March 31, 1974

Dear Deligne:

For several reasons I have not responded to Dieudonné's request. I was in particular apprehensive that the result of collecting together the problems of twenty-five mathematicians would be tasteless hodge-podge. However, if several qualified mathematicians would take it upon themselves to give coherent accounts of the problems in the areas with which they are most concerned, the result might be very useful. You, for example, might think of doing this for L-functions, for it is not too great an exaggeration to suggest that it is a topic which contains the outstanding problems in algebraic geometry, in particular, those you describe to Dieudonné.

Although the present letter is principally a response to your invitation to comment on your list of problems, I have also included, as an appendix, some remarks on the Shimura varieties which I hoped you might find amusing.

I begin with an attempt to give a precise form to the problems of my Washington lectures.

Local theory. In *On the classification of irreducible representations of real algebraic groups* it is verified that to every class of mappings φ of the Weil group W_F ($F = \mathbf{R}$ or $\mathbf{C}$) satisfying some simple formal condition is associated a finite set $\Pi(\varphi)$ of classes of irreducible quasi-simple representations of $G(F)$. The sets $\Pi(\varphi)$ are disjoint and exhaustive.

There should be a similar statement for admissible representations of $G(F)$ when F is non-archimedean. Let me give a conjecture. Given the finite Galois extension of K of F build the associate group

$$\widehat{G} = \widehat{G}^0 \times \mathfrak{G}(K/F)$$

and consider pairs (φ, Y) consisting of

(i) A continuous homomorphism φ of $W_{K/F}$ into $\widehat{G}$—such that for all $w \in W_{K/F}$ the image $\varphi(w)$ acts on the Lie algebra $\mathfrak{g}$ of $\widehat{G}^0$—such that

$$\begin{array}{ccc} W_{K/F} & \xrightarrow{\varphi} & \widehat{G} \\ & \searrow & \downarrow \\ & & \mathfrak{G}(K/F) \end{array}$$

is commutative.

(ii) A nilpotent $Y \in \widehat{\mathfrak{g}}$ such that

$$\varphi(w)(Y) = q^{\alpha(w)} Y$$

for all w. Here α is the standard map $W_{K/F} \to \mathbf{Z}$ and q is the number of elements in the residue field of F.

The pair (φ_1, Y_1) associated to K_1 and (φ_2, Y_2) associated to K_2 will be regarded as equivalent if after lifting φ_1 and φ_2 to $W_{L/F}$, where $K_1 \subseteq L, K_2 \subseteq L$, we have

$$\varphi_2(w) = g\varphi_1(w)g^{-1}, \qquad Y_2 = \operatorname{Ad} g(Y_1)$$

with $g \in \widehat{G}^0$.

Recall that the family $\widehat{\mathfrak{p}}(G)$ of parabolic subgroups of $\widehat{G}$ was defined in the preprint referred to above. Let $\Phi(G) = \Phi(G/F)$ consist of those classes (φ, Y) which satisfy the following conditions.

If $\widehat{P}$ is a parabolic subgroup with Levi factor $\widehat{M}$, if $\varphi(W_{K/F}) \subseteq \widehat{M}$, and if Y lies in the Lie algebra of $\widehat{M}^0$, then $\widehat{P}$ belongs to $\widehat{\mathfrak{p}}(G)$.

It should be possible to associate to each (φ, Y) which satisfy this condition a finite set $\Pi(\varphi, Y)$ of classes of irreducible admissible representations of $G(F)$. The sets $\Pi(\varphi, Y)$ should be disjoint, and their union should contain every irreducible admissible representation.

There are several properties which one expects the map $(\varphi, Y) \to \Pi(\varphi, Y)$ to possess. For example, the elements of $\Pi(\varphi, Y)$ should be square-integrable modulo the centre if and only if (φ, Y) factors through a Levi factor of no proper parabolic subgroup of $\widehat{G}$. However, there is no point in describing these properties now. I remind you, however, of the important normalizing property.

