March 24, 1974

Dear Roger,

You observed to me several years ago that a cuspidal representation of a Chevalley group G over a finite field $\mathfrak{o}_F/\mathfrak{p}_F$ yielded by induction an absolutely cuspidal representation over the local field F itself. I was intrigued, for as you know I am always trying to understand my Washington problems better and, in particular, always looking for examples in which the suggestions I made there could be tested.

I decided that the representations obtained in the above manner should correspond to homomorphisms φ of the Weil group into the associate group with the following two properties:

(i) As usual φ is realized as a homomorphism of the Weil group $W_{K/F}$ at a finite level

$$1 \longrightarrow K^\times \longrightarrow W_{K/F} \longrightarrow \mathfrak{G}(K/F) \longrightarrow 1$$

and $\varphi(w)$ is semi-simple for all w. The first new property to be insisted upon is that φ be tamely ramified, that is, be trivial on $1+\mathfrak{p}_K$. Moreover K is to be unramified but arbitrarily large.

(ii) The image $\varphi(W_{K/F})$ is contained in no proper parabolic subgroup of $\widehat{G}$, which is, because G is a Chevalley group, a direct product of its connected component $G^{\widehat{o}}$ and $\mathfrak{G}(K/F)$.

Then I tried to check this for Sp(4) by using Mrs. Srinivasan's results. Everything was almost perfect. To each such homomorphism there corresponded, as I shall describe later, in a fairly natural way finitely many cuspidal representations of $\mathrm{Sp}(4,\kappa)$ where $\kappa = \mathfrak{o}_F/\mathfrak{p}_F$. There was, alas, one difficulty. It was not clear what to do with the anomalous representation. I have been puzzled by this representation ever since. Your recent letter suggests a way out. You probably know whether or not it is feasible; so I would appreciate your comments. However I have first to describe the difficulty. What I want to verify first is that the possible φ, which are of course determined only up to conjugacy by elements of $G^{\widehat{o}}$, correspond in a 1:1 manner to pairs consisting of an anisotropic torus T over ϕ, the residue field of F, and a "non-degenerate" character of $T(\phi)$. First of all let me deduce some consequences of (i) and (ii) and the other usual conditions on φ. Since G is a Chevalley group, $\widehat{G}$ is a direct product $G^{\widehat{o}} \times \mathfrak{G}(K/F)$. Since φ composed with $\widehat{G} \to \mathfrak{G}(K/F)$ must be the standard map $W_{K/F} \to \mathfrak{G}(K/F)$, we may regard φ as a homomorphism of $W_{K/F}$ into $G^{\widehat{o}}$.

We may divide by $1+\mathfrak{p}_K \subset K^\times$ to get an extension

$$1 \longrightarrow \kappa^\times \longrightarrow H \longrightarrow \mathbf{Z} \longrightarrow 1.$$

Here κ is the residue field of K. $1 \in \mathbf{Z}$ is the Frobenius. If q is the number of elements in ϕ then $z \in \mathbf{Z}$ acts on $\kappa^\times$ by $\theta \longrightarrow \theta^{q^z}$. The extension is split.

Since $\kappa^\times$ is cyclic Theorem E.5.16 of Borel et al., *Seminar on algebraic groups* together with its proof shows that there is a torus $\widehat{T} \subset G^{\widehat{o}}$ which contains $\varphi(\kappa^\times)$ and is normalized by $\varphi(\mathbf{Z})$. Set $\omega = \varphi(1)$. It is the image of $\varphi(1)$. (Observe: one usual demand is that the image of φ consist of semi-simple elements.)

