

Bill,

Let me try to clarify my question slightly by giving the classification of pairs  $(t', C)$  in the case  $G$  is the simply-connected form of  $C_2$ . Then  $G^{\widehat{0}}$  is the adjoint form of  $B_2 (= C_2)$ , that is the quotient of  $\mathrm{Sp}(2)$  by its centre.

Up to conjugacy these are the following possibilities for  $X$ .

(i)  $X = 0$

(ii) Rank  $X = 1$ , then we may take

$$X = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(iii) Rank  $X = 2$ , then we may take

$$X = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

(iv) Rank  $X = 3$ , then we may take

$$X = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

Let's look for the corresponding  $\widehat{t}$ , up to conjugacy.

(i)  $\widehat{t}$  is any diagonal matrix (modulo  $\pm 1$ ) up to the action of the Weyl group. If  $\widehat{\lambda} = \widehat{\lambda}(t)$  is defined by

$$|\lambda(t)| = q^{+\langle \lambda, \widehat{\lambda} \rangle} \quad (\text{I worry about the sign})$$

define  $\chi = \chi(\widehat{t})$  by

$$\chi(t) = \widehat{\lambda}(\widehat{t}).$$

The representation associated to  $(\widehat{t}, 0)$  are the constituents of  $\mathrm{PS}(\chi)$  which contain the trivial representation of a good maximal compact.

**Example.** Let  $\widehat{\alpha}(\widehat{t}) = q$  for every simple positive root  $\widehat{\alpha}$ . Then if  $\delta$  is one-half the sum of the positive roots of  $G$ ,

$$\chi(t) = q^{\langle \delta, \widehat{\lambda}(t) \rangle} = |\delta(t)|$$

so that the representation corresponding to  $(\widehat{\lambda}, 0)$  in this case is the trivial representation.

(ii) one possible choice of  $\widehat{t}$  is

$$\begin{pmatrix} 1 & & & \\ & q^{1/2} & & \\ & & 1 & \\ & & & q^{-1/2} \end{pmatrix}$$

By basic general facts any other possible  $\widehat{t}$  is conjugate to an element in the normalizer of

$$\begin{pmatrix} * & 0 & * & 0 \\ 0 & * & 0 & 0 \\ * & 0 & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix}$$

Any element in the normalizer of this group which maps  $X$  into a multiple of itself lies in the group. Thus up to conjugacy we may take

$$\widehat{t} = \begin{pmatrix} a & & & \\ & \pm q^{1/2} & & \\ & & a^{-1} & \\ & & & \pm q^{-1/2} \end{pmatrix}$$

Replacing  $a$  by  $a^{-1}$  does not change the conjugacy class of  $(\widehat{t}, X)$ . Recall also that  $\widehat{t}$  and  $-\widehat{t}$  are to be identified. Consider the parabolic subgroup of  $C$  given by

$$P = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}$$

$$H = \begin{pmatrix} * & * & & 0 \\ * & * & & \\ & & * & * \\ 0 & & * & * \end{pmatrix}$$

Note: both  $G$  and  $G^{\widehat{0}}$  have been taken in the  $C_2$  form. Thus we have to be careful about the pair in between  $L$  and  $\widehat{L}$ .

$$L = \mathbf{Z} \oplus \mathbf{Z}$$

$$\widehat{L} \subseteq \mathbf{Z} \oplus \mathbf{Z}$$

$$\Delta : \alpha = (1, -1) \quad \beta = (0, 2)$$

$$\widehat{\Delta} : \widehat{\beta} = (1, -1) \quad \widehat{\alpha} = (0, 2)$$

$$\begin{aligned}
(x, y) \in L \quad (u, v) \in \widehat{L} \quad \langle (x, y), (u, v) \rangle &= \frac{v(x-y)}{2} + u \left( \frac{x+y}{2} \right) \\
&= \frac{u+v}{2} \cdot x + \frac{u-v}{2} \cdot y
\end{aligned}$$

