Dear Harish-Chandra,

I am very pleased that you are willing to come here to speak. I have decided to come to Princeton on Monday, October 7 and to stay over till Tuesday. However I don’t want to give a lecture then because Weil and Jacquet are familiar with anything conclusive I did before January and I have not yet finished the things I have been doing since. I have already given you an idea what these things are but I can perhaps be more precise now.

If $F$ is a local field let $C_F$ be the multiplicative group of $F$ and if $F$ is a global field let $C_F$ be the idèle class group of $F$. As I said before if $K/F$ is normal the Weil group $G_{K/F}$ is an extension of $C_F$ by the Galois group of $K/F$. If one likes one can take projective limits and get an object called the Weil group of $F$. If $F$ is global, and $F_p$ is a completion of $F$, and $K_p$ is a completion of $K$ over $p$ then there is a map of $G_{K_p/F_p}$ into $G_{K/F}$ and every representation of $G_{K/F}$ determines a representation of $G_{K_p/F_p}$. A representation $\rho$ is a finite-dimensional representation over $\mathbb{C}$ such that $\rho(g)$ is semi-simple for all $g$.

For a non-archimedean local field I can attach to every equivalence class $\omega$ of representations a function $L(s,\omega)$ in just one way so that the following conditions are satisfied.

(i) If $\omega \sim \chi_F$ is one dimensional

$$L(s,\omega) = \frac{1}{1 - \chi_F(\pi_F)|\pi_F|^s} \quad \text{if } \chi_F \text{ is trivial on units}$$

$$= 1 \quad \text{if } \chi_F \text{ is not trivial on units.}$$

(ii) $L(s,\omega_1 \oplus \omega_1) = L(s,\omega_1)L(s,\omega_2)$.

(iii) If $\omega_1 \simeq \text{Ind}(G_{K/F},G_{K/E},\omega_2)$ then

$$L(s,\omega_1) = L(s,\omega_2).$$

For archimedean fields $L(s,\omega)$ is defined by the following conditions

(i) If $F = \mathbb{R}$ and $\omega \sim \chi_\mathbb{R}$ with $\chi_\mathbb{R}(x) = (\text{sgn } x)^m |x|^r$, $m = 0$ or 1, then

$$L(s,\omega) = \pi^{-\frac{1}{2}(s+r+m)} \Gamma \left( \frac{s + r + m}{2} \right).$$

(ii) If $F = \mathbb{C}$ and $\omega \sim \chi_\mathbb{C}$ with $\chi_\mathbb{C}(z) = |z|^{2r} \frac{z^n \bar{z}^m}{1 + |z|^{m+n}}, m + n \geq 0, mn = 0$ then

$$L(s,\omega) = 2(2\pi)^{-s+r+m+n}\Gamma(s + r + \frac{m+n}{2}).$$

(iii) $L(s,\omega_1 \oplus \omega_2) = L(s,\omega_1)L(s,\omega_2)$

(iv) If $\omega_1 \simeq \text{Ind}(G_{K/F},G_{K/E},\omega_2)$ then

$$L(s,\omega_1) = L(s,\omega_2).$$
If $K$ is a global field, $\omega$ is an equivalence class of representations of the Weil group and $\omega_p$ the induced equivalence class of the Weil group of $F_p$ the Artin $L$-function is

$$L(s, \omega) = \prod_p L(s, \omega_p)$$

Let me state the theorem which I have been working on since last January and then comment on its relation to the functional equations of the $L$-functions. By the way, as soon as I understand a lemma of Dwork I should be able to write out the proof of this theorem. I think I have all the parts of the proof except for some easily managed details. If $F = \mathbb{R}$ and $\chi_F(x) = (\text{sgn } x)^m |x|^r$ with $m = 0$ or $1$ and $\psi_R$ is the additive character $\psi_R(x) = e^{2\pi iux}$ I set

$$\Delta(\chi_R, \psi_R) = (i \text{ sgn } u)^m |u|^r.$$

If $F = \mathbb{C}$ and $\chi_C(z) = |z|^{2r} \frac{z^m}{|z|^{n+\pi}}, m + n \geq 0, mn = 0$ and $\psi_C$ is the additive character $\psi_C(z) = e^{4\pi i \text{Re}(wz)}$ I set

$$\Delta(\chi_C, \psi_C) = i^{m+n} \chi_C(w).$$

If $F$ is non-archimedean, if $\chi_F$ is a generalized character of $C_F$ with conductor $\mathcal{P}_F^m$, if $\psi_F$ is a non-trivial additive character of $F$ and $\mathcal{P}_F^{-n}$ is the largest ideal on which it vanishes and $O_F\gamma = \mathcal{P}_F^{m+n}$ I set

$$\Delta(\chi_F, \psi_F) = \chi_F(\gamma) \left| \int_{U_F} \psi_F(\frac{\alpha}{\gamma}) \chi_F^{-1}(\alpha) \, d\alpha \right|.$$

$U_F$ is the group of units. The right side is independent of $\gamma$.