Claim. *$\widehat{T}$ is the connected component of the centralizer of the image of $\varphi(\kappa^\times)$.*

Observe that, because of (ii), ω, which normalizes T, can fix no rational character of T. Let $\widehat{\mathfrak{g}}_1$ be the centralizer of $\varphi(\kappa^\times)$ in $\widehat{\mathfrak{g}}$, the Lie algebra of $G^{\widehat{o}}$. $\widehat{\mathfrak{g}}_1$ is reductive and is normalized by ω. By Gantmacher (Mat. Sb. (1939)) there is a Cartan subalgebra $\widehat{\mathfrak{t}}$ and a Borel subalgebra $\mathfrak{p}$ of $\widehat{\mathfrak{g}}_1$ normalized by ω. Since $\widehat{\mathfrak{g}}_1$ clearly has the same rank as $\widehat{\mathfrak{g}}$ we may suppose $\widehat{\mathfrak{t}}$ is the Lie algebra of $\widehat{T}$. Then ω fixes the sum of the simple roots of $\widehat{\mathfrak{t}}$ with respect to $\widehat{\mathfrak{p}}$. Thus the sum must be 0. That is, $\widehat{\mathfrak{g}}_1 = \widehat{\mathfrak{t}}$ as required.

Corollary. *If $\widehat{\alpha}$ is a root of $\widehat{T}$ there is a $\theta \in \kappa^\times$ such that $\widehat{\alpha}\big(\varphi(\theta)\big) \neq 1$.*

The Weyl group of $\widehat{T}$ is the same as the Weyl group of T_0, a split Cartan subgroup of G. Thus the image of ω in the Weyl group can be used to twist T_0 to T, a Cartan subgroup of G over ϕ. T is anisotropic.

L: lattice of rational characters of T_0
$\widehat{L}$: lattice of rational characters of $\widehat{T}$
$\widehat{L} = \mathrm{Hom}(L, \mathbf{Z})$.

Notice $T(\kappa) \simeq \widehat{L} \otimes \kappa^\times$. If θ is a fixed generator of $\kappa^\times$ then we write $\widehat{\lambda} \otimes \theta = \theta^{\widehat{\lambda}}$. This represents an arbitrary element of $T(\kappa)$. The Frobenius sends

$$\theta^{\widehat{\lambda}} \longrightarrow \theta^{q\omega\widehat{\lambda}}.$$

Thus if s is the order of $\kappa^\times$

$$T(\phi) = \left\{\theta^{\widehat{\lambda}} \,\middle|\, q\omega\widehat{\lambda} - \widehat{\lambda} \in s\widehat{L}\right\}.$$

Thus the characters of $T(\phi)$ are the characters of

$$(*) \qquad \left\{\widehat{\lambda} \,\middle|\, q\omega\widehat{\lambda} - \widehat{\lambda} \in s\widehat{L}\right\} \quad \text{modulo } s\widehat{L}.$$

On the other hand, $\widehat{T}$ and $\omega = \varphi(1)$ being given, consider all ways of defining φ on $\kappa^\times$. We have only to define $\varphi(\theta)$ or

$$\widehat{\lambda}\big(\varphi(\theta)\big), \qquad \theta \in \widehat{L}.$$

The condition is

$$\widehat{\lambda}\big(\varphi(\theta^q)\big) = \widehat{\lambda}\Big(\omega\big(\varphi(\theta)\big)\Big) = \omega^{-1}\widehat{\lambda}\big(\varphi(\theta)\big)$$

or

$$q\omega\widehat{\lambda}\big(\varphi(\theta)\big) = \widehat{\lambda}\big(\varphi(\theta)\big).$$

Thus the set of possible φ is, since $\widehat{\lambda}\big(\varphi(\theta)\big)$ must be an s^{th} root of unity, the set of characters of

$$(**) \qquad \widehat{L} \text{ modulo } (q\omega - 1)\widehat{L} + s\widehat{L}.$$

Since $n = \det(q\omega - 1)$ is prime to p we may choose K so large that it is divisible by s. Set $M = q\omega - 1 : \widehat{L} \longrightarrow \widehat{L}$. There is an N such that

$$MN = n.$$

If $\widehat{\lambda} \in \widehat{L}$ and $\frac{s}{n}N\widehat{\lambda} = \widehat{\mu}$ then $M\widehat{\mu} \in s\widehat{L}$. If $\widehat{\lambda} = M\widehat{\nu}$ then $\widehat{\mu} \in s\widehat{L}$. Thus $\frac{s}{n}N$ defines a map from the group $(**)$ to the group $(*)$. It is easily seen to be an isomorphism. The character groups are also isomorphic.