Thus  $\widehat{L}$  is associated to the following character:

$$\begin{aligned}
t = \begin{pmatrix} \alpha & & & \\ & \beta & & \\ & & \alpha^{-1} & \\ & & & \beta^{-1} \end{pmatrix} & \quad \widehat{\lambda}(t) = (m+n, m-n) \\
& \quad |\alpha| = q^{+m} \quad |\beta| = q^{+n} \\
& \quad \chi(t) = a^{m+n} q^{\frac{1}{2}(m-n)}
\end{aligned}$$

Since

$$M = \left\{ \begin{pmatrix} A & 0 \\ 0 & A'^{-1} \end{pmatrix} \right\} \xrightarrow{\sim} \{A\} = \mathrm{GL}(2)$$

and since  $\chi$  is a character which defines a special representation  $\sigma$  of  $\mathrm{GL}(2)$ , we may induce  $\sigma$  (with the usual modifications) from  $P$  up to  $G$ . If we take  $|a| = 1$ , the representation associated to  $(\widehat{t}, X)$  should be any constituent of the result—which is unitary. Otherwise we may choose  $|a| > 1$  and then peel off a representation from the top as for real groups. This would then be the representation associated to  $(\widehat{t}, X)$ .

(iii) One possible choice of  $\widehat{t}$  is

$$\begin{pmatrix} q^{1/2} & & & \\ & q^{-1/2} & & \\ & & q^{-1/2} & \\ & & & q^{1/2} \end{pmatrix}$$

Any possible choice of  $\widehat{t}$  is conjugate to something in the normalizer of

$$\left\{ \begin{pmatrix} a & & & \\ & b & & \\ & & a^{-1} & \\ & & & b^{-1} \end{pmatrix} \right\}$$

The possible choices of  $\widehat{t}$  are of the following types

(a)

$$\widehat{t} = \begin{pmatrix} aq^{1/2} & & & \\ & aq^{-1/2} & & \\ & & a^{-1}q^{-1/2} & \\ & & & a^{-1}q^{1/2} \end{pmatrix}$$

(b)

$$\widehat{t} = \begin{pmatrix} & & & aq^{1/2} \\ & & -aq^{-1/2} & \\ & \frac{q^{1/2}}{a} & & \\ \frac{-q^{1/2}}{a} & & & \end{pmatrix}$$

Any two matrices in (b) are conjugate to each other by something which centralizes  $X$ . Thus they should yield exactly one representation which should be *square-integrable*. Notice that in (i), (ii), (iii)(a) we can find a Levi factor  $M^{\widehat{0}}$  of a proper parabolic  $P^{\widehat{0}}$  such that  $\widehat{t} \in M^{\widehat{0}}$  and  $X$  lies in the Lie algebra of  $M^{\widehat{0}}$ . This is however not possible for (iii)(b).

Look at (iii)(a).  $a$  and  $a^{-1}$  may be interchanged. Choose the following parabolic of  $G$ .

$$P = \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ & & * & \\ & * & * & * \end{pmatrix}$$

$$M = \begin{pmatrix} * & & & \\ & * & & * \\ & & * & \\ * & & & * \end{pmatrix}$$

A torus in  $M$  is

$$\begin{pmatrix} \alpha & & & \\ & \beta & & \\ & & \alpha^{-1} & \\ & & & \beta^{-1} \end{pmatrix}.$$

$\widehat{t}$  defines the following quasi-character.  $|\alpha| = q^m$   $|\beta| = q^n$

$$\widehat{\lambda}(t) = (m + n, m - n)$$

$$\chi(t) = a^{2m} q^n$$

This character defines a special representation of  $M$ . Proceed now as in (ii).

(iv) By general facts there is only one possibility for  $\widehat{t}$ .

$$\widehat{t} = \pm \begin{pmatrix} q^{3/2} & & & \\ & q^{1/2} & & \\ & & q^{-3/2} & \\ & & & q^{-1/2} \end{pmatrix}$$

The corresponding representation is the Steinberg representation. Again  $(\widehat{t}, X)$  is contained in a Levi factor of no proper parabolic.

Yours,  
Bob

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