Suppose G is a quasi-split and splits over the unramified extension K. We consider pairs $(\varphi, 0)$ where φ is of the form

$$\varphi(w) = \varphi_{\widehat{g}}(w) = (\widehat{g})^{\alpha(w)}.$$

$\widehat{g}$ is some given semi-simple element of $\widehat{G}$ which maps to the Frobenius in $\mathfrak{G}(K/F)$. As in my Washington lecture $\widehat{g}$ determines a quasi-character χ of $T(F)$ if T is a Cartan subgroup contained in a Borel over F. χ in turn determines the normalized principal series $PS(\chi)$, which of course may not be unitary. $\Pi(\varphi, 0)$ consists of the **constituents** (not components) of $PS(\chi)$ which contain the trivial representation of some special maximal compact. Observe that $\Pi(\varphi, 0)$ may contain more than one class.

All the classes in a given $\Pi(\varphi, Y)$ would be called L-indistinguishable. There are several cases which may be accessible enough that the conjecture can be further tested. The conjecture is therefore a little premature, for it would be best to examine these cases before making it. It is, by the way, my personal opinion that the local conjecture is, if correct, not too difficult, certainly not of the same order of difficulty as the global conjecture or the other problems on your list. The local theory is progressing very well in the hands of Casselman, Harish-Chandra, and Howe; I hope to see it pretty much completed within five to ten years.

ℓ-adic motivation. Ordinarily one introduces $\widehat{G} = \widehat{G}^0 \times \mathfrak{G}(K/F)$ as a complex group, namely one takes $\widehat{G}^0 = \widehat{G}^0(\mathbf{C})$. However, we could take $\widehat{G} = \widehat{G}^0(E)$ where E is a finite extension of $\mathbf{Q}_\ell$, with ℓ prime to the residual characteristic of F. In the present paragraph we do this and write $\widehat{G} = \widehat{G}(E)$. There are moreover several ways to define $\widehat{G}$:

$$\widehat{G} = \widehat{G}^0 \times \mathfrak{G}(K/F) \qquad\qquad \widehat{G} = \widehat{G}^0 \times \varprojlim \mathfrak{G}(K/F)$$

$$\widehat{G} = \widehat{G}^0 \times W_{K/F} \qquad\qquad \widehat{G} = \widehat{G}^0 \times \varprojlim W_{K/F}$$

I shall use the first, a Galois form, which is simpler. However, it is sometimes important to use the last, a Weil form, because it is more flexible. In particular, non-trivial extensions of $\mathfrak{G}(K/F)$ often split when inflated to $W_{L/F}$, if L is suitably large.

In any case suppose we are given a continuous homomorphism $\psi : \mathfrak{G}(\overline{F}/F) \to \widehat{G}$ such that

$$\begin{tikzcd} \mathfrak{G}(\overline{F}/F) \arrow[r, "\psi"] \arrow[dr] & \widehat{G} \arrow[d] \\ & \mathfrak{G}(K/F) \end{tikzcd}$$

is commutative and such that $\psi(\sigma)$ is semi-simple if σ projects to the Frobenius. Let K^{un} be the maximal unramified extension of K and let K_ℓ^+ be the union of all tamely ramified extensions of K^{un} whose degree is a power of ℓ. $\mathfrak{G}(K_\ell^+/K)$ is an extension

$$1 \to Z_\ell \to \mathfrak{G}(K_\ell^+/K) \to \widehat{\mathbf{Z}} \to 1.$$

This extension splits. Let U be the generator of $\mathbf{Z}_\ell$ and Ψ the lifting of the generator of $\widehat{\mathbf{Z}}$. If q^m is the order of the residue field of K

$$\Psi U \Psi^{-1} = U^{q^m}.$$

We may, enlarging the finite Galois extension K if necessary, suppose that ψ factors through $\mathfrak{G}(K_\ell^+/F)$ and that $\psi(U)$ is unipotent

$$\psi(U) = \exp X, \qquad X \in \widehat{\mathfrak{g}}.$$

Then

$$\sigma U \sigma^{-1} = U^{\alpha(\sigma)}$$

if σ projects to the $\alpha(\sigma)$-power of the Frobenius.

As you know there is associated to X a parabolic subgroup $\widehat{P}^0$ of $\widehat{G}^0$. Since $\mathfrak{G}(K_\ell^+/K)$ and $\mathfrak{G}(K_\ell^+/K^{\mathrm{un}})$ are normal in $\mathfrak{G}(K_\ell^+/F)$, $\widehat{P}^0$ is stable under $\psi(\mathfrak{G}(K_\ell^+/F))$ and its normalizer $\widehat{P}$ in $\widehat{G}$ is parabolic. Let $\widehat{P}$ have the Levi factor $\widehat{M}$ and the unipotent radical $\widehat{N}$. We may suppose $\psi(\Psi) \in \widehat{M}$.

