October 1973

Dear Deligne,

Thanks very much for your letter. What follows is only a provisional response to it. You will see however that your results yield, in conjunction with the trace formula, the desired relation between zeta functions for a totally indefinite quaternion algebra over a real quadratic field. You will also see, if you hadn't already observed it, that I was a little naive when talking to you at Bures. The present letter is I hope an improvement on my remarks at that time.

I too shall ignore the difficulties caused by the cusps. The sheaves play only a formal role; so I don't exclude them. Their effect will however hardly be perceptible.

I recall that the program is to use the trace formula to reduce the conjecture about the form taken by the zeta-function of a Shimura variety to a conjecture about the structure of its set of geometric points over the algebraic closure of the appropriate residue fields and then to solve the second conjecture by exploiting the relation of the variety to certain moduli problems.

This may not work in all cases. At the moment however we are interested in a relatively simple group.

- F: totally real field of degree n over $\mathbf{Q}$.
- B: quaternion algebra over E—B may be split.
- G: multiplicative group of B regarded as an algebraic group over $\mathbf{Q}$. The difficulties caused by problems connected with L-indistinguishability do not occur for this group.
- K_f: open compact subgroup of $G(\mathbf{A}_f)$.

Because I find it useful, I introduced, at the risk of offending your sensibilities, imbeddings $\overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$, $\overline{\mathbf{Q}} \to \overline{\mathbf{Q}}_p$. This all [text covered by paper] **[2]** first of all, to identify $\mathfrak{G}(\overline{\mathbf{Q}}/\mathbf{Q})/\mathfrak{G}(\overline{\mathbf{Q}}/F)$ with the collection of imbeddings $F \xrightarrow{\varphi} \mathbf{R}$. I write this collection as follows

$$\times\times\cdot\cdot\times\cdot\times$$

φ is represented by a point or a cross according as

$$B \otimes_\varphi \mathbf{R}$$

is or is not split. I refer to the imbeddings designated by a cross as marked. $\mathfrak{G}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ acts on this set. The orbits correspond to primes of F dividing p. Let a be the total number of marked imbeddings.

From now on we assume that E does not ramify over p, and that if $v_1, \ldots, v_r$ are the places of F dividing p then B splits at $v_1, \ldots, v_r$ so that

$$G(\mathbf{Q}_p) \simeq \prod_i \mathrm{GL}(2, F_{v_i}).$$

we also assume that the isomorphism can be so chosen that $K_f = K_f^p K_p$ with

$$K_p \simeq \prod_i \mathrm{GL}(2, O_{F_{v_i}}).$$

The zeta function should be expressed in terms of $L(s, \pi, \rho)$ where ρ is a certain representation of the associated group $\widehat{G}$ obtained as follows. The maps $\mathrm{GL}(1) \to S \to G$ yield an extreme, and hence dominant, weight of $G^{\widehat{0}}$, the connected component of $\widehat{G}$. Let ρ_0 be the corresponding representation. This highest weight is invariant under $\mathfrak{G}(\overline{\mathbf{Q}}/E)$. Extend ρ_0 to the subgroup

$$G^{\widehat{0}} \times \mathfrak{G}(\overline{\mathbf{Q}}/E)$$

[**3**] of $\widehat{G}$ in such a way that $\mathfrak{G}(\overline{\mathbf{Q}}/E)$ acts trivially on the highest weight vector. Then induce to $\widehat{G}$ to obtain ρ.

ρ restricted to the local associated group $\widehat{G}_p$ decomposes into a direct sum

$$\rho_p \simeq \bigoplus \rho_p^i$$

whose summands are parametrized by the double cosets

$$\mathfrak{G}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)\backslash\mathfrak{G}(\overline{\mathbf{Q}}/\mathbf{Q})/\mathfrak{G}(\overline{\mathbf{Q}}/E)$$

that is by the places $w_1, \dots, w_r$ of E dividing p. $L(s, \pi_p, \rho)$ decomposes accordingly into a product

$$\prod L(s, \pi_p, \rho_p^i)$$

and I suppose naturally that the factor $L(s, \pi, \rho_p^i)$ will correspond to the contribution of w_i to the L-function of the variety. There is no loss of generality in assuming that we are dealing with the coset containing 1. (We need only choose the original imbedding correctly.) Let ρ_p^1 be the corresponding representation.

