September 22, 1969
New Haven, CT

Dear Harish-Chandra,

I am very pleased that you are willing to come here to speak. I have decided to come to Princeton on Monday, October 7 and to stay over till Tuesday. However I don't want to give a lecture then because Weil and Jacquet are familiar with anything conclusive I did before January and I have not yet finished the things I have been doing since. I have already given you an idea what these things are but I can perhaps be more precise now.

If F is a local field let C_F be the multiplicative group of F and if F is a global field let C_F be the idèle class group of F. As I said before if K/F is normal the Weil group $G_{K/F}$ is an extension of C_F by the Galois group of K/F. If one likes one can take projective limits and get an object called the Weil group of F. If F is global, and $F_\mathfrak{p}$ is a completion of F, and $K_\mathfrak{P}$ is a completion of K over $\mathfrak{p}$ then there is a map of $G_{K_\mathfrak{P}/F_\mathfrak{p}}$ into $G_{K/F}$ and every representation of $G_{K/F}$ determines a representation of $G_{K_\mathfrak{P}/F_\mathfrak{p}}$. A representation ρ is a finite-dimensional representation over $\mathbf{C}$ such that $\rho(g)$ is semi-simple for all g.

For a non-archimedean local field I can attach to every equivalence class ω of representations a function $L(s,\omega)$ in just one way so that the following conditions are satisfied.

(i) If $\omega \sim \chi_F$ is one-dimensional

$$L(s,\omega) = \frac{1}{1-\chi_F(\pi_F)|\pi_F|^s} \qquad \text{if } \chi_F \text{ is trivial on units}$$
$$= 1 \qquad \text{if } \chi_F \text{ is not trivial on units.}$$

(ii) $L(s,\omega_1 \oplus \omega_1) = L(s,\omega_1)L(s,\omega_2)$.

(iii) If $\omega_1 \simeq \operatorname{Ind}(G_{K/F}, G_{K/E}, \omega_2)$ then

$$L(s,\omega_1) = L(s,\omega_2).$$

For archimedean fields $L(s,\omega)$ is defined by the following conditions

(i) If $F = \mathbf{R}$ and $\omega \sim \chi_\mathbf{R}$ with $\chi_\mathbf{R}(x) = (\operatorname{sgn} x)^m |x|^r$, $m = 0$ or 1, then

$$L(s,\omega) = \pi^{-\frac{1}{2}(s+r+m)}\Gamma\left(\frac{s+r+m}{2}\right).$$

(ii) If $F = \mathbf{C}$ and $\omega \sim \chi_\mathbf{C}$ with $\chi_\mathbf{C}(z) = |z|^{2r}\frac{z^n\bar{z}^n}{|z|^{m+n}}$, $m+n \geqslant 0$, $mn = 0$ then

$$L(s,\omega) = 2(2\pi)^{-\left(s+r+\frac{m+n}{2}\right)}\Gamma\left(s+r+\frac{m+n}{2}\right)$$

(iii) $L(s,\omega_1 \oplus \omega_2) = L(s,\omega_1)L(s,\omega_2)$

(iv) If $\omega_1 \simeq \operatorname{Ind}(G_{K/F}, G_{K/E}, \omega_2)$ then

$$L(s,\omega_1) = L(s,\omega_2).$$

If K is a global field, ω is an equivalence class of representations of the Weil group and $\omega_{\mathfrak{p}}$ the induced equivalence class of the Weil group of $F_{\mathfrak{p}}$ the Artin L-function is

$$L(s,\omega)=\prod_{\mathfrak{p}} L(s,\omega_{\mathfrak{p}})$$

Let me state the theorem which I have been working on since last January and then comment on its relation to the functional equations of the L-functions. By the way, as soon as I understand a lemma of Dwork I should be able to write out the proof of this theorem. I think I have all the parts of the proof except for some easily managed details. If $F=\mathbf{R}$ and $\chi_{\mathbf{R}}(x)=(\operatorname{sgn} x)^m|x|^r$ with $m=0$ or 1 and $\psi_{\mathbf{R}}$ is the additive character $\psi_{\mathbf{R}}(x)=e^{2\pi iux}$ I set

$$\Delta(\chi_{\mathbf{R}},\psi_{\mathbf{R}})=(i\operatorname{sgn} u)^m|u|^r.$$

If $F=\mathbf{C}$ and $\chi_{\mathbf{C}}(z)=|z|^{2r}\frac{z^m\overline{z}^n}{|z|^{m+n}}$, $m+n\geqslant 0$, $mn=0$ and $\psi_{\mathbf{C}}$ is the additive character $\psi_{\mathbf{C}}(z)=e^{4\pi i\operatorname{Re}(wz)}$ I set

$$\Delta(\chi_{\mathbf{C}},\psi_{\mathbf{C}})=i^{m+n}\chi_{\mathbf{C}}(w).$$

If F is non-archimedean, if χ_F is a generalized character of C_F with conductor $\mathfrak{P}_F^m$, if ψ_F is a non-trivial additive character of F and $\mathfrak{P}_F^{-n}$ is the largest ideal on which it vanishes and $O_F\gamma=\mathfrak{P}_F^{m+n}$ I set

$$\Delta(\chi_F,\psi_F)=\chi_F(\gamma)\frac{\int_{U_F}\psi_F\left(\frac{\alpha}{\gamma}\right)\chi_F^{-1}(\alpha)\,d\alpha}{\left|\int_{U_F}\psi_F\left(\frac{\alpha}{\gamma}\right)\chi_F^{-1}(\alpha)\,d\alpha\right|}.$$

U_F is the group of units. The right side is independent of γ.

