November 9, 1962
Princeton, N.J.

Dear Professor Harish-Chandra,

Thank you very much for your kindness in allowing me to read your paper on spherical functions. I have waited until now to return it because I wanted to send you my paper on automorphic forms at the same time. If you don't mind I would like to ask you question.

Suppose G is semi-simple (with finite center for simplicity) and K a maximal compact subgroup. Let $\sigma(K)$ be a matrix representation of K and $F(g_1, q_2)$ a function on $G \times G$ such that $F(g_1 q, q_2 g) = F(g_1, g_2)$ and $F(k_1 g_1, k_2 g_2) = \sigma(k_1) F(g_1, q_2) = \sigma(k_1) F(g_1, q_2) \sigma^{-1}(k_2)$. In order to apply the Selberg trace formula to concrete problems one has to be able to express

$$\phi_F(\gamma) = \int_{G \backslash G_\gamma} \operatorname{tr}\bigl(F(g\gamma g^{-1}, 1)\bigr)\, ds_\gamma$$

in terms of the "Fourier transform" of F. $\gamma = \exp(H)$ is semi-simple and G_γ is the centralizer of γ in G. If π is a unitary representation of G let $\{x_\ell^k\}$, $1 \leqslant \ell \leqslant d(\sigma)$, $1 \leqslant k \leqslant d(\pi, \sigma)$ be vectors such that, for each k, $(x_1^k, \ldots, x_{d(\sigma)}^k)$ transform under $\pi(k)$ according to σ. Thus

$$\int_G \sum_{\ell=1}^{d(\sigma)} \overline{F}_{i\ell}(1, g) \pi^*(g) x_\ell^k \, dg = \sum_j \overline{\pi}_{jk}(F) x_i^j.$$

$\bigl(\pi_{jk}(F)\bigr)$ is the Fourier transform of F at π. If there is a Plancherel formula then

$$F_{pq}(g, 1) = \int \operatorname{tr}\bigl(\pi(k)\omega_{pq}(g, \pi, \sigma)\bigr)\, d\omega(\pi).$$

where the $\bigl(\omega_{pq}(g, \pi, \sigma)\bigr)$ are a kind of elementary spherical function. Roughly speaking, $\phi_F(\gamma)$ is a distribution on the functions $\pi(F)$. (π varying over the representations occurring in the Plancherel formula.) It will be necessary to prove that this distribution is a function so that

$$\phi_F(\gamma) = \int \operatorname{tr}\bigl(\pi(F) T(\pi, \gamma)\bigr)\, d\mu(\pi).$$

where for convenience ω is replaced by a measure μ simpler than the Plancherel measure. For the applications I have in mind $\pi(F)$ can be given explicitly so the problem is to determine $T(\pi, \gamma)$ explicitly. For the discrete series you described in 1956 this is essentially the problem discussed in the paper I am sending you. I have an idea, which I will describe in a moment, for calculating it for the various continuous series. However there is no point in my trying to carry this through if, as is quite possible, you already know what $T(\pi, \gamma)$ is. I would appreciate it if, when I come to Columbia, you could tell me whether or not this is so.

It is enough to calculate $\phi_F(\gamma)$ when γ is regular so that G_γ can be replaced by the connected component of the identity in the centralizer B of a Cartan subalgebra j. Also

$\phi_F(\gamma)$ can be replaced by

$$\psi_F(\gamma) = \prod_{\alpha \in P} \left(e^{\frac{1}{2}\alpha(H)} - e^{-\frac{1}{2}\alpha(H)} \right) \phi_F(\gamma).$$

If p is a polynomial on j invariant under the Weyl group then

$$\partial(p)\psi_F(\gamma) = \phi_{Z_p F}(\gamma) \qquad \text{(formula for semi-simple groups).}$$

But $Z_p f(g,1) = \int \operatorname{tr}\bigl(\phi(K) Z_p \omega(g,\pi,\sigma)\bigr)\, d\mu(\pi)$. For each π there is H_π so that

$$Z_p \omega(g,\pi,\sigma) = p(H_\pi)\omega(g,\pi,\sigma).$$

Consequently

$$\partial(p) T(\pi,\gamma) = p(H_\pi) T(\pi,\gamma)$$

and

$$T(\pi,\gamma) = \sum_{s \in W} c(s,\pi) e^{\langle sH_\pi, H \rangle}.$$

It should be enough to determine $c(s,\pi)$ when B is compact. There are a number of boundary conditions that $c(s,\pi)$ must satisfy. These are probably no use for the discrete series but then $T(\pi,\gamma)$ is closely related to the trace. It is to be hoped however that for the principal series they will determine $c(\pi,\gamma)$ Then one will have to combined the two extremes to obtain $c(\pi,\gamma)$ for the other representations. The boundary conditions that I can see at present are

(i) $T(\pi,\gamma)$ is skew-symmetric under the Weyl group of K_c if $j \subseteq k$.

