

March 24, 1974

Dear Roger,

You observed to me several years ago that a cuspidal representation of a Chevalley group G over a finite field $\mathfrak{o}_F/\mathfrak{p}_F$ yielded by induction an absolutely cuspidal representation over the local field F itself. I was intrigued, for as you know I am always trying to understand my Washington problems better and, in particular, always looking for examples in which the suggestions I made there could be tested.

I decided that the representations obtained in the above manner should correspond to homomorphisms φ of the Weil group into the associate group with the following two properties:

- (i) As usual φ is realized as a homomorphism of the Weil group $W_{K/F}$ at a finite level

$$1 \longrightarrow K^\times \longrightarrow W_{K/F} \longrightarrow \mathfrak{G}(K/F) \longrightarrow 1$$

and $\varphi(w)$ is semi-simple for all w . The first new property to be insisted upon is that φ be tamely ramified, that is, be trivial on $1 + \mathfrak{p}_K$. Moreover K is to be unramified but arbitrarily large.

- (ii) The image $\varphi(W_{K/F})$ is contained in no proper parabolic subgroup of $\widehat{G}$, which is, because G is a Chevalley group, a direct product of its connected component G° and $\mathfrak{G}(K/F)$.

Then I tried to check this for $\mathrm{Sp}(4)$ by using Mrs. Srinivasan's results. Everything was almost perfect. To each such homomorphism there corresponded, as I shall describe later, in a fairly natural way finitely many cuspidal representations of $\mathrm{Sp}(4, \kappa)$ where $\kappa = \mathfrak{o}_F/\mathfrak{p}_F$. There was, alas, one difficulty. It was not clear what to do with the anomalous representation. I have been puzzled by this representation ever since. Your recent letter suggests a way out. You probably know whether or not it is feasible; so I would appreciate your comments. However I have first to describe the difficulty. What I want to verify first is that the possible φ , which are of course determined only up to conjugacy by elements of G° , correspond in a 1:1 manner to pairs consisting of an anisotropic torus T over ϕ , the residue field of F , and a "non-degenerate" character of $T(\phi)$. First of all let me deduce some consequences of (i) and (ii) and the other usual conditions on φ . Since G is a Chevalley group, $\widehat{G}$ is a direct product $G^\circ \times \mathfrak{G}(K/F)$. Since φ composed with $\widehat{G} \rightarrow \mathfrak{G}(K/F)$ must be the standard map $W_{K/F} \rightarrow \mathfrak{G}(K/F)$, we may regard φ as a homomorphism of $W_{K/F}$ into G° .

We may divide by $1 + \mathfrak{p}_K \subset K^\times$ to get an extension

$$1 \longrightarrow \kappa^\times \longrightarrow H \longrightarrow \mathbf{Z} \longrightarrow 1 \quad .$$

Here κ is the residue field of K . $1 \in \mathbf{Z}$ is the Frobenius. If q is the number of elements in ϕ then $z \in \mathbf{Z}$ acts on $\kappa^\times$ by $\theta \longrightarrow \theta^{q^z}$. The extension is split.

Since $\kappa^\times$ is cyclic Theorem E.5.16 of Borel et al., *Seminar on algebraic groups* together with its proof shows that there is a torus $\widehat{T} \subset G^\circ$ which contains $\varphi(\kappa^\times)$ and is normalized

by $\varphi(\mathbf{Z})$. Set $\omega = \varphi(1)$. It is the image of $\varphi(1)$. (Observe: one usual demand is that the image of φ consist of semi-simple elements.)

Claim. $\hat{T}$ is the connected component of the centralizer of the image of $\varphi(\kappa^\times)$.

