

March 24, 1974

Dear Roger,

You observed to me several years ago that a cuspidal representation of a Chevalley group  $G$  over a finite field  $\mathfrak{o}_F/\mathfrak{p}_F$  yielded by induction an absolutely cuspidal representation over the local field  $F$  itself. I was intrigued, for as you know I am always trying to understand my Washington problems better and, in particular, always looking for examples in which the suggestions I made there could be tested.

I decided that the representations obtained in the above manner should correspond to homomorphisms  $\varphi$  of the Weil group into the associate group with the following two properties:

- (i) As usual  $\varphi$  is realized as a homomorphism of the Weil group  $W_{K/F}$  at a finite level

$$1 \longrightarrow K^\times \longrightarrow W_{K/F} \longrightarrow \mathfrak{G}(K/F) \longrightarrow 1$$

and  $\varphi(w)$  is semi-simple for all  $w$ . The first new property to be insisted upon is that  $\varphi$  be tamely ramified, that is, be trivial on  $1 + \mathfrak{p}_K$ . Moreover  $K$  is to be unramified but arbitrarily large.

- (ii) The image  $\varphi(W_{K/F})$  is contained in no proper parabolic subgroup of  $\widehat{G}$ , which is, because  $G$  is a Chevalley group, a direct product of its connected component  $G^{\widehat{\circ}}$  and  $\mathfrak{G}(K/F)$ .

Then I tried to check this for  $\mathrm{Sp}(4)$  by using Mrs. Srinivasan's results. Everything was almost perfect. To each such homomorphism there corresponded, as I shall describe later, in a fairly natural way finitely many cuspidal representations of  $\mathrm{Sp}(4, \kappa)$  where  $\kappa = \mathfrak{o}_F/\mathfrak{p}_F$ . There was, alas, one difficulty. It was not clear what to do with the anomalous representation. I have been puzzled by this representation ever since. Your recent letter suggests a way out. You probably know whether or not it is feasible; so I would appreciate your comments. However I have first to describe the difficulty. What I want to verify first is that the possible  $\varphi$ , which are of course determined only up to conjugacy by elements of  $G^{\widehat{\circ}}$ , correspond in a 1:1 manner to pairs consisting of an anisotropic torus  $T$  over  $\phi$ , the residue field of  $F$ , and a "non-degenerate" character of  $T(\phi)$ . First of all let me deduce some consequences of (i) and (ii) and the other usual conditions on  $\varphi$ . Since  $G$  is a Chevalley group,  $\widehat{G}$  is a direct product  $G^{\widehat{\circ}} \times \mathfrak{G}(K/F)$ . Since  $\varphi$  composed with  $\widehat{G} \rightarrow \mathfrak{G}(K/F)$  must be the standard map  $W_{K/F} \rightarrow \mathfrak{G}(K/F)$ , we may regard  $\varphi$  as a homomorphism of  $W_{K/F}$  into  $G^{\widehat{\circ}}$ .

We may divide by  $1 + \mathfrak{p}_K \subset K^\times$  to get an extension

$$1 \longrightarrow \kappa^\times \longrightarrow H \longrightarrow \mathbf{Z} \longrightarrow 1 \quad .$$

Here  $\kappa$  is the residue field of  $K$ .  $1 \in \mathbf{Z}$  is the Frobenius. If  $q$  is the number of elements in  $\phi$  then  $z \in \mathbf{Z}$  acts on  $\kappa^\times$  by  $\theta \rightarrow \theta^{q^z}$ . The extension is split.

Since  $\kappa^\times$  is cyclic Theorem E.5.16 of Borel et al., *Seminar on algebraic groups* together with its proof shows that there is a torus  $\widehat{T} \subset G^{\widehat{\circ}}$  which contains  $\varphi(\kappa^\times)$  and is normalized by  $\varphi(\mathbf{Z})$ . Set  $\omega = \varphi(1)$ . It is the image of  $\varphi(1)$ . (Observe: one usual demand is that the image of  $\varphi$  consist of semi-simple elements.)