I recall that every representation π_p of $G(\mathbf{Q}_p)$ which contains the trivial representation of K_p corresponds to a map φ_p of the Weil group $W_F \to \widehat{G}_p$. φ_p factors through the usual map $W_F \to \mathbf{Z}$. Let Φ be the Frobenius at p. If m_0 is the smallest power of Φ which fixes E and m is a positive integer then

$$\operatorname{trace} \rho_p^1(\Phi^m) = 0$$

unless m_0 divides m. We are trying to show that if m_0 divides m and [**4**] lies in $G(\mathbf{A}_f^p)$ then the trace of the operation on the cohomology, [text too light to read] the middle dimension, of the variety $K_f M/\overline{k}_{w_1}$ which is defined by g and Φ^m is

$$p^{ma/2} \sum_\pi \operatorname{trace} \pi_f^p(f_g) \operatorname{trace} \rho_p^1\big(\varphi_p(\Phi^m)\big).$$

Here k_{w_1} is the residue field at w_1 and f_g is the characteristic function of $K_f^p g K_f^p$ divided by the measure of K_f^p. π runs over all representations occurring in the space of cusp forms for which $??_\infty$[1] lies in a given class, which depends on the sheaf with respect to which the cohomology is taken.

For any $m > 0$ there is a spherical function f_p^m on $K_p\backslash G(\mathbf{Q}_p)/K_p$ [such] that

$$\operatorname{trace} \pi_p(f_p^m) = p^{ma/2} \operatorname{trace} \rho_p^1\big(\varphi_p(\Phi^m)\big)$$

[1] Text too light to read.

if π_p corresponds to φ_p. For the group under consideration $G(\mathbf{Q}_p)$ is a product

$$G(\mathbf{Q}_p) \simeq \prod_i \mathrm{GL}(2, F_{v_i})$$

and f_p^m is of the form

$$f_p^m(g) = \prod_i f_{v_i}^m(g_i).$$

Let me now introduce the data needed to describe f_{v_i}. I write [text too light to read] the ith orbit

$$\times\times\cdot\times\cdot$$

that Φ acts cyclically as a shift to the right by one.

- n_i: Number of elements in the orbit (under $\Phi^{\mathbf{Z}}$)
- ℓ_i: greatest common divisor of m and n_i
- a_i: number of [text cut off] orbits under $\Phi^{m\mathbf{Z}}$

[**5**] Note that $a_i/\ell_i = b_i/n_i$ where b_i is the number of marked elements in the full orbit. As usual the spherical functions on $\mathrm{GL}(2, F_{v_i})$ are isomorphic to the symmetric polynomials in two variables s_i and t_i. The function $f_{v_i}^m$ is defined by

$$\begin{aligned} f_{v_i}^m &\sim p^{mb_i/2}(s_i^{m/\ell_i} + t^{m/\ell_i})^{a_i} \\ &= \left\{(p^{n_i/2}s_i)^{m/\ell_i} + (p^{n_i/2}t_i)^{m/\ell_i}\right\}^{a_i} \\ &= \sum_{u=0}^{a_i} \binom{a_i}{u} (p^{n_i/2}s_i)^{mu/\ell_i}(p^{n_i/2}t_i)^{\frac{m}{\ell_i}(a_i-u)} \end{aligned}$$

Observe that it is now assumed that $m_0|m$. Otherwise the function is f_p^m. [text too light to read] zero. This is an interesting case; what we exclude from consideration. [Text too light to read] set

$$f_{v_i}^m = \sum_{j=0}^{[b_i/2]} f_{v_i}^{m,j}$$

where

$$f_{v_i}^{m,j} = 0.$$

If b_i/a_i does not divide j but where

$$f_{v_i}^{m,j} \sim \binom{a_i}{a_i j/b_i}\left\{(p^{n_i/2}s_i)^{mj/n_i}(p^{n_i/2}t)^{\frac{m}{n_i}(b_i-j)} + (p^{n_i/2}s_i)^{\frac{m}{n_i}(b_i-j)}(p^{n_i/2}t_i)^{nj/n_i}\right\}$$

if it does. (If $j = b_i/2$ and $2|a_i$ ignore the second summand.) I then set

$$f_p^m = \sum_{j_1=0}^{b_1} \cdots \sum_{j_r=0}^{b_r} f_p^{m,j_1,\dots,j_r}.$$

The trace formula is to be applied to functions of the form[**6**]

$$f_\infty(h_\infty)f_g(h_f^p)f_p^m(h_p)$$

where f_∞ is chosen appropriately. We may apply it to the functions

$$f_\infty(h_\infty)f_g(h_f^p)f_p^{m,j_1,\dots,j_r}(h_p)$$

individually.

