

April 18, 1968  
Ankara, Turkey

Dear Harish-Chandra,

I have been wanting to write you for some time to thank you for your letter and to assure you that everything is satisfactory here. However, I had begun to form some suspicions about representations of reductive algebraic groups over local fields and I wanted at least to check these carefully in the case of  $GL_2$  before writing to you. I see now that I will have left here before I finish so I started to write you anyway and voice the suspicions in a premature form.

I remind you that if  $k$  is a local field and  $K$  a normal extension then  $G_{K,k}$  (the Weil group of  $K/k$ ) is a certain extension of  $K^*$  by the Galois group of  $K/k$ . If  $k \subseteq K \subseteq L$  with  $K$  and  $L$  normal there is a homomorphism  $\varphi$  of  $G_{L,k}$  into  $G_{K,k}$ . This homomorphism is not uniquely determined but if  $\rho$  is a representation of  $G_{K,k}$  the equivalence class of  $\rho \circ \varphi$  is uniquely determined. Thus if  $\rho_1$  and  $\rho_2$  are representations of  $G_{K_1,k}$  and  $G_{K_2,k}$  respectively I can call them equivalent if they become equivalent when lifted to  $G_{K_1K_2,k}$ . Also I can speak of the representations of the Weil group of  $k$  without being explicit about the choice of  $K$ . Now in my old letter to Weil I made a rough attempt to define the dual group of a reductive group. The attempt was not satisfactory but provides a basis for thought. I have come to believe that associated to almost every equivalence class of continuous representations of the Weil group in this dual group (a complex group) which is such that  $\rho(\sigma)$  is semi-simple for all  $\sigma$  there should correspond an equivalence class of representations of the algebraic group. In particular to *every* unitary representation of the Weil group in this dual group should correspond a unitary representation of the algebraic group over  $k$ . To give some basis to this belief and to complete the things I was doing with Jacquet I wanted to check this out completely for  $GL_2$ . In this case the dual group is  $GL(2, \mathbf{C})$ .

If the local field is non-archimedean and the characteristic of the residue class field is different from 2 then there are basically only two ways of getting a representation of the Weil group in  $GL(2, \mathbf{C})$ . Either take  $K = k$  so that  $G_{K,k} = k^*$  and send  $\alpha \rightarrow \begin{pmatrix} \chi_1(\alpha) & 0 \\ 0 & \chi_2(\alpha) \end{pmatrix}$  where  $\chi_1$  and  $\chi_2$  are two characters of  $k^*$  or take  $K$  to be a quadratic extension of  $k$ , take a character  $\chi$  of  $K^*$ , and let  $\chi$  induce a representation of  $G_{K,k}$ . Now given  $\chi_1, \chi_2$  or  $\chi$  we know how to construct a representation of  $GL(2, k)$ . The only thing to check is that if the representations give equivalent representations of the Weil group in the sense mentioned above then the representations of  $GL(2, k)$  are equivalent. This I have done, although I should probably look at the proof again. It can, for example, happen that  $K$  and  $K'$  are distinct quadratic extensions while the representations induced from  $\chi$  and  $\chi'$  give equivalent representations of the Weil group. Archimedean fields are even simpler.

However, it is very likely that if the characteristic of the residue class field is 2 there are two-dimensional representations of the Weil group which are not abelian and are not associated to a character of a quadratic extension of  $k$ . This is why I told Jacquet that I considered it unlikely that we had exhausted the representations in this case. Although my

work in this is proceeding at a reasonably steady pace it will be a while before it is finished. I hope to be able to tell you something definite when I return in August.

All the best,

Bob Langlands

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