**Claim.**  $\widehat{T}$  is the connected component of the centralizer of the image of  $\varphi(\kappa^\times)$ .

Observe that, because of (ii),  $\omega$ , which normalizes  $T$ , can fix no rational character of  $T$ . Let  $\widehat{\mathfrak{g}}_1$  be the centralizer of  $\varphi(\kappa^\times)$  in  $\widehat{\mathfrak{g}}$ , the Lie algebra of  $G^{\widehat{o}}$ .  $\widehat{\mathfrak{g}}_1$  is reductive and is normalized by  $\omega$ . By Gantmacher (Mat. Sb. (1939)) there is a Cartan subalgebra  $\widehat{\mathfrak{t}}$  and a Borel subalgebra  $\widehat{\mathfrak{p}}$  of  $\widehat{\mathfrak{g}}_1$  normalized by  $\omega$ . Since  $\widehat{\mathfrak{g}}_1$  clearly has the same rank as  $\widehat{\mathfrak{g}}$  we may suppose  $\widehat{\mathfrak{t}}$  is the Lie algebra of  $\widehat{T}$ . Then  $\omega$  fixes the sum of the simple roots of  $\widehat{\mathfrak{t}}$  with respect to  $\widehat{\mathfrak{p}}$ . Thus the sum must be 0. That is,  $\widehat{\mathfrak{g}}_1 = \widehat{\mathfrak{t}}$  as required.

**Corollary.** If  $\widehat{\alpha}$  is a root of  $\widehat{T}$  there is a  $\theta \in \kappa^\times$  such that  $\widehat{\alpha}(\varphi(\theta)) \neq 1$ .

The Weyl group of  $\widehat{T}$  is the same as the Weyl group of  $T_0$ , a split Cartan subgroup of  $G$ . Thus the image of  $\omega$  in the Weyl group can be used to twist  $T_0$  to  $T$ , a Cartan subgroup of  $G$  over  $\phi$ .  $T$  is anisotropic.

$$\begin{aligned} L &: \text{lattice of rational characters of } T_0 \\ \widehat{L} &: \text{lattice of rational characters of } \widehat{T} \\ \widehat{L} &= \text{Hom}(L, \mathbf{Z}). \end{aligned}$$

Notice  $T(\kappa) \simeq \widehat{L} \otimes \kappa^\times$ . If  $\theta$  is a fixed generator of  $\kappa^\times$  then we write  $\widehat{\lambda} \otimes \theta = \theta^{\widehat{\lambda}}$ . This represents an arbitrary element of  $T(\kappa)$ . The Frobenius sends

$$\theta^{\widehat{\lambda}} \longrightarrow \theta^{q\omega\widehat{\lambda}}.$$

Thus if  $s$  is the order of  $\kappa^\times$

$$T(\phi) = \left\{ \theta^{\widehat{\lambda}} \mid q\omega\widehat{\lambda} - \widehat{\lambda} \in s\widehat{L} \right\}.$$

Thus the characters of  $T(\phi)$  are the characters of

$$(*) \quad \left\{ \widehat{\lambda} \mid q\omega\widehat{\lambda} - \widehat{\lambda} \in s\widehat{L} \right\} \text{ modulo } s\widehat{L}.$$

On the other hand,  $\widehat{T}$  and  $\omega = \varphi(1)$  being given, consider all ways of defining  $\varphi$  on  $\kappa^\times$ . We have only to define  $\varphi(\theta)$  or

$$\widehat{\lambda}(\varphi(\theta)), \quad \theta \in \widehat{L}.$$

The condition is

$$\widehat{\lambda}(\varphi(\theta^q)) = \widehat{\lambda}(\omega(\varphi(\theta))) = \omega^{-1}\widehat{\lambda}(\varphi(\theta))$$

or

$$q\omega\widehat{\lambda}(\varphi(\theta)) = \widehat{\lambda}(\varphi(\theta)).$$

Thus the set of possible  $\varphi$  is, since  $\widehat{\lambda}(\varphi(\theta))$  must be an  $s^{\text{th}}$  root of unity, the set of characters of

$$(**) \quad \widehat{L} \text{ modulo } (q\omega - 1)\widehat{L} + s\widehat{L}.$$

Since  $n = \det(q\omega - 1)$  is prime to  $p$  we may choose  $K$  so large that it is divisible by  $s$ . Set  $M = q\omega - 1 : \widehat{L} \longrightarrow \widehat{L}$ . There is an  $N$  such that

$$MN = n.$$

If  $\widehat{\lambda} \in \widehat{L}$  and  $\frac{s}{n}N\widehat{\lambda} = \widehat{\mu}$  then  $M\widehat{\mu} \in s\widehat{L}$ . If  $\widehat{\lambda} = M\widehat{\nu}$  then  $\widehat{\mu} \in s\widehat{L}$ . Thus  $\frac{s}{n}N$  defines a map from the group  $(**)$  to the group  $(*)$ . It is easily seen to be an isomorphism. The character groups are also isomorphic.