If E is a separable extension of F and ψ_F is given then

$$\psi_{E/F}(X)=\psi_F(\operatorname{Tr}_{E/F}X).$$

Theorem. *Suppose F is a given local field and ψ_F a given non-trivial additive character of F. It is possible in exactly one way to assign to each separable extension E of F a complex number $\rho(E/F,\psi_F)$ and to each equivalence class ω of representations of the Weil group of E a complex number $\epsilon(\omega,\psi_{E/F})$ so that*

(*i*) *If $\omega\simeq\chi_E$ then $\epsilon(\omega,\psi_{E/F})=\Delta(\chi_E,\psi_{E/F})$*
(*ii*) *$\epsilon(\omega_1\oplus\omega_2,\psi_{E/F})=\epsilon(\omega_1,\psi_{E/F})\epsilon(\omega_2,\psi_{E/F})$*
(*iii*) *If $\omega_1\simeq\operatorname{Ind}(G_{K/F},G_{K/E},\omega_2)$ then*

$$\epsilon(\omega_1,\psi_F)=\rho(E/F,\psi_F)^{\dim\omega_2}\epsilon(\omega_2,\psi_{E/F}).$$

If A_F^s is the generalized character $\alpha\to|\alpha|_F^s$ set $\epsilon(s,\omega,\psi_F)=\epsilon(A_F^{s-1/2}\otimes\omega,\psi_F)$. If F is a global field and ψ_F a non-trivial character of $\mathbf{A}_F/F$ let $\psi_{F_{\mathfrak{p}}}$ be the restriction of ψ_F to $F_{\mathfrak{p}}$. If ω is an equivalence class of representations of the Weil group of F and

$$\epsilon(s,\omega)=\prod_{\mathfrak{p}}\epsilon(s,\omega_{\mathfrak{p}},\psi_{F_{\mathfrak{p}}})$$

the functional equation of the L-function can, on the basis of the previous theorem, be shown to be

$$L(s,\omega)=\epsilon(s,\omega)L(1-s,\widetilde{\omega})$$

if $\widetilde{\omega}$ is contragredient to ω.

Once I have this theorem I should very quickly be able to derive a consequence of interest for group representations. However I have not yet carried out the computations. After some preliminaries I shall state the consequence.

If F is a non-archimedean local field a two-dimensional equivalence class ω of representations of the Weil group of F will be called special if ω is the direct sum of two one-dimensional representations μ_F and ν_F and $\mu_F\nu_F^{-1} = A_F^1$ or A_F^{-1}. If $F = \mathbf{R}$ a two-dimensional equivalence class ω is special if $\omega \simeq \mu_F \oplus \nu_F$, and $\mu_F(x) = |x|^{s_1}\left(\frac{x}{|x|}\right)^{m_1}$, $\nu_F(x) = |x|^{s_1}\left(\frac{x}{|x|}\right)^{m_2}$ and $(s_1 - s_2) - (m_1 - m_2)$ is an odd integer. If $F = \mathbf{C}$, ω is special if $\omega \simeq \mu_F \oplus \nu_F$

$$\mu_F(z) = |z|^{2s_1}\left(\frac{z}{|z|}\right)^{m_1}$$

$$\nu_F(z) = |z|^{2s_2}\left(\frac{z}{|z|}\right)^{m_2}$$

and one of $\frac{s_1-s_2}{2} - \left(1 + \frac{|m_1-m_2|}{2}\right)$ and $\frac{s_2-s_1}{2} - \left(1 + \frac{|m_1-m_2|}{2}\right)$ is a non-negative integer.

$L(\psi_F)$ is the space of functions on $\mathrm{GL}(2, F)$ satisfying

$$\varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) \equiv \psi_F(x)\varphi(g)$$

together with certain conditions on smoothness and rate of growth. Here is the consequence I mentioned.

Theorem. *Suppose that for every global field F and every two-dimensional irreducible equivalence class of representations of the Weil group of F the function $L(s, \omega)$ is entire and bounded in vertical strips. Then if F is a local field, ω a two-dimensional equivalence class of representations of the Weil group of F which is not special, and ψ_F a non-trivial additive character of F there is a unique simple representation π_ω of $\mathrm{GL}(2, F)$ satisfying*

(i) π_ω acts on $L \subseteq L(\psi_F)$

(ii) If φ belongs to L and χ_F is a generalized character of C_F the integral

$$\int_{C_F} \varphi\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} g\right) \chi_F(\alpha)|\alpha|_F^s \, d\alpha$$

converges for $\mathrm{Re}\, s$ sufficiently large. Denote its value by $\Phi(g, s, \varphi, \chi_F)$ and set

$$\Phi(g, s, \varphi, \chi_F) = L\left(s + \frac{1}{2}, \omega \otimes \chi_F\right)\Phi'(g, s, \varphi, \chi_F).$$

$\Phi'(g, s, \varphi, \chi_F)$ is an entire function of s bounded in vertical strips. Moreover

$$\Phi'\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g, -s, \varphi(\det\omega\chi_F)^{-1}\right) = \epsilon(\chi_F A_F^s \otimes \omega, \psi_F)\Phi'(g, s, \varphi, \chi_F)$$

if $\det\omega$ is the 1-dimensional representation obtained from ω by taking determinants.

It will follow that π_{ω_1} equivalent to π_{ω_2} implies $\omega_1 = \omega_2$. This theorem makes the existence of the representations I mentioned to you earlier pretty much certain. If they do not exist the L-series behave in an entirely unexpected manner.

Yours truly,

Bob Langlands

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