If $E$ is a separable extension of $F$ and $\psi_F$ is given then

$$\psi_{E/F}(X) = \psi_F(\text{Tr}_{E/F} X).$$

**Theorem.** Suppose $F$ is a given local field and $\psi_F$ a given non-trivial additive character of $F$. It is possible in exactly one way to assign to each separable extension $E$ of $F$ a complex number $\rho(E/F, \psi_F)$ and to each equivalence class $\omega$ of representations of the Weil group of $E$ a complex number $\epsilon(\omega, \psi_{E/F})$ so that

(i) If $\omega \simeq \chi_E$ then $\epsilon(\omega, \psi_{E/F}) = \Delta(\chi_E, \psi_{E/F})$

(ii) $\epsilon(\omega_1 \otimes \omega_2, \psi_{E/F}) = \epsilon(\omega_1, \psi_{E/F})\epsilon(\omega_2, \psi_{E/F})$

(iii) If $\omega_1 \simeq \text{Ind}(G_{K/F}, G_{K/F}, \omega_2)$ then

$$\epsilon(\omega_1, \psi_F) = \rho(E/F, \psi_F)^{\text{dim} \omega_2} \epsilon(\omega_2, \psi_{E/F}).$$

If $A^s_F$ is the generalized character $\alpha \rightarrow |\alpha|^s_F$ set $\epsilon(s, \omega, \psi_F) = \epsilon(A^{s-1/2}_F \otimes \omega, \psi_F)$. If $F$ is a global field and $\psi_F$ a non-trivial character of $A_F/F$ let $\psi_{F_p}$ be the restriction of $\psi_F$ to $F_p$. If $\omega$ is an equivalence class of representations of the Weil group of $F$ and

$$\epsilon(s, \omega) = \prod_p \epsilon(s, \omega_p, \psi_{F_p})$$

the functional equation of the $L$-function can, on the basis of the previous theorem, be shown to be

$$L(s, \omega) = \epsilon(s, \omega)L(1 - s, \tilde{\omega})$$

if $\tilde{\omega}$ is contragredient to $\omega$. 
Once I have this theorem I should very quickly be able to derive a consequence of interest for group representations. However I have not yet carried out the computations. After some preliminaries I shall state the consequence.

If $F$ is a non-archimedean local field a two-dimensional equivalence class $\omega$ of representations of the Weil group of $F$ will be called special if $\omega$ is the direct sum of two one-dimensional representations $\mu_F$ and $\nu_F$ and $\mu_F \nu_F^{-1} = A_F^1$ or $A_F^{-1}$. If $F = \mathbb{R}$ a two-dimensional equivalence class $\omega$ is special if $\omega \simeq \mu_F \oplus \nu_F$, and $\mu_F(x) = |x|^{s_1} \left( \frac{x}{|x|} \right)^{m_1}$, $\nu_F(x) = |x|^{s_2} \left( \frac{x}{|x|} \right)^{m_2}$ and $(s_1 - s_2) - (m_1 - m_2)$ is an odd integer. If $F = \mathbb{C}$, $\omega$ is special if $\omega \simeq \mu_F \oplus \nu_F$

$$\mu_F(z) = |z|^{2s_1} \left( \frac{z}{|z|} \right)^{m_1}$$

$$\nu_F(z) = |z|^{2s_2} \left( \frac{z}{|z|} \right)^{m_2}$$

and one of $\frac{s_1 - s_2}{2} - (1 + \frac{|m_1 - m_2|}{2})$ and $\frac{s_2 - s_1}{2} - (1 + \frac{|m_1 - m_2|}{2})$ is a non-negative integer.

$L(\psi_F)$ is the space of functions on $GL(2, F)$ satisfying

$$\varphi \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) \equiv \psi_F(x) \varphi(g)$$

together with certain conditions on smoothness and rate of growth. Here is the consequence I mentioned.

**Theorem.** Suppose that for every global field $F$ and every two-dimensional irreducible equivalence class of representations of the Weil group of $F$ the function $L(s, \omega)$ is entire and bounded in vertical strips. Then if $F$ is a local field, $\omega$ a two-dimensional equivalence class of representations of the Weil group of $F$ which is not special, and $\psi_F$ a non-trivial additive character of $F$ there is a unique simple representation $\pi_\omega$ of $GL(2, F)$ satisfying

(i) $\pi_\omega$ acts on $L \subseteq L(\psi_F)$

(ii) If $\varphi$ belongs to $L$ and $\chi_F$ is a generalized character of $C_F$ the integral

$$\int_{C_F} \varphi \left( \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} g \right) \chi_F(\alpha) |\alpha|_F^s \, d\alpha$$

converges for $\text{Re } s$ sufficiently large. Denote its value by $\Phi(g, s, \varphi, \chi_F)$ and set

$$\Phi(g, s, \varphi, \chi_F) = L(s + \frac{1}{2}, \omega \otimes \chi_F) \Phi'(g, s, \varphi, \chi_F).$$

$\Phi'(g, s, \varphi, \chi_F)$ is an entire function of $s$ bounded in vertical strips. Moreover

$$\Phi' \left( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g, -s, \varphi(\det \omega \chi_F)^{-1} \right) = \epsilon(\chi_F A_F^s \otimes \omega, \psi_F) \Phi'(g, s, \varphi, \chi_F)$$

if $\det \omega$ is the 1-dimensional representation obtained from $\omega$ by taking determinants.

It will follow that $\pi_{\omega_1}$ equivalent to $\pi_{\omega_2}$ implies $\omega_1 = \omega_2$. This theorem makes the existence of the representations I mentioned to you earlier pretty much certain. If they do not exist the $L$-series behave in an entirely unexpected manner.

Yours truly,

Bob Langlands