The  $\varphi$  associated to a character of (\*\*) will satisfy (ii) if and only if the character is 1 on no root  $\hat{\alpha}$ . A character of (\*) will therefore be called non-degenerate if it is 1 on no  $\hat{\beta} = \frac{s}{n}N\hat{\alpha}$ ,  $\hat{\alpha}$  a root. Observe also that  $\hat{T}$  being given  $\omega$  is only determined up to conjugacy within the normalizer and that only the image of  $\omega$  in the Weyl group matters for  $\omega$  can be replaced by  $t\omega t^{-1} = t\omega(t^{-1})\omega$ ,  $t \in \hat{T}$  and  $t\omega(t^{-1})$  is arbitrary because  $\omega$  fixes no rational character. The image of  $\omega$  in the Weyl group being given,  $\varphi(\theta)$  is determined only up to the action of the centralizer of  $\omega$  in the Weyl group. This means that the character of  $T(\phi)$  is only determined up to the action of the Weyl group of  $T$  over  $\phi$ .

**The group  $\text{Sp}(4)$ .** There are two possibilities for  $\omega$ .

- (i) Rotation through  $90^\circ$ . The centralizer has order 4.
- (ii) Rotation through  $180^\circ$ . The centralizer has order 8.

If we represent the roots of  $T$  as  $(x, y) \rightarrow x - y, x + y, 2x, 2y$ , then the dual roots may be represented as

$$\begin{array}{cccc} \hat{\alpha}_1 & \hat{\alpha}_2 & \hat{\alpha}_3 & \hat{\alpha}_4 \\ (1, -1) & (1, 1), & (1, 0), & (0, 1). \end{array}$$

These roots generate  $\hat{L}$ .

- (i) Choosing  $\hat{\alpha}_3$  and  $\hat{\alpha}_4$  as a basis

$$q\omega - 1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - 1 = \begin{pmatrix} -1 & q \\ -q & -1 \end{pmatrix}$$

has determinant  $q^2 + 1$  and

$$(q\omega - 1)\hat{L} = \{(u, v) \mid q^2 + 1 \mid qu - v\}.$$

The quotient of  $\hat{L}$  by this is cyclic of order  $q^2 + 1$ . It is generated by  $\hat{\alpha}_3, \hat{\alpha}_4$  and  $\hat{\alpha}_1, \hat{\alpha}_2$  generate the subgroups of index 2 for  $\text{gcd}(q^2 + 1, q + 1) = \text{gcd}(q^2 + 1, q - 1) = 2$ . (We are taking  $q$  odd.) Thus a character is non-degenerate if and only if it is not of order 2. There are  $q^2 - 1$  such characters which break up into  $\frac{q^2 - 1}{4}$  orbits under the action of the Weyl group.

- (ii) Here

$$q\omega - 1 = -\begin{pmatrix} q + 1 & 0 \\ 0 & q + 1 \end{pmatrix}.$$

We now break the characters of the group (\*\*) into two classes.

- (a) Those which do not take  $\hat{\alpha}_3$  or  $\hat{\alpha}_4$  into  $\pm 1$ .
- (b) Those which do.

These are easily seen to be  $(q - 1)(q - 3)$  non-degenerate characters of the first type which break up into  $\frac{(q-1)(q-3)}{8}$  orbits under the Weyl group. There are  $2(q - 1)$  non-degenerate characters of the second type. They break up into  $\frac{q-1}{2}$  orbits.

**Comparison with Mrs. Srinivasan's classification** (cf. also p. D-44–D-45 of Borel et al.).

- (i) The  $\varphi$ 's which correspond to an  $\omega$  of type (i) correspond in a 1:1 fashion with the cuspidal characters  $\chi_1(j)$  of Mrs. Srinivasan.
- (ii) (a) These  $\varphi$ 's correspond in a 1:1 fashion to the cuspidal characters  $\chi_4(k, \ell)$ .

- (b) To each of these  $\varphi$ 's correspond **two** cuspidal representations of  $G$ , one of type  $\xi'_{21}(k)$ , one of type  $\xi'_{22}(k)$ .

That one  $\varphi$  should correspond to more than one representation is not surprising. This happens already over  $\mathbf{R}$ .

We have now accounted for every cuspidal representation but one, the anomalous representation of Mrs. Srinivasan.

**Difficulty:** *How does the general prediction account for the anomalous representation?*

Four possibilities present themselves.

