and βp. Here β is the inverse temperature, U runs over the possible interactions, μ is the chemical potential for gases and the magnetic field for magnetic lattices, p is the pressure. As far as I can tell, μ is a purely theoretical . . . [Rest of page missing.]

We are interested in the points at which the surface is not smooth because there are points at which a phase transition takes place. What is fascinating to a mathematician is that there appear to be severe restraints on the structure of the surface in the neighborhood of a singular point. There are conjectures and, I believe, a vast amount of experimental evidence, but so far as I see no theorems.

Suppose for example that there is only one possibility for U so that the linear parameters are β, $\nu = \beta\mu$, and $q = \beta p$. Then the hypersurface is a surface of three dimensions, and is given by writing q as a convex function of β and ν, $\beta > 0$.

This function frequently takes a special form locally. There is a value $\beta_0 > 0$ so that, for $\beta > \beta_0$, but close to β_0, and ν close to 0, one has

$$q(\epsilon, \nu) = A_0\epsilon^{2-\alpha'} + A_1\epsilon^{\overline{\beta}}|\nu| + A_2\epsilon^{-\gamma'}\nu^2 + \text{higher order terms in } \nu$$

Here A_0, A_1, A_2 are smooth non-vanishing functions and $\epsilon = \beta_0 - \beta$. Here α', $\overline{\beta}$ (the bar is to distinguish it from the inverse temperature; I'm not responsible for the conflicting notation), and γ' are critical exponents. They are defined in a far less transparent but physically more meaningful manner in Stanley, *Introduction to Phase Transitions + Critical Phenomena.* They are all non-negative and $\overline{\beta} > 0$, $2 > \alpha$.

The final pages of this letter from Robert Langlands to William Casselman are missing.

Compute the matrix of second order partial derivatives.

$$\begin{pmatrix} \frac{\partial^2 q}{\partial \epsilon^2} & \frac{\partial^2 q}{\partial \epsilon\, \partial \nu} \\ \frac{\partial^2 q}{\partial \nu\, \partial \epsilon} & \frac{\partial^2 q}{\partial \nu^2} \end{pmatrix}$$

to obtain at $\nu = 0$

$$\begin{pmatrix} (2-\alpha')(1-\alpha')A_0\epsilon^{-\alpha'} & \pm\beta A_1\epsilon^{\overline{\beta}-1} \\ \pm\beta A_1\epsilon^{\overline{\beta}-1} & 2A_2\epsilon^{-\gamma'} \end{pmatrix}$$

It must be non-negative definite because of convexity. Letting $\epsilon \to 0$ we see that this can be so only if

$$\alpha' + \gamma' \geqslant 2 - 2\overline{\beta}$$

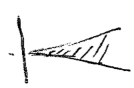
This is the Rushbrooke inequality. It is believed to be an equality, but no theorem is available, not even for specific models, such as the two-dimensional Ising model.

At the risk of making this letter so long that you won't even begin to read it I give you one other example. The surface introduced above is often also such that on the line $\epsilon = 0$ it has the form

$$q(0,\nu) = B_0(\nu)\nu^{\frac{1+\delta}{\delta}} \qquad \delta > 0$$

Here ν is another critical exponent and B_0 is continuous and non-zero at $\nu = 0$.

This q is now given in a set of the form



Above this set it looks roughly like

Otherwise there is no sharp bend.

Suppose we join the point $(\epsilon, 0)$ to the point $(0, \nu)$, $\nu > 0$ by a segment and then take a point $\big(x\epsilon, (1-x)\nu\big)$ on this segment. For x close to 1 we may use convexity to infer that

$$\begin{aligned} xA_0(\epsilon,0)\epsilon^{2-\alpha'} + (1-x)B_0(0,\nu)\nu^{\frac{1+\delta}{\delta}} &\geqslant A_0\big(x\epsilon,(1-x)\nu\big)x^{2-\alpha'}\epsilon^{2-\alpha'} \\ &+ A_1\big(x\epsilon,(1-x)\nu\big)x^{\overline{\beta}}(1-x)\epsilon^{\overline{\beta}}\nu + A_2\big(x\epsilon,(1-x)\nu\big)x^{-\gamma'}(1-x)^2\epsilon^{-\gamma'}\gamma^2 \\ &+ O(1-x)^3 \end{aligned}$$

We expand $A_0\big(x\epsilon,(1-x)\nu\big)$ in powers of $1-x$, drop the term γ on both sides independent of $1-x$, then divide by $1-x$, and finally let $1-x \to 0$ to obtain an inequality in ϵ and ν. It is valid for small ϵ and ν. Thus we may replace ϵ by $\lambda^{\frac{1}{2-\alpha'}}\epsilon$ and ν by $\lambda^{\delta/1+\delta}\nu$, $\lambda > 0$. The result is an inequality of the form

$$C\lambda \geqslant \lambda\Big(D\lambda^{\frac{1}{2-\alpha'}} + E\lambda^{\delta/1+\delta} + F\Big) + G\lambda^{\frac{\overline{\beta}}{2-\alpha'}+\frac{\delta}{1+\delta}}$$

The coefficients depend on ϵ and ν and $G > 0$. Letting $\lambda \to 0$ we see that this inequality is possible only if

$$\frac{\overline{\beta}}{2-\alpha'} + \frac{\delta}{1+\delta} \geqslant 1$$

A little juggling and this becomes

$$\alpha' + \overline{\beta}(1 + \delta) \geqslant 2,$$

which is the Griffiths inequality, believed to be an equality.

Since the critical exponents turn out to be experimentally measurable, it is of considerable interest to have some theoretical understanding of the circumstances under which the surface has the form described above, as well as a real proof that the equalities are satisfied.

But I won't keep you any longer.

[Additional pages are unavailable.]

Compiled on May 7, 2024.