Observe that, because of (ii), ω , which normalizes T , can fix no rational character of T . Let $\hat{\mathfrak{g}}_1$ be the centralizer of $\varphi(\kappa^\times)$ in $\hat{\mathfrak{g}}$, the Lie algebra of G^ϕ . $\hat{\mathfrak{g}}_1$ is reductive and is normalized by ω . By Gantmacher (Mat. Sb. (1939)) there is a Cartan subalgebra $\hat{\mathfrak{t}}$ and a Borel subalgebra $\hat{\mathfrak{p}}$ of $\hat{\mathfrak{g}}_1$ normalized by ω . Since $\hat{\mathfrak{g}}_1$ clearly has the same rank as $\hat{\mathfrak{g}}$ we may suppose $\hat{\mathfrak{t}}$ is the Lie algebra of $\hat{T}$. Then ω fixes the sum of the simple roots of $\hat{\mathfrak{t}}$ with respect to $\hat{\mathfrak{p}}$. Thus the sum must be 0. That is, $\hat{\mathfrak{g}}_1 = \hat{\mathfrak{t}}$ as required.

Corollary. If $\hat{\alpha}$ is a root of $\hat{T}$ there is a $\theta \in \kappa^\times$ such that $\hat{\alpha}(\varphi(\theta)) \neq 1$.

The Weyl group of $\hat{T}$ is the same as the Weyl group of T_0 , a split Cartan subgroup of G . Thus the image of ω in the Weyl group can be used to twist T_0 to T , a Cartan subgroup of G over ϕ . T is anisotropic.

$$\begin{aligned} L &: \text{lattice of rational characters of } T_0 \\ \hat{L} &: \text{lattice of rational characters of } \hat{T} \\ \hat{L} &= \text{Hom}(L, \mathbf{Z}). \end{aligned}$$

Notice $T(\kappa) \simeq \hat{L} \otimes \kappa^\times$. If θ is a fixed generator of $\kappa^\times$ then we write $\hat{\lambda} \otimes \theta = \theta^{\hat{\lambda}}$. This represents an arbitrary element of $T(\kappa)$. The Frobenius sends

$$\theta^{\hat{\lambda}} \longrightarrow \theta^{q\omega\hat{\lambda}}.$$

Thus if s is the order of $\kappa^\times$

$$T(\phi) = \left\{ \theta^{\hat{\lambda}} \mid q\omega\hat{\lambda} - \hat{\lambda} \in s\hat{L} \right\}.$$

Thus the characters of $T(\phi)$ are the characters of

$$(*) \quad \left\{ \hat{\lambda} \mid q\omega\hat{\lambda} - \hat{\lambda} \in s\hat{L} \right\} \quad \text{modulo } s\hat{L}.$$

On the other hand, $\hat{T}$ and $\omega = \varphi(1)$ being given, consider all ways of defining φ on $\kappa^\times$. We have only to define $\varphi(\theta)$ or

$$\hat{\lambda}(\varphi(\theta)), \quad \theta \in \hat{L}.$$

The condition is

$$\hat{\lambda}(\varphi(\theta^q)) = \hat{\lambda}(\omega(\varphi(\theta))) = \omega^{-1}\hat{\lambda}(\varphi(\theta))$$

or

$$q\omega\hat{\lambda}(\varphi(\theta)) = \hat{\lambda}(\varphi(\theta)).$$

Thus the set of possible φ is, since $\hat{\lambda}(\varphi(\theta))$ must be an s th root of unity, the set of characters of

$$(**) \quad \hat{L} \quad \text{modulo } (q\omega - 1)\hat{L} + s\hat{L}.$$

Since $n = \det(q\omega - 1)$ is prime to p we may choose K so large that it is divisible by s . Set $M = q\omega - 1 : \hat{L} \longrightarrow \hat{L}$. There is an N such that

$$MN = n.$$

If $\hat{\lambda} \in \hat{L}$ and $\frac{s}{n}N\hat{\lambda} = \hat{\mu}$ then $M\hat{\mu} \in s\hat{L}$. If $\hat{\lambda} = M\hat{\nu}$ then $\hat{\mu} \in s\hat{L}$. Thus $\frac{s}{n}N$ defines a map from the group $(**)$ to the group $(*)$. It is easily seen to be an isomorphism. The character groups are also isomorphic.