If $\sigma \in \mathfrak{G}(K_\ell^+/F)$ write

$$\psi(\sigma) = n(\sigma)m(\sigma), \qquad n(\sigma) \in \widehat{N}, \quad m(\sigma) \in \widehat{M}.$$

Let

$$\sigma \Psi \sigma^{-1} = \Psi U^{x(\sigma)}, \qquad x(\sigma) \in \mathbf{Z}_\ell.$$

Then

$$\Psi^{-1} \sigma \Psi = U^{x(\sigma)} \sigma$$

and

$$\begin{aligned} \psi(\Psi^{-1}) n(\sigma) \psi(\Psi) &= U^{x(\sigma)} n(\sigma) \\ \psi(\Psi)^{-1} m(\sigma) \psi(\Psi) &= m(\sigma). \end{aligned}$$

It follows easily that

$$\exp\left(\frac{-x(\sigma)}{q^m - 1}\right) n(\sigma)$$

commutes with $\psi(\Psi)$.

Enlarging K once again if necessary, we may suppose that any eigenvalue of $\psi(\Psi)$ on $\widehat{\mathfrak{G}}$ of which some power is an integral power of q is already an integral power of q and that the centralizer of $\psi(\Psi)$ in $\widehat{G}^0$ is connected. Let $\widehat{\mathfrak{g}}_1$ be the Lie algebra spanned by

$$\{V \in \widehat{\mathfrak{G}} \mid \psi(\Psi)V = q^r V,\ r \in \mathbf{Z}\}.$$

and let $\widehat{G}_1$ be the normalizer of $\widehat{\mathfrak{g}}_1$ in $\widehat{G}$. Let $\widehat{\mathfrak{p}}_1$ be the parabolic sub-algebra of $\widehat{\mathfrak{g}}_1$ spanned by

$$\{V \in \widehat{\mathfrak{g}} \mid \psi(\Psi)V = q^r V,\ r \in \mathbf{Z}, r \geqslant 0\}.$$

Let $\widehat{P}_1$ be the normalizer of $\widehat{\mathfrak{p}}_1$ in $\widehat{G}_1$. The calculations above show that $\psi(\mathfrak{G}(K_\ell^+/F)) \subseteq \widehat{P}_1$. Let $\widehat{M_1}$ be a Levi factor of $\widehat{P}_1$ and $\widehat{N}_1$ its unipotent radical. We may suppose that $\psi(\Psi) \in \widehat{M_1}$. Since $\widehat{P}_1 = \widehat{N}_1\widehat{M_1}$, we may write

$$\psi(\sigma) = n_1(\sigma)m_1(\sigma).$$

Since $\exp bX \in \widehat{N}_1, b \in \mathbf{Q}_\ell$, and since the centralizer of $\psi(\Psi)$ in $\widehat{N}_1$ is trivial we conclude by the same argument as before that

$$n_1(\sigma) = \exp\Big(\frac{x(\sigma)}{q^m - 1}X\Big).$$

The map

$$\sigma \longrightarrow m_1(\sigma)$$

is clearly a representation of $\mathfrak{G}(K_\ell^+/F)$ by semi-simple matrices. Moreover

$$m_1(\sigma)(X) = q^{\alpha(\sigma)}X$$

and $m_1(\sigma)$ factors through $\mathfrak{G}(K^{\mathrm{un}}/F)$.

If we have an imbedding $\eta : E \to \mathbf{C}$ and if $w \to \sigma$ is the standard map $W_{K/F} \to \mathfrak{G}(K^{\mathrm{un}}/F)$ we may define φ, Y by

$$\varphi : w \to \eta(m_1(\sigma)), \qquad Y = \eta(X).$$

Conclusion: An ℓ-adic representation of $\mathfrak{G}(\overline{F}/F)$ in $\widehat{G}(E)$ plus an imbedding $E \to \mathbf{C}$ yields a pair (φ, Y).