- (1) To one of the  $\varphi$  above there corresponds an extra representation, the anomalous one.
- (2) The anomalous representation corresponds either to some homomorphism of the Weil group into  $\widehat{G}$  which does not satisfy (i) and (ii) or to some homomorphism of the Galois group into the  $\ell$ -adic  $\widehat{G}$  (note:  $\widehat{G}$  can be defined over any field and in particular over  $\overline{\mathbf{Q}}_\ell$ ). Thus the anomalous representation could be special.
- (3) There are algebro-geometric objects (motives) over  $\mathbf{Q}_p$  which do not yield  $\ell$ -adic representations into  $\widehat{G}$  over  $\overline{\mathbf{Q}}_\ell$  but yet correspond to representations of  $G(\mathbf{Q}_p)$ .
- (4) There are representations of  $G(\mathbf{Q}_p)$  which do not correspond to algebro-geometric objects.

The last two possibilities entail such complications that one fervently hopes they do not occur. The first *seems* to be excluded on grounds of symmetry. There is no obvious way to guess the appropriate  $\varphi$ . This leaves the second possibility. There is an experiment which can be performed to test this assumption. You are I suppose in a position to perform it. Let me describe the experiment.

**Experiment:** Consider  $G = \mathrm{Sp}(2n)$  the symplectic group on  $2n$  variables.  $G^\circ$  is the orthogonal group in  $2n + 1$  variables. Consider an orthogonal group  $H$  in  $2n$  variables.  $H^\circ$  is also the orthogonal group in  $2n$  variables. There is an obvious imbedding  $H^\circ \hookrightarrow G^\circ$ .  $\widehat{G}$  is a direct product  $G^\circ \times \mathfrak{G}(K/F)$ . Suppose  $H$  is an outer form. Then  $\widehat{H}$  is a semi-direct product  $H^\circ \times \mathfrak{G}(K/F)$ . We can imbed  $\widehat{H} \hookrightarrow \widehat{G}$  extending  $H^\circ \hookrightarrow G^\circ$ . Namely realize  $G^\circ$  as the adjoint group of the orthogonal group of

$$\begin{pmatrix} 0 & I & & \\ I & 0 & & \\ & & & 1 \end{pmatrix}$$

We map  $1 \times \sigma \in H^\circ \times \mathfrak{G}(K/F)$  onto  $1 \times \sigma$  or onto

$$\begin{pmatrix} I & 0 & & \\ & 0 & 1 & \\ 0 & I & & \\ & 1 & 0 & \\ & & & -1 \end{pmatrix} \times \sigma$$

according as  $\sigma$  does or does not act trivially on the Dynkin diagram of  $H$ .

According to the *expected* functoriality this map  $\psi : \widehat{H} \hookrightarrow \widehat{G}$  should carry with it a map from  $L$ -indistinguishable classes of representations of  $H$  to  $L$ -indistinguishable classes of representations of  $G$ .

According to Gelbart's paper *Holomorphic Discrete Series for the Real Symplectic Group* this functoriality can over the reals be realized in the following concrete manner.

Take the Weil representation in  $L^2(M_{2m,m})$  ( $M_{2m,m}$  are the  $2m \times m$  matrices) and decompose according to the action of  $\mathrm{SO}(2m)$ . The representation of  $\mathrm{Sp}(2m)$  associated to a representation  $\rho$  of  $\mathrm{SO}(2m)$  in this way lies in the  $L$ -indistinguishable class  $\Pi_{\psi,\eta}$  if  $\rho$  lies in  $\Pi_\eta$  (notation of my preprint *On the classification ...*). In any case to get at least one element of the  $L$ -indistinguishable class of representations of  $G$  corresponding to  $\rho$  one works with the Weil representation in the usual way.

Presumably the same is true over a  $p$ -adic field. Thus the difficulty could be resolved by an answer to the following question.

**Question:** *Does the anomalous representation or rather the corresponding induced representation occur in the Weil representation of  $\mathrm{Sp}(4, \mathbf{Q}_p)$  defined by an anisotropic quadratic form in four variables? If so, for what forms, and for which representations of the special orthogonal group of the form?*

I hazard the guess that it is a one-dimensional representation of the special orthogonal group which is relevant. I could make further guesses but I prefer to wait for your response, for I believe you are able to answer the question.

Deinen jüngsten Brief habe ich gestern bekommen. Es würde mich freuen, dein Manuskript lesen zu dürfen.\*

Mit herzlichem GruÙe

Dein  
Bob

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\*Roger Howe had just spent a year in Bonn.

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