(ii) According to note at the end of the last 1957 paper in the American Journal $c_1 = \cdots = c_r$. The value of the Plancherel measure itself would give another boundary condition but it is to be hoped that it is not necessary to use this.

(iii) For brevity I will be a bit careless in describing this condition. If $G_0 = \mathrm{SL}(2,\mathbf{R})$ and $B_0 = \{u_\theta\}$, $B_{1,0} = \{h_t\}$ in the notation of your note on the Plancherel formula then

$$\lim_{t \to 0} \sinh t \int_{G_0 \backslash B_{1,0}} f(g_0 h_t g_0^{-1}) = c \lim_{\theta \to 0} \sin\theta \left\{ \int_{G \backslash B_0} \bigl\{ f(g u_\theta g^{-1}) + f(g u_{-\theta} g^{-1}) \bigr\} \right\}$$

Suppose $\gamma \in B$ is semi regular and is contained in a non-compact B_1. Then there is a G_0 (or a covering group of G_0) contained in the centralizer of γ. Choose $\gamma' \in G_0 \cap B$ so that $\gamma\gamma'$ is regular and $\gamma'' \in G_0 \cap B_1$ so that $\gamma\gamma''$ is regular; then

$$\text{(a)} \quad \int_{G \backslash B} \operatorname{tr}\bigl\{ F(g\gamma\gamma'' g^{-1}, 1) + F(g\gamma\gamma'^{-1} g^{-1}, 1) \bigr\}$$
$$= c \int_{G \backslash G_\gamma} ds_\gamma \int_{G \backslash B_0} \operatorname{tr}\bigl\{ F(g\gamma g_0 \gamma' g_0^{-1} g^{-1}, 1) + F(g\gamma g_0 \gamma'^{-1} g_0^{-1} g^{-1}, 1) \bigr\}$$

$$\text{(b)} \qquad \int_{G \backslash B_1} \operatorname{tr}\bigl\{ F(g\gamma\gamma'' g^{-1}, 1) \bigr\} = c \int_{G \backslash G_\gamma} ds_\gamma \int_{G \backslash B_{1,0}} \operatorname{tr}\bigl\{ F(g\gamma g_0 \gamma'' g_0^{-1} g^{-1}, 1) \bigr\}$$

Multiply (a) and (b) by the appropriate factors and send γ', γ'' to 1. Then the right hand sides will differ only by a constant. If we use induction on the dimension of j the left hand side of (a) can be assumed known in terms of $\pi(F)$ and the left-hand side of (b) is expressed in terms of $c(s,\pi)$.

The simplest example is $\mathrm{SL}(2,\mathbf{R})/\{\pm 1\}$. Then if $d\mu(\pi) = d\lambda$, $\pi = z_\lambda^+$.

$$\sin\theta/2 \; T(z_\lambda^+, u_\theta) = c_1 e^{\lambda\theta} + c_2 e^{-\lambda\theta}$$

The first condition gives nothing. The second and third give

$$\lambda(c_1 - c_2) = -\lambda(c_1 e^{2\pi\lambda} - c_2 e^{-2\pi\lambda}); \quad e_1(1 + e^{2\pi\lambda}) + c_2(1 + e^{-2\pi\lambda}) = \alpha.$$

α is a constant. Solving we obtain $c_1 = \frac{e^{-\pi\lambda}}{4\cosh\pi\lambda}\alpha$, $c_2 = \frac{e^{\pi\lambda}}{4\cosh\pi\lambda}\alpha$ so that the Plancherel measure is $\beta\lambda\frac{\sinh\pi\lambda}{\cosh\pi\lambda}$. These values of c_1 and c_2 agree with the formula in Selberg's paper.

I have worked out other examples but at present have no general way of utilizing these boundary conditions.

Yours truly,

R. P. Langlands

Compiled on July 3, 2024.