You will find the imbedding $\eta : E \to \mathbf{C}$ repugnant. I do not find it repugnant, merely perplexing. What I thoroughly detest is the introduction of ℓ-adic representations of $G(F)$. The study of elliptic modular forms shows us already that we cannot pass from the ℓ-adic theory to the complex theory, and therefore to L-functions, without imbeddings $\overline{\mathbf{Q}} \hookrightarrow \mathbf{C}, \overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_\ell$. One point which troubles me, but which I have not yet seriously pondered, is the following. If the eigenvalues of all $\psi(\sigma)$, for any linear representation of $\widehat{G}(E)$, lie in $\overline{\mathbf{Q}}$ is (φ, Y) independent of η, provided η is the identity on $\overline{\mathbf{Q}} \cap E$.

A basic question, which you are best qualified to discuss, is: **What is a motive of type $\widehat{G}$ over a local field?**

Observe that it is clear how to define the local L-function $L(s, \pi, \rho)$ for $\pi \in \Pi(\varphi, Y)$.

Global theory. If G is a connected reductive group over the global field F two representations $\pi = \bigotimes \pi_v$ and $\pi' = \bigotimes \pi'_v$ which occur as **constituents** of the space of automorphic forms on $G(\mathbf{A})$ would be called L-indistinguishable if π_v and π'_v are L-indistinguishable for each v. The basic object in the theory of automorphic forms is a class of L-indistinguishable representations.

Basic problem. Suppose that H and G are given over the global field F. Take $\widehat{H}$ and $\widehat{G}$ in the Weil form. Suppose an L-homomorphism ψ (cf. the preprint referred to earlier) $\widehat{H} \to \widehat{G}$ is given. Suppose $\pi = \bigotimes \pi_v$ occurs as a constituent of the space of automorphic forms on H. Let $\pi_v \in \Pi(\varphi_v, Y_v)$. If ψ_v is the restriction of ψ to the associate group over F_v, suppose $(\psi_v \circ \varphi_v, \psi_v(Y_v))$ lie in $\Phi(G/F_v)$ for each v. Is there then a constituent $\pi' = \bigotimes \pi'_v$ of the space of automorphic forms on G such that $\pi'_v \in \Pi(\psi_v \circ \varphi_v, \psi_v(Y_v))$ for all v?

An affirmative solution to this problem entails of course the solution to Artin's conjecture. The question can be motivated entirely within the theory of automorphic forms by the following two considerations:

(a) **The Artin principle.** The only "natural" way to prove analytic continuation and the functional equation for an L-function is that of Hecke and Tate. This is effective only for certain L-functions associated to simple algebras. To handle any other L-function, one must prove a reciprocity law which shows it equals one of these.

(b) **Examples.** A number of examples were known.
   (i) $H = 1, G = \mathrm{GL}(1)$—Class field theory.
   (ii) H, G abelian—ditto
   (iii) H the Levi factor of a parabolic subgroup of G over F and $\psi : \widehat{H} \to \widehat{G}$ the associated imbedding—this is part of the theory of Eisenstein series.
   (iv) H : the multiplicative group of a quadratic extension, $G = \mathrm{GL}(2)$, and $\psi : \widehat{H} \to \widehat{G}$ the obvious imbedding—this is treated by Hecke and Maass.
   (v) $H = \mathrm{GL}(2), G$ obtained from $\mathrm{GL}(2)$ over a quadratic extension by restriction of scalars, $\psi : \widehat{H} \to \widehat{G}$, the diagonal map on $\widehat{H}^0$—this is Doi, Naganuma, Jacquet.

Two vague problems. Suppose π occurs in the space of automorphic forms on G.

(i) Give a "reasonable" necessary and sufficient condition that $L(s, \pi, \rho)$ be holomorphic for $\mathrm{Re}\, s > 1$ and all representations ρ of the Galois form of $\widehat{G}$. As you know, these are the π which will satisfy Ramanujan's conjecture.

(ii) Give a "reasonable" necessary and sufficient condition that there exist a group H, a $\psi : \widehat{H} \to \widehat{G}$, and a constituent π' of the space of automorphic forms on H such that π lies in the L-indistinguishable class on G obtained from ψ and π' and such that, for every representation ρ of $\widehat{H}$, $L(s, \pi', \rho)$ is holomorphic for $\mathrm{Re}\, s > 1$ with a pole at $s = 1$ of order equal to the multiplicity with which the trivial representation occurs in ρ. These are the π for which one can expect to handle the Sato-Tate conjecture. Namely for almost all $v, \pi'_v \in \Pi(\varphi_{g_v}, 0)$ with g_v in a maximal compact $\widehat{H}_{\mathrm{comp}}$ of $\widehat{H}(\mathbf{C})$. The conjugacy classes $\{g_v\}$ should be uniformly distributed in $\widehat{H}_{\mathrm{comp}}$.