The φ associated to a character of $(**)$ will satisfy (ii) if and only if the character is 1 on no root $\widehat{\alpha}$. A character of $(*)$ will therefore be called non-degenerate if it is 1 on no $\widehat{\beta} = \frac{s}{n}N\widehat{\alpha}, \widehat{\alpha}$ a root. Observe also that $\widehat{T}$ being given ω is only determined up to conjugacy within the normalizer and that only the image of ω in the Weyl group matters for ω can be replaced by $t\omega t^{-1} = t\omega(t^{-1})\omega, t \in \widehat{T}$ and $t\omega(t^{-1})$ is arbitrary because ω fixes no rational character. The image of ω in the Weyl group being given, $\varphi(\theta)$ is determined only up to the action of the centralizer of ω in the Weyl group. This means that the character of $T(\phi)$ is only determined up to the action of the Weyl group of T over ϕ.

The group $\mathrm{Sp}(4)$**.** There are two possibilities for ω.

(i) Rotation through 90°. The centralizer has order 4.

(ii) Rotation through 180°. The centralizer has order 8.

If we represent the roots of T as $(x, y) \longrightarrow x - y, x + y, 2x, 2y$, then the dual roots may be represented as

$$\begin{matrix} \widehat{\alpha}_1 & \widehat{\alpha}_2 & \widehat{\alpha}_3 & \widehat{\alpha}_4 \\ (1,-1) & (1,1), & (1,0), & (0,1). \end{matrix}$$

These roots generate $\widehat{L}$.

(i) Choosing $\widehat{\alpha}_3$ and $\widehat{\alpha}_4$ as a basis

$$q\omega - 1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - 1 = \begin{pmatrix} -1 & q \\ -q & -1 \end{pmatrix}$$

has determinant $q^2 + 1$ and

$$(q\omega - 1)\widehat{L} = \{(u, v) \mid q^2 + 1 | qu - v\}.$$

The quotient of $\widehat{L}$ by this is cyclic of order $q^2 + 1$. It is generated by $\widehat{\alpha}_3, \widehat{\alpha}_4$ and $\widehat{\alpha}_1, \widehat{\alpha}_2$ generate the subgroups of index 2 for gcd $(q^2 + 1, q + 1) =$ gcd $(q^2 + 1, q - 1) = 2$. (We are taking q odd.) Thus a character is non-degenerate if and only if it is not of order 2. There are $q^2 - 1$ such characters which break up into $\frac{q^2-1}{4}$ orbits under the action of the Weyl group.

(ii) Here

$$q\omega - 1 = -\begin{pmatrix} q+1 & 0 \\ 0 & q+1 \end{pmatrix}.$$

We now break the characters of the group $(**)$ into two classes.

(a) Those which do not take $\widehat{\alpha}_3$ or $\widehat{\alpha}_4$ into ± 1.

(b) Those which do.

These are easily seen to be $(q - 1)(q - 3)$ non-degenerate characters of the first type which break up into $\frac{(q-1)(q-3)}{8}$ orbits under the Weyl group. There are $2(q - 1)$ non-degenerate characters of the second type. They break up into $\frac{q-1}{2}$ orbits.

Comparison with Mrs. Srinivasan's classification (cf. also p. D-44–D-45 of Borel et al.).

(i) The φ's which correspond to an ω of type (i) correspond in a 1:1 fashion with the cuspidal characters $\chi_1(j)$ of Mrs. Srinivasan.

(ii) (a) These φ's correspond in a 1:1 fashion to the cuspidal characters $\chi_4(k, \ell)$.

(b) To each of these φ's correspond **two** cuspidal representations of G, one of type $\xi'_{21}(k)$, one of type $\xi'_{22}(k)$.

That one φ should correspond to more than one representation is not surprising. This happens already over $\mathbf{R}$.

We have now accounted for every cuspidal representation but one, the anomalous representation of Mrs. Srinivasan.

Difficulty: *How does the general prediction account for the anomalous representation?*

Four possibilities present themselves.