I shall group the terms as follows. Fix a subset S of $\{1, \ldots, r\}$ and for each $i \in S$ a k_i, $0 \leqslant k_i \leqslant b_i/2$, with $k_i \neq b_i/2$. Suppose first that S is not empty. Then fix a totally imaginary quadratic extension L of F which splits at v_i, $i \in S$. The group corresponding to S, $\{k_i\}$, and L consists of all terms corresponding to a regular class $\{\gamma\}$ whose eigenvalues generate over F a field isomorphic to L and to

$$f_p^{m,j_1,\ldots,j_r}$$

with $j_i = k_i$, $i \in S$ and $j_i = b_i/2$ if $i \notin S$ and L splits at v_i. There is one other group, that corresponding to the empty set. It consists of all terms corresponding to a class $\{\gamma\}$ and to

$$f_p^{m,j_1,\ldots,j_r}$$

such that $j_i = b_i/2$ whenever γ has two distinct eigenvalues in F_{v_i}.

In order to interpret the results of the trace formula I associate to each group of terms some additional objects. First of all an algebraic [text cut off] over $\mathbf{Q}$. If S is not empty it is the group defined in the usual way by $L^\times$. If S is empty it is the group defined by the quaternion algebra which is ramified at every infinite place of F, at every finite place at which B is ramified, and at the places v_i for which b_i is odd, but nowhere else. I also introduce a group $\overline{G}$ over $\mathbf{Q}_p$.

$$\overline{G} = \prod_i \overline{G}_i$$

[**7**] where $\overline{G}_i$ is obtained by restriction of scalars from F_{v_i} to $\mathbf{Q}_p$. If $i \in S$ then $\overline{G}_i(\mathbf{Q}_p) = L_{v_i}^\times$, with $L_{v_i} = L \otimes_F F_{v_i}$. If $i \notin S$ then

$$\overline{G}_i(\mathbf{Q}_p) = \mathrm{GL}(2, F_v)$$

if b_i is even but

$$\overline{G}_i(\mathbf{Q}_p) = G'(F_v)$$

where G' is the multiplicative group of a quaternion algebra if b_i is odd. I also introduce some phantom sets X. These are sets whose existence at this stage of the discussion is doubtful; I shall explain later what they should be. For now I list the properties as I want

$$X = \prod_{i=1}^{-} X_i \qquad \{\text{This property will be slightly modified later}\}$$

where $\overline{G}_i(\mathbf{Q}_p)$ acts on X_i to the right and $\mathbf{Z}$ acts to the left. Then $\overline{G}(\mathbf{Q}_p)$ acts on X to the right and $\mathbf{Z}$ acts to the left. I build

$$G(\mathbf{A}_f^p) \times X$$

in which $G(\mathbf{A}_f^p)$ acts to the right and $m_0\mathbf{Z}$ to the left. I let $H(\mathbf{Q})$ act by

$$\gamma : (g, x) \to (\gamma g, x^{\gamma^{-1}}).$$

Observe that $H(\mathbf{Q})$ maps to $G(\mathbf{A}_f^p)$ and to $\overline{G}(\mathbf{Q}_p)$. Let the quotient by this action be Y. $G(\mathbf{A}_f^p)$ and $\mathbf{Z}$ still act on Y. In particular [text cut off] may divide Y by K_f^p to obtain Y/K_f^p.