Motives. Would you be willing to try to define a motive of type $\widehat{G}$ over a global field? One will of course want to know which L-indistinguishable classes of global π correspond to motives. I assume you believe that for a function field the necessary and sufficient condition is that for almost all v, $\pi_v \in \Pi(\varphi_{g_v}, 0)$ where $\rho(g_v)$ has eigenvalues in $\overline{\mathbf{Q}}$ for all representations ρ of the Galois form of $\widehat{G}$.

Notice that if F is $\mathbf{R}$ or $\mathbf{C}$ then W_F contains $\mathbf{C}^\times = S(\mathbf{R})$. A homomorphism $\varphi : W_F \to \widehat{G}$ with

$$\begin{tikzcd} W_F \arrow[r, "\varphi"] \arrow[dr, "\mathrm{id}"'] & \widehat{G} \arrow[d] \\ & W_F \end{tikzcd}$$

commutative could be said to be of type A_0 if the restriction of φ to $\mathbf{C}^\times$ factored as

$$\mathbf{C}^\times = S(\mathbf{R}) \hookrightarrow S(\mathbf{C}) \xrightarrow{\eta} \widehat{G}$$

where η is algebraic over $\mathbf{C}$.

Obvious guess. For a number field, π corresponds to a motive if and only if for each infinite place $v, \pi_v \in \Pi(\varphi_v)$ where φ_v is of type A_0.

I suppose that the problem of defining a motive of type $\widehat{G}$ is intimately related to the Tate conjectures and therefore to such problems as the isogeny of two elliptic curves with the same zeta-function. Lang hoped to apply the methods of transcendental numbers to this problem. He has not yet succeeded. None the less it is tantalizing to think that such methods may one day play a role.

All the best.
Yours,

R. Langlands
RL:MMM

Appendix

Recall the setup of your Bourbaki talk:

- G over $\mathbf{Q}$ is given. I take it to be connected.
- $h : S \to G$ over $\mathbf{R}$.
- $E = E(G, h)$
- $K \subseteq G(\mathbf{A}_f)$ open, compact

Suppose V is a finite set of primes and

$$K = K_V \prod_{\ell \notin V} K_\ell$$

with

$$K \subseteq \prod_{\ell \notin V} G(\mathbf{Q}_\ell)$$

and where K_ℓ, $\ell \notin V$, is a special maximal compact, that is, the intersection of the stabilizer of a special vertex in the building of $G_{sc}(\mathbf{Q}_p)$ with

$$\{g \in G(\mathbf{Q}_\ell) \mid |\chi(g)|_\ell = 1 \text{ for every rational character } \chi \text{ of } G\}.$$

G_{sc} is the simply connected covering group of the derived group G_{der}. I also suppose that, outside of V, G is quasi-split over $\mathbf{Q}_\ell$ and split over an unramified extension of $\mathbf{Q}_\ell$.

Then the Shimura variety S_K should be realizable as a smooth scheme over $O_E[1/\ell, \ell \in V]$ if O_E is the ring of integers in E. I am not sure how one can characterize this scheme or exactly what additional properties it should have, especially when $S_K(\mathbf{C})$ is not compact. In any case if $\mathfrak{p}$ is a prime of E dividing p and $p \notin V$ and if $k_\mathfrak{p}$ is the residue field at $\mathfrak{p}$ one should be able to speak of $S_K(\overline{k}_\mathfrak{p})$, the set of geometrical points over $\overline{k}_\mathfrak{p}$.

What I want to describe now is the way it seems to be possible to predict the structure of $S_K(\overline{k}_\mathfrak{p})$ in terms of h and G alone. One of my ambitions for the immediate future is to prove that the prediction is correct for a large number of cases. I think that if one uses the word "large" in a sufficiently modest sense this should not be too difficult. For you it would probably be a simple exercise, but I need to acquire more technique.