The φ associated to a character of $(**)$ will satisfy (ii) if and only if the character is 1 on no root $\hat{\alpha}$. A character of $(*)$ will therefore be called non-degenerate if it is 1 on no $\hat{\beta} = \frac{s}{n}N\hat{\alpha}$, $\hat{\alpha}$ a root. Observe also that $\hat{T}$ being given ω is only determined up to conjugacy within the normalizer and that only the image of ω in the Weyl group matters for ω can be replaced by $t\omega t^{-1} = t\omega(t^{-1})\omega$, $t \in \hat{T}$ and $t\omega(t^{-1})$ is arbitrary because ω fixes no rational character. The image of ω in the Weyl group being given, $\varphi(\theta)$ is determined only up to the action of the centralizer of ω in the Weyl group. This means that the character of $T(\phi)$ is only determined up to the action of the Weyl group of T over ϕ .

The group $\text{Sp}(4)$. There are two possibilities for ω .

- (i) Rotation through 90° . The centralizer has order 4.
- (ii) Rotation through 180° . The centralizer has order 8.

If we represent the roots of T as $(x, y) \longrightarrow x - y, x + y, 2x, 2y$, then the dual roots may be represented as

$$\begin{array}{cccc} \hat{\alpha}_1 & \hat{\alpha}_2 & \hat{\alpha}_3 & \hat{\alpha}_4 \\ (1, -1) & (1, 1), & (1, 0), & (0, 1). \end{array}$$

These roots generate $\hat{L}$.

- (i) Choosing $\hat{\alpha}_3$ and $\hat{\alpha}_4$ as a basis

$$q\omega - 1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - 1 = \begin{pmatrix} -1 & q \\ -q & -1 \end{pmatrix}$$

has determinant $q^2 + 1$ and

$$(q\omega - 1)\hat{L} = \{ (u, v) \mid q^2 + 1 \mid qu - v \}.$$

The quotient of $\hat{L}$ by this is cyclic of order $q^2 + 1$. It is generated by $\hat{\alpha}_3, \hat{\alpha}_4$ and $\hat{\alpha}_1, \hat{\alpha}_2$ generate the subgroups of index 2 for $\gcd(q^2 + 1, q + 1) = \gcd(q^2 + 1, q - 1) = 2$. (We are taking q odd.) Thus a character is non-degenerate if and only if it is not of order 2. There are $q^2 - 1$ such characters which break up into $\frac{q^2 - 1}{4}$ orbits under the action of the Weyl group.

- (ii) Here

$$q\omega - 1 = -\begin{pmatrix} q + 1 & 0 \\ 0 & q + 1 \end{pmatrix}.$$

We now break the characters of the group $(**)$ into two classes.

- (a) Those which do not take $\hat{\alpha}_3$ or $\hat{\alpha}_4$ into ± 1 .
- (b) Those which do.

These are easily seen to be $(q - 1)(q - 3)$ non-degenerate characters of the first type which break up into $\frac{(q-1)(q-3)}{8}$ orbits under the Weyl group. There are $2(q - 1)$ non-degenerate characters of the second type. They break up into $\frac{q-1}{2}$ orbits.

Comparison with Mrs. Srinivasan's classification (cf. also p. D-44–D-45 of Borel et al.).

- (i) The φ 's which correspond to an ω of type (i) correspond in a 1:1 fashion with the cuspidal characters $\chi_1(j)$ of Mrs. Srinivasan.
- (ii) (a) These φ 's correspond in a 1:1 fashion to the cuspidal characters $\chi_4(k, \ell)$.
 (b) To each of these φ 's correspond **two** cuspidal representations of G , one of type $\xi'_{21}(k)$, one of type $\xi'_{22}(k)$.

That one φ should correspond to more than one representation is not surprising. This happens already over $\mathbf{R}$.

We have now accounted for every cuspidal representation but one, the anomalous representation of Mrs. Srinivasan.