The union of the various Y/K_f^p is supposed to turn out to be the set of points on $k_f M$ with coordinates in the algebraic closure of the residue field. The action of Φ^m, $m_0|m$, on these points is supposed to correspond to the action of $m \in m_0\mathbf{Z}$ on Y/K_f^p. The actions of $G(\mathbf{A}_f^p)$ on Y and on

$$\varprojlim k_f M(\overline{k}_{w_1})$$

[**8**] should be compatible. Because of this I write, if $x_i \in X_i$, and $m \in m_0$

$$mx_i = \Phi^m x_i.$$

I can copy §5 of my Antwerp notes. If μ is a representation of $G(\mathbf{Q})$ over $\mathbf{Q}$ I can associate to it a sheaf on Y/K_f^p and if $g \in G(\mathbf{A}_f^p)$ and $m_0|m$ I can introduce a correspondence on Y/K_f^p which also acts on the sheaf and then sum up the traces at the fixed points.

To make sense of this sum, another condition must be imposed on X. It should only contain finitely many $\overline{G}(\mathbf{Q}_p)$ orbits and the stabilizer $G_x(\overline{\mathbf{Q}}_p)$ of a point x should be open. Let

$$T_x^m = \left\{ \overline{g} \in \overline{G}(\mathbf{Q}_p) \;\middle|\; \Phi^m x\overline{g} = x \right\}$$

As usual $H_\gamma(\mathbf{Q}_p)$ and $\overline{G}_\gamma(\mathbf{Q}_p)$ are the centralizers of γ in $H(\mathbf{Q}_p)$ and $\overline{G}(\mathbf{Q}_p)$. I also demand that if $\gamma \in H(\mathbf{Q})$ lies in T_x^m for some x and some positive m then

$$H_\gamma(\mathbf{Q}_p) = \overline{G}_\gamma(\mathbf{Q}_p).$$

For each x I form

$$\frac{1}{\operatorname{meas} \overline{G}_x(\mathbf{Q}_p)} \int_{\overline{G}_\gamma(\mathbf{Q}_p)\backslash\overline{G}(\mathbf{Q}_p)} \delta_x^m(h^{-1}\gamma h)\, dh$$

Here δ_x^m is the characteristic function of T_x^m. This expression, as a function of γ, is a class function on $\overline{G}(\mathbf{Q}_p)$. Moreover it depends only on the orbit to which x belongs. Taking the sum over orbits we obtain a class function $\varphi^m(\gamma)$ on $\overline{G}(\mathbf{Q}_p)$. It is clear that φ^m is of the form [**9**]

$$\varphi^m(\gamma) = \prod_i \varphi_i^{m_i}(\gamma_i)$$

if $\gamma = \prod_{i=1}^r \gamma_i$. The sum of the traces of the fixed points is equal to

$$\sum_{\{\gamma\}} \operatorname{trace} \mu(\gamma) \frac{\operatorname{meas}\Big(Z_f^0 H(\gamma,\mathbf{Q})\backslash H(\gamma,\mathbf{A}_f)\Big)}{\operatorname{meas} Z_f^0} \varphi^m(\gamma) \int_{H(\gamma,\mathbf{A}_f^p)\backslash G(\mathbf{A}_f^p)} f_g(h^{-1}\gamma h)\, dh$$

Z_0^f is a small open subgroup in the centre of $G(\mathbf{A}_f)$. In a moment I shall compare this with the trace formula. I observe first the somewhat disappointing circumstance that the trace formula can predict the function φ^m—nothing more. (Added—this is not so bad.)

It is the functions φ_i^m which we want to predict. They only depend on the data at the place v_i, that is, on $\overline{G}_i$, j_i, a_i, and so on. As I remarked earlier the conditions on X would have to be slightly modified. I would rather preserve them, at the cost of associating to each grouping of terms in the trace formula several X and hence several Y. Let s be the number of elements in S. If S is not empty there will be 2^{s-1} spaces X. If S is empty there will be 1.

(a) $0 \leqslant j_i < b_i/2$. $\overline{G}_i(\mathbf{Q}_p) = L_{v_i}^\times$. If $H(\mathbf{Z}_p)$ is the group of integral elements in $L_{v_i}^\times$ then

$$\operatorname{meas} H(\mathbf{Z}_p)\varphi_i^m(\gamma) = \begin{cases} 0 & b_i/a_i \nmid j \\ \binom{a_i}{a_i j/b_i} p^{md_i} \chi\left(\gamma^{-1}\begin{pmatrix} \varpi^{mj/m_i} & 0 \\ 0 & \varpi^{\frac{m(b_i-j)}{n_i}} \end{pmatrix}\right) & \text{otherwise} \end{cases}$$

Added, i. e. $d_i = j_i$.