I fix $\mathfrak{p}$ and p and write

$$K = K^p K_p \qquad K^p \subseteq G(\mathbf{A}_f^p).$$

Rather than $S_K(\overline{k}_\mathfrak{p})$ consider

$$\varprojlim_{K^p} S_K(\overline{k}_\mathfrak{p}) = S_{K_p}(\overline{k}_\mathfrak{p}).$$

The group $G(\mathbf{A}_f^p)$ acts on $S_{K_p}(\overline{K}_\mathfrak{p})$ to the right. To obtain $S_K(\overline{k}_\mathfrak{p})$ back again, I have only to divide by K^p. If Φ is the Frobenius over $k_\mathfrak{p}$, then Φ acts on $S_K(\overline{k}_\mathfrak{p})$ to the left.

The set $S_{K_p}(\overline{k}_\mathfrak{p})$ is the union of certain subsets invariant under $G(\mathbf{A}_f^p)$ and Φ. It is these subsets that I want to describe. Each of them is constructed from the following data:

(i) A group H over $\mathbf{Q}$ and an imbedding $H(\mathbf{A}_f^p) \hookrightarrow G(\mathbf{A}_f^p)$.
(ii) A group $\overline{G}$ over $\mathbf{Q}_p$ and an imbedding $H(\mathbf{Q}_p) \hookrightarrow \overline{G}(\mathbf{Q}_p)$.
(iii) A space X on which $\overline{G}(\mathbf{Q}_p)$ and Φ act, the two actions commuting with each other.

The imbeddings $H(\mathbf{A}_f^p) \hookrightarrow G(\mathbf{A}_f^p)$, $H(\mathbf{Q}_p) \hookrightarrow \overline{G}(\mathbf{Q}_p)$ define an action of $H(\mathbf{Q})$ on $G(\mathbf{A}_f^p) \times X$. The subsets of $S_{K_p}(\overline{k}_p)$ to which I referred have the form

$$Y = H(\mathbf{Q})\backslash G(\mathbf{A}_f^p) \times X$$

$G(\mathbf{A}_f^p)$ acts in the obvious way and Φ acts through its action on X.

The first step is to construct the various H and $\overline{G}$. I suppose $\overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$ is fixed, in order that E be defined and then I fix $\overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$ so that the prime of E it defines is $\mathfrak{p}$.

Suppose γ belongs to $G(\mathbf{Q})$ and is semi-simple. Suppose moreover that all the eigenvalues of γ have absolute value 1 away from ∞ and p. Recall that G is a linear group, so that it makes sense to speak of eigenvalues. Let

$$H^0 = \{g \in G \mid g\gamma^m = \gamma^m g \text{ for some } m \neq 0 \text{ in } \mathbf{Z}.\}.$$

H^0 is connected. Suppose $h^0 : S \to H^0$ is defined over $\mathbf{R}$ and the composition $S \xrightarrow{h^0} H^0 \hookrightarrow G$ is conjugate under $G(\mathbf{R})$ to h.

$h^0 \circ r$ defines $\mathrm{GL}(1) \to H^0$ over $\mathbf{C}$. If T is a Cartan subgroup of H^0 through which it factors we obtain an element of

$$\widehat{L}(T) = \mathrm{Hom}(\mathrm{GL}(1), T).$$

By conjugation we obtain for any T in H^0 an orbit $\{\widehat{\mu}\}$ of the Weyl group (of T in H^0) in $\widehat{L}(T)$.

If T is a Cartan subgroup in H^0 over $\mathbf{Q}_p$, define $\widehat{\lambda}(\gamma) \in \widehat{L}(T)$ by

$$|\lambda(\gamma)|_p = p^{-\langle \lambda, \widehat{\lambda}(\gamma)\rangle}.$$

Let M be the set of rational characters of H^0 defined over $\mathbf{Q}_p$. I say that the pair (γ, h^0) is of Frobenius type if there is an $r > 0$ in $\mathbf{Q}$ so that $\widehat{\lambda}(\gamma) - r\widehat{\mu}$ is orthogonal to M. Here $\widehat{\mu}$ lies in the orbit defined above. The pairs (γ_1, h_1^0), (γ_2, h_2^0) are said to be equivalent if there exists $m > 0$, $n > 0$, $\delta \in G(\mathbf{Q})$, and $g \in H_2^0(\mathbf{R})$ so that

$$\gamma_2^m = \delta\gamma_1^{n\delta - 1} \qquad h_2^0 = \mathrm{ad}(g\delta) \circ h_1^0.$$

The sets Y in $S_{K_p}(\overline{k}_{\mathfrak{p}})$ are indexed by equivalence classes of pairs of Frobenius type.

