May 18, 1965
Berkeley, CA

Dear Harish-Chandra,

Thank you for paper No. 1. I have been slow in responding to your letter because there is something I want to mention to you that I had to clarify it for myself.

Unfortunately I do not have my notes for the talks I gave in 62/63 with me so I cannot be sure exactly what I did prove then. Let me tell you what I still remember how to prove. Suppose G is connected, γ is regular, the Cartan subalgebra j which γ centralizes is non-compact, and the integral

$$\text{(a)} \qquad \int_{G_\gamma\backslash G} f(g^{-1}\gamma g)\, d\bar{g}$$

is absolutely convergent. These are the conditions of which I am not sure; besides they could be weakened or at least changed by applying results available at the time and Th 3 of your paper No. 5 (Inv. Eig.-Dist. on a S.S.L.G)[1] however it was also necessary to suppose (and this is the real weakness of my result) that f is a finite linear combination of functions of the form $\big(\pi(g)\phi,\phi\big)$ where π is an irreducible *integrable* representation of G and ϕ and ψ transform according to some finite-dimensional representation of K. The conclusion was of course that under these conditions the integral (a) vanished. As I recall the proof was as follows.

Let $\mathfrak{a}$ be the non-compact part of j, let A be the group with Lie algebra $\mathfrak{a}$ and as usual let $G = ANMK$. It is enough to show that

$$\int_N f(ng)\, dn = 0 \qquad \text{for all } g \in G$$

and it is not hard to show that this is a consequence of the vanishing almost everywhere of

$$\text{(b)} \qquad \int_G \varphi(gh) f(h)\, dh$$

for all infinitely differentiable functions φ on $N\backslash G$ with compact support. If $f(g) = \big(\pi(g)\phi,\phi\big)$ this will follow if it is shown that π does not occur discretely in the regular representation of G on $L^2(N\backslash G)$. To show this one shows that the Casimir operator has no eigenfunctions in $L^2(N\backslash G)$ and if this is done by utilizing the Fourier transform with respect to A, which acts on $N\backslash G$ to the left. I once mentioned this procedure to you and you indicated that you were familiar with it.

Finally I should like to make a comment, which might interest you, about the kind of integrals which occur in the process of getting a trace formula. If

$$\lambda(f)\phi(g) = \int_G \phi(gh) f(h)\, dh \qquad \big(\phi \in L^2(\Gamma\backslash G)\big)$$

[1] *Invariant eigendistributions on a semisimple Lie group,* 1964, Trans. Am. Math. Soc. 119:457–508

the problem is to express the trace $T(f)$ of the restriction of $\lambda(f)$ to the space of cusp forms in terms of $T_\pi(f)$, or better $\pi(f)$, π irreducible and unitary. $T(f)$ is of course an invariant "distribution" and if $\Gamma\backslash G$ one is led to express it in terms of integrals (a), also invariant, and the problem is to calculate the Fourier transforms of the latter distributions. If, however, $\Gamma\backslash G$ is not compact the process envisaged by Selberg is such that one is led to the problem of expressing certain elementary but *not* invariant "distributions" in terms of the various $\pi(f)$. I am still unable to reduce the contribution from the non-semi-simple elements to the trace form to the elementary form; however, I believe I am now in a position to reduce the contribution from the semi-simple elements to an elementary form. Let me describe to you the kind of integral that results. To understand the following replace percuspidal by minimal parabolic/$\mathbf{Q}$.

Let P be a percuspidal subgroup, let $\mathfrak{a}$ be a split component of P, and let Ω be the Weyl group of $\mathfrak{a}$ (normalizer/centralizer). A distinguished subspace of $\mathfrak{a}$ is a subspace defined by the vanishing of a subset of the set of simple roots. If $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are two distinguished subspaces $\Omega(\mathfrak{a}_1, \mathfrak{a}_2)$ is the set of all linear transformations from $\mathfrak{a}_1$ to $\mathfrak{a}_2$ obtained by restricting elements of Ω to $\mathfrak{a}_1$. $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are called associate if $\Omega(\mathfrak{a}_1, \mathfrak{a}_2)$ is not empty. To each distinguished subspace there corresponds a cuspidal (read parabolic/$\mathbf{Q}$) subgroup. Let $\mathfrak{a}^0$ be a distinguished subspace; let $\mathfrak{a}^1, \ldots, \mathfrak{a}^r$ be the distinguished subspaces associate to $\mathfrak{a}^0$. Let P^i, $0 \leqslant i \leqslant r$, be the cuspidal subgroup corresponding to $\mathfrak{a}^i$; if A^i is the connected group with Lie algebra $\mathfrak{a}^i$ then $P^i = A^i S^i$ ($S^i = M^i N^i$ is a normal subgroup of P^i). Let K be a maximal compact subgroup of G (suppose G is semi-simple with finite centre) and suppose there is a Cartan involution with $\theta(H) = -H$, $H \in \mathfrak{a}$, $\theta(X) = X$, $X \in k$.

Suppose $\sigma \in \Omega(\mathfrak{a}^0, \mathfrak{a}^i)$; let $s(\sigma)$ be a representative of σ in the normalizer of $\mathfrak{a}$. If $s(\sigma)g = bsk$, $b \in A^i$, $s \in S^i$, $k \in K$ let $H_\sigma(g)$ in $\mathfrak{a}^0$ be such that

$$\exp H\sigma(g) = s^{-1}(\sigma) b^{-1} s(\sigma)$$

$H_\sigma(g)$ depends only on σ and g. The points $H_\sigma(g)$, $\sigma \in \bigcup_{i=1}^r \Omega(\mathfrak{a}^0, \mathfrak{a}^i)$ are the vertices of a convex polyhedron in $\mathfrak{a}^0$. Let the volume of this polyhedron be $\xi(g)$. $\xi(g)$ is a function on $A^0 M^0 \backslash G$. Suppose γ is such that $A^0 \subseteq G_\gamma \subseteq A^0 M^0$. The integral I want to draw to your attention is

$$\int_{G_\gamma\backslash G} f(g^{-1}\gamma g)\xi(g)\, d\bar{g}.$$

Yours truly,

Bob Langlands

Compiled on December 22, 2023.