Undated

Bill,

Let me try to clarify my question slightly by giving the classification of pairs (t', C) in the case G is the simply-connected form of C_2. Then $\widehat{G^0}$ is the adjoint form of $B_2(= C_2)$, that is the quotient of $\mathrm{Sp}(2)$ by its centre.

Up to conjugacy these are the following possibilities for X.

(i) $X = 0$

(ii) Rank $X = 1$, then we may take

$$X = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ & & & \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(iii) Rank $X = 2$, then we may take

$$X = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ & & & \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

(iv) Rank $X = 3$, then we may take

$$X = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ & & & \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

Let's look for the corresponding $\widehat{t}$, up to conjugacy.

(i) $\widehat{t}$ is any diagonal matrix (modulo ± 1) up to the action of the Weyl group. If $\widehat{\lambda} = \widehat{\lambda}(t)$ is defined by

$$\left|\lambda(t)\right| = q^{+\langle \lambda, \widehat{\lambda} \rangle} \qquad \text{(I worry about the sign)}$$

define $\chi = \chi(\widehat{t})$ by

$$\chi(t) = \widehat{\lambda}(\widehat{t}).$$

The representation associated to $(\widehat{t}, 0)$ are the constituents of $\mathrm{PS}(\chi)$ which contain the trivial representation of a good maximal compact.

Example. Let $\widehat{\alpha}(\widehat{t}) = q$ for every simple positive root $\widehat{\alpha}$. Then if δ is one-half the sum of the positive roots of G,

$$\chi(t) = q^{\langle \delta, \widehat{\lambda}(t) \rangle} = \big|\delta(t)\big|$$

so that the representation corresponding to $(\widehat{\lambda}, 0)$ in this case is the trivial representation.

(ii) one possible choice of $\widehat{t}$ is

$$\begin{pmatrix} 1 & & & \\ & q^{1/2} & & \\ & & 1 & \\ & & & q^{-1/2} \end{pmatrix}$$

By basic general facts any other possible $\widehat{t}$ is conjugate to an element in the normalizer of

$$\begin{pmatrix} * & 0 & * & 0 \\ 0 & * & 0 & 0 \\ * & 0 & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix}$$

Any element in the normalizer of this group which maps X into a multiple of itself lies in the group. Thus up to conjugacy we may take

$$\widehat{t} = \begin{pmatrix} a & & & \\ & \pm q^{1/2} & & \\ & & a^{-1} & \\ & & & \pm q^{-1/2} \end{pmatrix}$$

Replacing a by a^{-1} does not change the conjugacy class of $(\widehat{t}, X)$. Recall also that $\widehat{t}$ and $-\widehat{t}$ are to be identified. Consider the parabolic subgroup of C given by

$$P = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ & & & \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}$$

$$H = \begin{pmatrix} \begin{matrix} * & * \\ * & * \end{matrix} & 0 \\ 0 & \begin{matrix} * & * \\ * & * \end{matrix} \end{pmatrix}$$

Note: both G and $\widehat{G^0}$ have been taken in the C_2 form. Thus we have to be careful about the pair in between L and $\widehat{L}$.

$$L = \mathbf{Z} \oplus \mathbf{Z} \qquad \Delta : \alpha = (1, -1) \quad \beta = (0, 2)$$

$$\widehat{L} \subseteq \mathbf{Z} \oplus \mathbf{Z} \qquad \widehat{\Delta} : \widehat{\beta} = (1, -1) \quad \widehat{\alpha} = (0, 2)$$

$$(x,y)\in L \quad (u,v)\in \widehat{L} \qquad\qquad \big\langle (x,y),(u,v)\big\rangle = \frac{v(x-y)}{2} + u\left(\frac{x+y}{2}\right) = \frac{u+v}{2}\cdot x + \frac{u-v}{2}\cdot y$$