Fix one such pair. The group H will be obtained from H^0 by an inner twisting. Choose a Cartan subgroup T of H^0 defined over $\mathbf{Q}$ so that $T \cap H^0_{\mathrm{der}}$ is anisotropic at ∞ and p. Replacing h^0 by $\mathrm{ad}\, g \circ h^0$ if necessary, we suppose that h^0 factors through T. To be definite take $\widehat{\mu}$ in $\widehat{L}(T) = \mathrm{Hom}(\mathrm{GL}(1), T)$ to be $h^0 \circ r$. Let T_{ad} be the image of T in H^0_{ad} and suppose T splits over k.

The twisting giving H lies in $H^1(\mathfrak{G}(\overline{\mathbf{Q}}/\mathbf{Q}), H^0_{\mathrm{ad}}(\overline{\mathbf{Q}}))$. I shall give it as an element of

$$H^1(\mathfrak{G}(k/\mathbf{Q}), T_{\mathrm{ad}}(k)),$$

but omit the verification, which is a matter of standard techniques in Galois cohomology, that this actually yields an element of $H^1(\mathfrak{G}(\overline{\mathbf{Q}}/\mathbf{Q}), H^0_{\mathrm{ad}}(\overline{\mathbf{Q}}))$, which does not depend on the choice of T, K, or h^0, provided of course the above conditions are satisfied. I shall actually define an element of $H^1(\mathfrak{G}(k/\mathbf{Q}), T_{\mathrm{ad}}(\mathbf{A}(k)))$. Tate-Nakayama theory shows readily that it lies in the image of

$$H^1(\mathfrak{G}(k/\mathbf{Q}), T_{\mathrm{ad}}(k)) \to H^1(\mathfrak{G}(k/F), T_{\mathrm{ad}}(\mathbf{A}(k))).$$

Since the Hasse principle is valid for H^0_{ad}, it will not matter how I pull back.

The element of $H^1(\mathfrak{G}(k/\mathbf{Q}), T_{\mathrm{ad}}(\mathbf{A}(k)))$ I am about to define will be trivial at every finite prime except p. To define it at infinity and p, I observe that for any place v

$$T(k_v) = \widehat{L}(T) \otimes k_v^\times.$$

Suppose $a_{\sigma,\tau}$ is the fundamental 2-cocycle for $K_v/\mathbf{Q}_v$

$$\sigma \to a_\sigma = \sum_{\tau \in \mathfrak{G}(K_v/\mathbf{Q}_v)} \sigma\tau\widehat{\mu} \otimes a_{\sigma,\tau}$$

is not a cocycle in $T(k_v)$. However, its image $\{b_\sigma\}$ in $T_{\mathrm{ad}}(k_v)$ is. At ∞ and p the element I want is the class of $\{b_\sigma\}$.

I have still to define $\overline{G}$. It will be a twisted form over $\mathbf{Q}_p$ of a group $\overline{G}^0$. $\overline{G}^0$ is the connected group whose Lie algebra is spanned by vectors V satisfying

$$\mathrm{Ad}\,\gamma(V) = \epsilon V \qquad \epsilon \in \overline{\mathbf{Q}}_p$$

with $|\epsilon|_p = 1$. $\overline{G}^0$ contains H^0. Thus $T \subseteq \overline{G}^0$. Let $\overline{T}_{\mathrm{ad}}$ be the image of T in $\overline{G}^0_{\mathrm{ad}}$. One verifies that the image $\{\bar{b}_\sigma\}$ of $\{a_\sigma\}$ in $\overline{T}_{\mathrm{ad}}$ is a 1-cocycle. This is the cocycle one uses to twist $\overline{G}^0$ and hence to obtain $\overline{G}$.

