November 9, 2007

To Toby Gee:

I am a little embarrassed about yesterday's question, but on reflection I understand that implicit in it there is a very simple question that I had never properly clarified for myself. Rather there are two questions that I had never properly distinguished. I comment on them and the difference between them now, but more for my own sake than for yours.

The group is GL(2) over $\mathbf{Q}$ for which there are automorphic representations π and associated Galois representations $\sigma = \sigma(\pi)$.

(1) *Is the answer to the question whether π is associated to a σ dependent on the class of π_∞ alone?*

(2) *Let μ be an algebraic representation of the group* GL(2) *over* $\mathbf{Q}$. *Associated to μ there is a formal construction in the étale cohomology theory—first employed by Deligne—that attaches to a certain class of π the desired representation $\sigma = \sigma(\pi)$. This construction is discussed again in my Antwerp paper, where a start was made on showing that for all primes p the representation σ_p is really the local correspondent of π_p. That was finally shown by Carayol.*

Notice that π becomes π' below! These are presumably two different problems! There is a third question and it is—in some sense—one of the questions that you asked.

(3) *Which of the representations, either π or σ, appearing in* (1) *are obtained by the construction of* (2)*?*

The answer to this is not a question at infinity alone. *A priori* there may be representations that are correct at infinity in the sense of (1) but that do not arise from (2). *A posteriori* it is clear what they are. I omit, by the way, the one-dimensional automorphic representations. They are not pertinent here.

There is a fourth question.

(4) *Is the construction of* (2) *possible for* SL(2)*?*

The answer of course is no, because there is no Shimura variety for SL(2).

It is the question (3) that I want to answer, but for infinite-dimensional π. I will answer it in terms of my Antwerp talk,[1] but that is equivalent to the usual construction in the theory of modular forms. For my references to pages of my Antwerp notes, I use the version of the text that is on the web. Let μ be the representation of GL(2) with character

$$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \to \frac{\alpha^n\beta^m - \alpha^m\beta^n}{\alpha - \beta}, \qquad n > m.$$

Then $\pi = \pi(\mu)$ on p. 21 is such that

$$g \to \pi'_\infty(g) = |\det g|^{-1/2}\pi_\infty(g) \tag{1}$$

[1]`https://publications.ias.edu/rpl/paper/77`

is the representation of GL(2) to which the attached representation of the Weil group is induced from $z \to z^{-n}\overline{z}^{-m}$.

It is proved in the paper—as a reformulation of well-understood ideas—that to every automorphic representation satisfying (1), there is an associated Galois representation $\sigma = \sigma(\pi)$. The conjecture stated on p. 33 and partially proved in the paper is that for every π and every $\pi'_p = \sigma_p$. Thus σ is the Galois representation associated to π'.[2] In this formulation, the function of the cyclotomic character is not so prominent. The central character has a unique decomposition as a Dirichlet character times a cyclotomic character and this cyclotomic character is given a special importance in the classical treatment that it does not have in a more adelic treatment.

Of course, we could for the statement of the theorem work with π' alone. The representation π is a simple twisting of π' by a character. The cohomological construction of σ depends, however, on π. This is what you were telling me is true in general. The mystery—or rather the delicate point—lies in the definition of Shimura varieties and the analysis of their cohomology. This is an interesting issue, perhaps even an important issue, but not necessarily an intrinsic part of any general conjecture. I never before clearly understood that, as a consequence of the theorem, for all π' with π'_∞ of algebraic (??) type, there is indeed a Galois representation. We have of course excluded at least one case, that for which σ_∞ has $m = n$.

It would be useful to have an analysis along the above lines for a general Shimura variety. It is perhaps implicit in Clozel's paper.

[2]Editorial comment: See author's comments on this letter.

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