April 18, 1968
Ankara, Turkey

Dear Harish-Chandra,

I have been wanting to write you for some time to thank you for your letter and to assure you that everything is satisfactory here. However, I had begun to form some suspicions about representations of reductive algebraic groups over local fields and I wanted at least to check these carefully in the case of GL_2 before writing to you. I see now that I will have left here before I finish so I started to write you anyway and voice the suspicions in a premature form.

I remind you that if k is a local field and K a normal extension then $G_{K,k}$ (the Weil group of K/k) is a certain extension of K^* by the Galois group of K/k. If $k \subseteq K \subseteq L$ with K and L normal there is a homomorphism φ of $G_{L,k}$ into $G_{K,k}$. This homomorphism is not uniquely determined but if ρ is a representation of $G_{K,k}$ the equivalence class of $\rho \circ \varphi$ is uniquely determined. Thus if ρ_1 and ρ_2 are representations of $G_{K_1,k}$ and $G_{K_2,k}$ respectively I can call them equivalent if they become equivalent when lifted to $G_{K_1K_2,k}$. Also I can speak of the representations of the Weil group of k without being explicit about the choice of K. Now in my old letter to Weil I made a rough attempt to define the dual group of a reductive group. The attempt was not satisfactory but provides a basis for thought. I have come to believe that associated to almost every equivalence class of continuous representations of the Weil group in this dual group (a complex group) which is such that $\rho(\sigma)$ is semi-simple for all σ there should correspond an equivalence class of representations of the algebraic group. In particular to *every* unitary representation of the Weil group in this dual group should correspond a unitary representation of the algebraic group over k. To give some basis to this belief and to complete the things I was doing with Jacquet I wanted to check this out completely for GL_2. In this case the dual group is $\mathrm{GL}(2, \mathbf{C})$.

If the local field is non-archimedean and the characteristic of the residue class field is different from 2 then there are basically only two ways of getting a representation of the Weil group in $\mathrm{GL}(2, \mathbf{C})$. Either take $K = k$ so that $G_{K,k} = k^*$ and send $\alpha \to \begin{pmatrix} \chi_1(\alpha) & 0 \\ 0 & \chi_2(\alpha) \end{pmatrix}$ where χ_1 and χ_2 are two characters of k^* or take K to be a quadratic extension of k, take a character χ of K^*, and let χ induce a representation of $G_{K,k}$. Now given χ_1, χ_2 or χ we know how to construct a representation of $\mathrm{GL}(2, k)$. The only thing to check is that if the representations give equivalent representations of the Weil group in the sense mentioned above then the representations of $\mathrm{GL}(2, k)$ are equivalent. This I have done, although I should probably look at the proof again. It can, for example, happen that K and K' are distinct quadratic extensions while the representations induced from χ and χ' give equivalent representations of the Weil group. Archimedean fields are even simpler.

However, it is very likely that if the characteristic of the residue class field is 2 there are two-dimensional representations of the Weil group which are not abelian and are not associated to a character of a quadratic extension of k. This is why I told Jacquet that I considered it unlikely that we had exhausted the representations in this case. Although my work in this is proceeding at a reasonably steady pace it will be a while before it is finished. I hope to be able to tell you something definite when I return in August.

All the best,

Bob Langlands

Compiled on November 12, 2024.