It remains to define the space X. I first define a cocycle of $W_{k_p/\mathbf{Q}_p}$ with values in $\overline{G}^0(\mathbf{Q}_p)$. I use the cocycle $a_{\sigma,\tau}$ to define $W_{k_p/\mathbf{Q}_p}$ as an extension of $\mathfrak{G}(k_p/\mathbf{Q}_p)$ by $k_p^\times$. I write $\widehat{\lambda} \otimes x \in \widehat{L}(T) \otimes k_p^\times = T(k_p)$ as $x^{\widehat{\lambda}}$. Let

$$\nu = \sum_{\tau \in \mathfrak{G}(k_p/\mathbf{Q}_p)} \tau\widehat{\mu}$$

and if $w = (x, \sigma) \in W_{k_p/\mathbf{Q}_p}$ set

$$b_w = x^{\widehat{\nu}} a_\sigma$$

$w \to b_w$ is in fact a cocycle.

I restrict it to W_{k_p/E_p}. Note: $E_p = E_\mathfrak{p}$ is the closure of E in $\overline{\mathbf{Q}}_p$. We have assumed that G is quasi-split at p and split over an unramified extension. One verifies that this is also valid for $\overline{G}^0$. Choose a Borel subgroup $\overline{B}$ of $\overline{G}^0$ and a Cartan subgroup $\overline{T}$ in it, both defined over $\mathbf{Q}_p$. By conjugation, $\widehat{\mu}$ defines an orbit in $\widehat{L}(\overline{T})$. Let $\widehat{\lambda}$ be that element of the orbit which lies in the Weyl chamber positive with respect to $\overline{B}$. $\mathfrak{G}(k_p/E_p)$ is by definition the stabilizer of $\widehat{\lambda}$. The parabolic subgroup $\overline{P} \supseteq \overline{B}$ of $\overline{G}^0$ whose Lie algebra is spanned by that of $\overline{T}$ and the set of all root vectors X_α with $\langle \alpha, \widehat{\lambda} \rangle \geq 0$ is defined over E_p.

On the other hand K_p was taken to be special which means under the present circumstances (according to my understanding of Bruhat-Tits) that it is defined by a Chevalley basis of $\mathfrak{G}_{sc}$, the Lie algebra of G_{sc}. This in turn yields a Chevalley basis for $\overline{\mathfrak{G}}_{sc}$ and hence a special maximal compact $\overline{K}_p$ of $\overline{G}^0(E_p)$.

$\overline{P}$ also defines a parabolic subgroup over $k_\mathfrak{p}$ and hence a parahoric subgroup $\overline{I}$ in $\overline{K}_p$. A parahoric subgroup is the set of g in $\overline{G}^0(E_p)$ which fix the points of the polysimplex associated to a parahoric in $\overline{G}^0_{sc}(E_p)$ and for which

$$|\chi(g)|_p = 1$$

if χ is a rational character of $\overline{G}^0$.

$\overline{I}$ yields then a parahoric $\overline{I}(k_p^{un})$ in $\overline{G}^0(k_p^{un})$, if k_p^{un} is the maximal unramified extension of k_p. The mapping $W_{k_p/E_p} \to \mathfrak{G}(k_p^{un}/E_p)$ yields an action of W_{k_p/E_p} in $\overline{G}^0(k_p^{un})$. Since $\overline{I}(k_p^{un})$ is fixed it also yields an action $\eta'(w)$ of W_{k_p/E_p} on $\overline{G}^0(k_p^{un})/\overline{I}(k_p^{un})$. $\overline{G}^0(k_p^{un})$ also acts. Let $\eta(w)$ be the action $b_w\eta'(w)$. Since $\overline{G}$ is obtained from $\overline{G}^0$ by a twisting we made regard $\overline{G}(\mathbf{Q}_p)$

as a subgroup of $\overline{G}^0(k_p)$. It is clear that the actions of $\overline{G}(\mathbf{Q}_p)$ and W_{k_p/E_p} commute. Let W^0 be the kernel of $W_{k_p/E_p} \to \mathfrak{G}(k_p^{un}/E_p)$ and let X' be the set of points in $\overline{G}^0(k_p^{un})/\overline{I}(k_p^{un})$ fixed by W^0. On X' both $\overline{G}(\mathbf{Q}_p)$ and Φ act. Φ of course acts through its inverse image in W_{k_p/E_p}.

Any point x' in X' determines by projection a parahoric of $\overline{G}^0_{sc}(k_p^{un})$ and hence a polysimplex x in the Bruhat-Tits building of $\overline{G}^0_{sc}(k_p^{un})$. My candidate for X is the set of all $x' \in X'$ such that if $y' = \Phi x'$ then the polysimplices x and y have a point in common.

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