Here $d_i = mm\{j_i, b_i - j_i\}$ and χ is the characteristic function of $H(\mathbf{Z}_p)$. ϖ is a generator of the maximal ideal of $O_{F_{v_i}}$.

[10] The 2^{s-1} is connected, as you know, with the possible orderings of the places L over v_i which are implicit in the above formula.

(b) $\overline{G}_i(\mathbf{Q}_p) = \mathrm{GL}(2, F_{v_i})$ or $G'(F_{v_i})$. Recall that G' is the multiplicative group of a quaternion algebra. There are several cases.

(i) γ elliptic and regular. If a_i is odd then $\varphi_i^m(\gamma)$ is 0 unless $\mathrm{Nm}\,\gamma$ is a unit times ϖ^{ma_i/ℓ_i} when it is

$$\sum_{\substack{0\leqslant j<b_i/2\\ \frac{b_i}{j_i}|j}} \binom{a_i}{a_ij/b_i} p^{md_i} \frac{\mathrm{meas}\, G'(\gamma, F_{v_i})\backslash G'(F_{v_i})}{\mathrm{meas}\, G'(O_{F_{v_i}})} \tag{1}$$

If a_i is even it is the sum of this and the product of

$$\binom{a_i}{a_i/2} p^{ma_i(b_i?)/2} \begin{cases} 0 & \mathrm{Nm}\,\gamma\varpi^{-ma_i/\ell_i} \text{ not a unit} \\ 1 & \mathrm{Nm}\,\gamma\varpi^{-ma_i/\ell_i} \text{ a unit} \end{cases}$$

with

(α) γ unramified at v_i: $\frac{q^{\omega(\gamma)+1}+q^{\omega(\gamma)}-2}{q-1}$
(β) γ ramified at v_i: $\frac{q^{\omega(\gamma)+1}-1}{q-1}$

$\frac{1}{\mathrm{meas}\,\overline{G}_i(O_{F_{p_i}})} \int_{\overline{G}_{i,\gamma}(\mathbf{Q}_{p_i})\backslash \overline{c}_i} \eta$ [text too light to read]

η character of $\overline{G}_i(O_{F_{[\text{text too light to read}]}})$

$\omega(\gamma)$ is defined on p. 252 of Jacquet-Langlands. q is the number of elements in the residue field of $O_{F_{v_i}}$, that is, $q = p^{n_i}$. **[11]**

(ii) γ hyperbolic and regular. If a_i is odd then $\varphi_i^m(\gamma) = 0$. Otherwise it is

$$\binom{a_i}{a_i/2} p^{mb_i/2} \frac{|\alpha\beta|^{1/2}}{|\alpha-\beta|} \chi\left(\gamma^{-1}\varpi^{\frac{mb_i}{2n_i}}\right)$$

There is a normalization of measures implicit here

χ is here the characteristic function of $\mathrm{GL}(2, O_{F_{v_i}})$.

(iii) γ central in $\mathrm{GL}(2, F_{v_i})$. If a_i is odd then $\varphi_i^m(\gamma)$ is given by [illegible] (1). If a_i is even it is the sum of minus (1) and

$$\frac{\binom{a_i}{a_i/2} p^{mb_i/2} \chi\left(\gamma^{-1}\varpi^{\frac{mb_i}{2n_i}}\right)}{\mathrm{meas}\,\mathrm{GL}(2, O_{F_{v_i}})}.$$

So that you will have a clear idea of the meaning of the spaces X I consider the case that G is GL(2) over F. I use the notation of Demazure *Lectures on p-divisible groups* except that, to avoid confusion, I write his F as $\mathbf{F}$. To each grouping in the trace formula and to each i I associate an $\mathbf{F}$-space X_i on which $O_{F_{v_i}}$ acts. $\overline{G}_i(\mathbf{Q}_p)$ will be the group of endomorphisms of N which commute with $O_{F_{v_i}}$. If M is an $\mathbf{F}$-lattice in N its dual M' may be regarded as an F-lattice in the dual N' of N. X_i will consist of all pairs (M, θ), where M is an $\mathbf{F}_{i,v}$-lattice in N invariant under $O_{F_{v_i}}$ and θ is an $O_{F_{v_i}}$ isomorphism from the $\mathbf{F}$-lattice M to the $\mathbf{F}$-lattice M'. An element $g \in \overline{G}(\mathbf{Q}_p)$ acts by sending M to Mg and θ to the bottom arrow in $e.\check{N}(-1)$ [text unclear]