Thus $\widehat{L}$ is associated to the following character:

$$t = \begin{pmatrix} \alpha & & & \\ & \beta & & \\ & & \alpha^{-1} & \\ & & & \beta^{-1} \end{pmatrix} \qquad \begin{array}{c} \widehat{\lambda}(t) = (m+n, m-n) \\ |\alpha| = q^{+m} \quad |\beta| = q^{+n} \end{array}$$

$$\chi(t) = a^{m+n} q^{\frac{1}{2}(m-n)}$$

Since

$$M = \left\{ \begin{pmatrix} A & 0 \\ 0 & A'^{-1} \end{pmatrix} \right\} \xrightarrow{\sim} \{A\} = \mathrm{GL}(2)$$

and since χ is a character which defines a special representation σ of $\mathrm{GL}(2)$, we may induce σ (with the usual modifications) from P up to G. If we take $|a| = 1$, the representation associated to $(\widehat{t}, X)$ should be any constituent of the result—which is unitary. Otherwise we may choose $|a| > 1$ and then peel off a representation from the top as for real groups. This would then be the representation associated to $(\widehat{t}, X)$.

(iii) One possible choice of $\widehat{t}$ is

$$\begin{pmatrix} q^{1/2} & & & \\ & q^{-1/2} & & \\ & & q^{-1/2} & \\ & & & q^{1/2} \end{pmatrix}$$

Any possible choice of $\widehat{t}$ is conjugate to something in the normalizer of

$$\left\{ \begin{pmatrix} a & & & \\ & b & & \\ & & a^{-1} & \\ & & & b^{-1} \end{pmatrix} \right\}$$

The possible choices of $\widehat{t}$ are of the following types

(a)

$$\widehat{t} = \begin{pmatrix} aq^{1/2} & & & \\ & aq^{-1/2} & & \\ & & a^{-1}q^{-1/2} & \\ & & & a^{-1}q^{1/2} \end{pmatrix}$$

(b)

$$\widehat{t} = \begin{pmatrix} & & & aq^{1/2} \\ & & -aq^{-1/2} & \\ & \frac{q^{1/2}}{a} & & \\ \frac{-q^{1/2}}{a} & & & \end{pmatrix}$$

Any two two matrices in (b) are conjugate to each other by something which centralizes X. Thus they should yield exactly one representation which should be *square-integrable.* Notice that in (i), (ii), (iii)(a) we can find a Levi factor $M^{\widehat{0}}$ of a proper parabolic $P^{\widehat{0}}$ such that $\widehat{t} \in M^{\widehat{0}}$ and X lies in the Lie algebra of $M^{\widehat{0}}$. This is however not possible for (iii)(b).

Look at (iii)(a). a and a^{-1} may be interchanged. Choose the following parabolic of G.

$$P = \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ & & & \\ & & * & \\ & * & * & * \end{pmatrix}$$

$$M = \begin{pmatrix} * & & & \\ & * & & * \\ & & & \\ & & * & \\ & * & & * \end{pmatrix}$$

A torus in M is

$$\begin{pmatrix} \alpha & & & \\ & \beta & & \\ & & \alpha^{-1} & \\ & & & \beta^{-1} \end{pmatrix}.$$

$\widehat{t}$ defines the following quasi-character. $|\alpha| = q^m \quad |\beta| = q^n$

$$\widehat{\lambda}(t) = (m+n, m-n)$$
$$\chi(t) = a^{2m} q^n$$

This character defines a special representation of M. Proceed now as in (ii).

(iv) By general facts there is only one possibility for $\widehat{t}$.

$$\widehat{t} = \pm \begin{pmatrix} q^{3/2} & & & \\ & q^{1/2} & & \\ & & q^{-3/2} & \\ & & & q^{-1/2} \end{pmatrix}$$

The corresponding representation is the Steinberg representation. Again $(\widehat{t}, X)$ is contained in a Levi factor of no proper parabolic.

Yours,
Bob

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