(1) To one of the φ above there corresponds an extra representation, the anomalous one.
(2) The anomalous representation corresponds either to some homomorphism of the Weil group into $\widehat{G}$ which does not satisfy (i) and (ii) or to some homomorphism of the Galois group into the ℓ-adic $\widehat{G}$ (note: $\widehat{G}$ can be defined over any field and in particular over $\overline{\mathbf{Q}}_\ell$). Thus the anomalous representation could be special.
(3) There are algebro-geometric objects (motives) over $\mathbf{Q}_p$ which do not yield ℓ-adic representations into $\widehat{G}$ over $\overline{\mathbf{Q}}_\ell$ but yet correspond to representations of $G(\mathbf{Q}_p)$.
(4) There are representations of $G(\mathbf{Q}_p)$ which do not correspond to algebro-geometric objects.

The last two possibilities entail such complications that one fervently hopes they do not occur. The first *seems* to be excluded on grounds of symmetry. There is no obvious way to guess the appropriate φ. This leaves the second possibility. There is an experiment which can be performed to test this assumption. You are I suppose in a position to perform it. Let me describe the experiment.

Experiment: Consider $G = \mathrm{Sp}(2n)$ the symplectic group on $2n$ variables. $G^{\widehat{o}}$ is the orthogonal group in $2n+1$ variables. Consider an orthogonal group H in $2n$ variables. $H^{\widehat{o}}$ is also the orthogonal group in $2n$ variables. There is an obvious imbedding $H^{\widehat{o}} \hookrightarrow G^{\widehat{o}}$. $\widehat{G}$ is a direct product $G^{\widehat{o}} \times \mathfrak{G}(K/F)$. Suppose H is an outer form. Then $\widehat{H}$ is a semi-direct product $H^{\widehat{o}} \times \mathfrak{G}(K/F)$. We can imbed $\widehat{H} \hookrightarrow \widehat{G}$ extending $H^{\widehat{o}} \hookrightarrow G^{\widehat{o}}$. Namely realize $G^{\widehat{o}}$ as the adjoint group of the orthogonal group of

$$\begin{pmatrix} 0 & I & \\ I & 0 & \\ & & 1 \end{pmatrix}$$

We map $1 \times \sigma \in H^{\widehat{o}} \times \mathfrak{G}(K/F)$ onto $1 \times \sigma$ or onto

$$\begin{pmatrix} I & & 0 & & \\ & 0 & & 1 & \\ 0 & & I & & \\ & 1 & & 0 & \\ & & & & -1 \end{pmatrix} \times \sigma$$

according as σ does or does not act trivially on the Dynkin diagram of H.

According to the *expected* functoriality this map $\psi : \widehat{H} \hookrightarrow \widehat{G}$ should carry with it a map from *L-indistinguishable* classes of representations of H to L-indistinguishable classes of representations of G.

According to Gelbart's paper *Holomorphic Discrete Series for the Real Symplectic Group* this functoriality can over the reals be realized in the following concrete manner.

Take the Weil representation in $L^2(M_{2m,m})$ ($M_{2m,m}$ are the $2m \times m$ matrices) and decompose according to the action of $\mathrm{SO}(2m)$. The representation of $\mathrm{Sp}(2m)$ associated to a representation ρ of $\mathrm{SO}(2m)$ in this way lies in the L-indistinguishable class $\Pi_{\psi\cdot\eta}$ if ρ lies in Π_η (notation of my preprint *On the classification* ...). In any case to get at least one element of the L-indistinguishable class of representations of G corresponding to ρ one works with the Weil representation in the usual way.

Presumably the same is true over a p-adic field. Thus the difficulty could be resolved by an answer to the following question.

Question: *Does the anomalous representation or rather the corresponding induced representation occur in the Weil representation of* $\mathrm{Sp}(4, \mathbf{Q}_p)$ *defined by an anisotropic quadratic form in four variables*? If so, for what forms, and for which representations of the special orthogonal group of the form?

I hazard the guess that it is a one-dimensional representation of the special orthogonal group which is relevant. I could make further guesses but I prefer to wait for your response, for I believe you are able to answer the question.

Deinen jüngsten Brief habe ich gestern bekommen. Es würde mich freuen, dein Manuskript lesen zu dürfen.*

Mit herzlichem Gruße

Dein
Bob

*Roger Howe had just spent a year in Bonn.

Compiled on December 22, 2023.