$$\begin{array}{ccc} M & \xrightarrow{\theta} & M' \\ \updownarrow{\scriptstyle g^{-1}} & & \downarrow{\scriptstyle g^{-1}} \\ Mg & \longrightarrow & Mg^{-1} \end{array}$$

—(contragradient)

(Maybe θ is required to be equal to minus its transpose.) **[12]** An element $n \in \mathbf{Z}$ acts by sending M to $F^m M$ and θ to the bottom arrow in

$$\begin{array}{ccc} M & \xrightarrow{\theta} & M' \\ \uparrow\wr & & \downarrow\wr \\ F^m M & \longrightarrow & p^m \mathbf{F}^m M' \end{array}$$

I choose N as follows

(a) $0 \leqslant j_i < b_i/2 = n_i/2 \quad \overline{G}_i(\mathbf{Q}_p) = L_{v_i}^{\times}$.

$$N = (E^{j_i/n_i} \oplus E^{n_i - j_i/n_i}) \otimes_{O_K} O_{F_{v_i}}.$$

K is the extension of $\mathbf{Q}_p$ of degree $n_i/\operatorname{g.c.d.}(n_i, j_i)$ which lies in F_{v_i}. I imbed O_K in the obvious way in the ring of endomorphisms of the first factor.

(b) $\overline{G}_i(\mathbf{Q}_p)$ is $\mathrm{GL}(2, F_{v_i})$.

$$N = (E^{1/2} \oplus E^{1/2}) \otimes_{O_K} O_{F_{v_i}}.$$

K is the quadratic extension of $\mathbf{Q}_p$ in F_{v_i}.

(c) $\overline{G}_i(\mathbf{Q}_p)$ is $G'(F_{v_i})$.

$$N = E^{1/2} \otimes_{\mathbf{Z}_p} O_{F_{v_i}}.$$

First of all I think that, with the remark made above but it is necessary to associate, under certain circumstances, to one grouping of the terms in the trace formula more than one X, the various Y correspond to the isogeny classes. Moreover the picture here presented should agree with yours.

To verify the assertion made at the beginning I should **[13]** check that the functions φ_i^m have the form predicted by the trace formula when $[F : \mathbf{Q}] = 2$. The only non-trivial case is (b) above with $n_i = 2$.

Since I want to include a couple of other examples, I start with some general remarks. I identify $E^{1-\lambda}$ with the dual of E^{λ} by identifying

$$\varphi \in \operatorname{Hom}\Big(E^{\lambda}, \widetilde{W}(-1)\Big)$$

with

$$\sum_{a=0}^{n-1} \varphi(\mathbf{F}^a) \frac{\mathbf{F}^a}{p^a} \qquad \lambda = s/r.$$

I also write the elements of

$$E^{\lambda} \otimes_{O_K} O_{F_{v_i}}$$

as

$$\sum_{a=0}^{r-1} \sum_{b=0}^{\frac{n_i}{r_0}-1} \alpha_{a,b} \mathbf{F}^c \qquad \alpha_{a,b} \in B(\overline{\mathbf{F}}_p),\ r_0 = [K : \mathbf{Q}_p].$$

In particular

$$\alpha \mathbf{F}^a \otimes \beta \leftrightarrow \sum_b \alpha \beta^{\sigma^{a+rb}} \mathbf{F}^a$$

σ is the Frobenius on $B(\overline{\mathbf{F}}_p)$ in which I imbed F_{v_i}.

Examples.

(1) $n_i = 2$ I write an element of N as

$$\epsilon = \alpha_0 + \alpha_1\mathbf{F} + \alpha_0' + \alpha_1'\mathbf{F}$$

and consider the following $\mathbf{F}$-lattices. **[14]**

(i) Choose $x \in \overline{\mathbf{F}}_p$. $\epsilon \in M_1(x) \iff \alpha_0, \alpha_1, \alpha_0', \alpha_1' \in \frac{1}{p}W(\overline{\mathbf{F}}_p)$ and

$$\alpha_0' \equiv \alpha_0 x \pmod{W(\overline{\mathbf{F}}_p)}$$

θ is given by

$$p^2\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{GL}(2, F_{v_i})$$

(ii) Choose $x \in \overline{\mathbf{F}}_p$. $\epsilon \in M_2(x) \iff \alpha_0,\ \alpha_0' \in W(\overline{\mathbf{F}}_p)$, $\alpha_1,\ \alpha_1' \in \frac{1}{p}W(\overline{\mathbf{F}}_p)$ and

$$\alpha_1' \equiv \alpha_1 x \pmod{W(\overline{\mathbf{F}}_p)}.$$

Clearly

$$\mathbf{F}M_1(x) = M_2(x^\sigma)$$
$$\mathbf{F}M_2(x) = M_1(x^\sigma)p.$$

All $M_1(x)$ and $M_2(x)$ with $x \in \mathbf{F}_p^2$ are in the same orbit. In particular

$$\mathbf{F}M_1(0) = M_1(0)\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}.$$

Also the group $\mathrm{GL}(2, O_{F_{v_i}})$ acts on $\{M_1(x)\}$ and $\{M_2(x)\}$. All this fits in with what you said. Since I would like to finish this later today I will not check, but just take for granted, that every $\overline{G}_i(\mathbf{Q}_p)$ orbit contains one of the lattices.

A little calculation shows that

(a) $\gamma \in \overline{G}_i(\mathbf{Q}_p)$ central. Then $\varphi_i^m(\gamma) = 0$ if m is odd or $(\mathrm{Nm}\,\gamma)\varpi^{-m/2}$ is not a unit. Otherwise it is

$$\frac{1}{\operatorname{meas}\mathrm{GL}(2, O_{F_{v_i}})}\left\{\overbrace{p^2+1}^{\text{From } M_1(0)} + \overbrace{p^2(p^4-1)\frac{2(p^m-p^2)}{p^2(p^4-1)}}^{\text{From others}}\right\} = \frac{1}{\operatorname{meas}\mathrm{GL}(2, O_{F_{v_i}})}\{2p^m - p^2 + 1\}.$$

[15] Since

$$\frac{\operatorname{meas}\mathrm{GL}(2, O_{F_{v_i}})}{\operatorname{meas} G'(O_{F_{v_i}})} = p^2 - 1$$

this is the required result.

(b) γ hyperbolic and regular. If m is odd or $\gamma^{-1}\varpi^{m/2}$ is not a unit matrix then $\varphi_i^m(\gamma) = 0$. Otherwise it is

$$\frac{1}{\operatorname{meas}\mathrm{GL}(2, O_{F_{v_i}})}\left\{\underbrace{\frac{|\alpha\beta|^{1/2}}{|\alpha-\beta|}\left(\frac{p^2+1}{p^2-1}\left(1-\frac{1}{p^2}\right)+\frac{p^2+1}{p^2}\right)}_{\text{from } M_1(0)} + \underbrace{\frac{|\alpha\beta|^{1/2}}{|\alpha-\beta|}\frac{p^2(p^4-1)}{p^2(p^4-1)}2(p^m-p^2)}_{\text{from the others}}\right\}$$

This is the required result.

There is no point in continuing. I have checked however—it is an easy calculation—that the elliptic elements are alright.

(2) Now consider the case that n_i is odd and $\overline{G}_i(\mathbf{Q}_p)$ is $G'(F_{v_i})$. There are idempotents in $W(\overline{\mathbf{F}}_p) \otimes O_{F_{v_i}}$ which project on

$$\sum_{a+b\ (\text{constant}=c)\ \text{mod}\ n_i} \alpha_{a,b}\mathbf{F}^a \qquad 0 \leqslant a \leqslant 1$$

$\mathbf{F}$ shifts the constant 1 to the right. One possible lattice M consists of sums of the above form with $\alpha_{a,b} \in W(\overline{\mathbf{F}}_p)$. θ is F. Also

$$\mathbf{F}M = M\mathbf{F}$$

The contribution of the orbit corresponding to this lattice is the term of (1) with $j = 0$.

Let $1 \leqslant b_0 \leqslant n_i$ [text cut off]. Choose $x \in \overline{\mathbf{F}}_p$. Define $M(b_0, x)$ to consist of all **[16]** sums of the above form with $\alpha_{0,b_0} \in W(\overline{\mathbf{F}}_p)$, $b + b_0$, $\alpha_{1,b-1} \in W(\overline{\mathbf{F}}_p)/p$, $b \neq b_0$, with both α_{0,b_0}, and α_{1,b_0-1} in $W(\overline{\mathbf{F}}_p)/p$, and with

$$\alpha_{1,b_0-1} - x\alpha_{0,b_0} \in W(\overline{\mathbf{F}}_p)$$

If $\delta \in W(\overline{\mathbf{F}}_p)^\times$ with $\delta^\sigma/\delta = -1$ then $\theta = \mathbf{F}\delta$. Also

$$\mathbf{F}M(b_0, x) = M(b_0 + 1, x^\sigma)\mathbf{F}.$$

The contribution of these lattices to $\varphi_i^m(\gamma)$ is 0 unless $n_i|m$, when it is the term of (1) with $j = 1$.

(3) Consider a case where $\overline{G}_i(\mathbf{Q}_p) = L_{v_i}^\times \simeq F_{v_i}^\times \times F_{v_i}^\times$. The case $j_i = 0$ is rather easy; so take $n_i = 3$, $j_i = 1$. I write an element of N as

$$\alpha + \beta\mathbf{F} + \gamma\mathbf{F}^2 + \alpha' + \beta'\mathbf{F} + \gamma'\mathbf{F}^2.$$

(a) $x \in \overline{\mathbf{F}}_p$. Define the lattice $M_1(x)$. α, α', β, β', $\gamma \in W(\overline{\mathbf{F}}_p)/p$, $\gamma' \in W(\overline{\mathbf{F}}_p)/p^2$

$$\alpha' - x\alpha \in W(\overline{\mathbf{F}}_p).$$

Then

$$\theta = \frac{1}{p}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(b) Define $M_2(x)$ by α, $\alpha' \in W(\overline{\mathbf{F}}_p)$, β, β', γ, $\gamma' \in W(\overline{\mathbf{F}}_p)/p$

$$\beta' - \beta x \in W(\overline{\mathbf{F}}_p).$$

Then

$$\theta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

[17]

(c) Define $M_3(x)$ by $\alpha \in W(\overline{\mathbf{F}}_p)/p$, $\alpha' \in W(\overline{\mathbf{F}}_p)$, β, $\beta' \in W(\overline{\mathbf{F}}_p)/p$, γ, $\gamma' \in W(\overline{\mathbf{F}}_p)/p^2$

$$\gamma' - x\gamma \in \frac{W(\overline{\mathbf{F}}_p)}{p}$$

$$\theta = \frac{1}{p}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

It is clear that

$$\mathbf{F}M_1(x) = M_2(x^\sigma) \qquad \mathbf{F}M_2(x) = M_3(x^\sigma)p$$
$$\mathbf{F}M_3(x) = M_1(x^\sigma)\begin{pmatrix} 1 & \\ & p \end{pmatrix}$$

Thus

$$\mathbf{F}^3 M_1(x) = M_1(x^{\sigma^3})\begin{pmatrix} p^2 & 0 \\ 0 & p \end{pmatrix}$$

Thus from this collection, which I have not verified meets every orbit, we get the right function $\varphi_i^m(\gamma)$.

Summary. The problem is to construct the sets X_i in general by a geometrical process and to show that the function $\varphi_i^m(\gamma)$ has the form predicted by the trace formula. The examples above indicate that it may not be vain to hope that this is so.

Remarks.

(a) Morita tells me he has been investigating the case that B splits at exactly one infinite place. I do not yet know the details of his results. I also do not know how seriously he has considered the general case.

(b) He showed me an old (1967) letter of Shimura to Tate **[18]** in which the case of a totally indefinite algebra over a real quadratic field is already treated.

I hope that this letter is of some use to you. Please exercise some care with it; it is easy to make a mistake with these kind of calculations.

All the best,
R Langlands

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