Difficulty: *How does the general prediction account for the anomalous representation?*

Four possibilities present themselves.

- (1) To one of the φ above there corresponds an extra representation, the anomalous one.
- (2) The anomalous representation corresponds either to some homomorphism of the Weil group into $\widehat{G}$ which does not satisfy (i) and (ii) or to some homomorphism of the Galois group into the ℓ -adic $\widehat{G}$ (note: $\widehat{G}$ can be defined over any field and in particular over $\overline{\mathbf{Q}}_\ell$). Thus the anomalous representation could be special.
- (3) There are algebro-geometric objects (motives) over $\mathbf{Q}_p$ which do not yield ℓ -adic representations into $\widehat{G}$ over $\overline{\mathbf{Q}}_\ell$ but yet correspond to representations of $G(\mathbf{Q}_p)$.
- (4) There are representations of $G(\mathbf{Q}_p)$ which do not correspond to algebro-geometric objects.

The last two possibilities entail such complications that one fervently hopes they do not occur. The first *seems* to be excluded on grounds of symmetry. There is no obvious way to guess the appropriate φ . This leaves the second possibility. There is an experiment which can be performed to test this assumption. You are I suppose in a position to perform it. Let me describe the experiment.

Experiment: Consider $G = \mathrm{Sp}(2n)$ the symplectic group on $2n$ variables. $G^\widehat{}$ is the orthogonal group in $2n + 1$ variables. Consider an orthogonal group H in $2n$ variables. $H^\widehat{}$ is also the orthogonal group in $2n$ variables. There is an obvious imbedding $H^\widehat{} \hookrightarrow G^\widehat{}$. $\widehat{G}$ is a direct product $G^\widehat{} \times \mathfrak{G}(K/F)$. Suppose H is an outer form. Then $\widehat{H}$ is a semi-direct product $H^\widehat{} \times \mathfrak{G}(K/F)$. We can imbed $\widehat{H} \hookrightarrow \widehat{G}$ extending $H^\widehat{} \hookrightarrow G^\widehat{}$. Namely realize $G^\widehat{}$ as the adjoint group of the orthogonal group of

$$\begin{pmatrix} 0 & I & \\ I & 0 & \\ & & 1 \end{pmatrix}$$

We map $1 \times \sigma \in H^\widehat{} \times \mathfrak{G}(K/F)$ onto $1 \times \sigma$ or onto

$$\begin{pmatrix} I & & 0 & \\ & 0 & & 1 \\ 0 & & I & \\ & 1 & & 0 & \\ & & & & -1 \end{pmatrix} \times \sigma$$

according as σ does or does not act trivially on the Dynkin diagram of H .

According to the *expected* functoriality this map $\psi : \widehat{H} \hookrightarrow \widehat{G}$ should carry with it a map from *L-indistinguishable* classes of representations of H to *L-indistinguishable* classes of representations of G .

According to Gelbart's paper *Holomorphic Discrete Series for the Real Symplectic Group* this functoriality can over the reals be realized in the following concrete manner.

Take the Weil representation in $L^2(M_{2m,m})$ ($M_{2m,m}$ are the $2m \times m$ matrices) and decompose according to the action of $\mathrm{SO}(2m)$. The representation of $\mathrm{Sp}(2m)$ associated to a representation ρ of $\mathrm{SO}(2m)$ in this way lies in the *L-indistinguishable* class $\Pi_{\psi,\eta}$ if ρ lies in Π_η (notation of my preprint *On the classification...*). In any case to get at least one element of the *L-indistinguishable* class of representations of G corresponding to ρ one works with the Weil representation in the usual way.

Presumably the same is true over a p -adic field. Thus the difficulty could be resolved by an answer to the following question.

Question: *Does the anomalous representation or rather the corresponding induced representation occur in the Weil representation of $\mathrm{Sp}(4, \mathbf{Q}_p)$ defined by an anisotropic quadratic form in four variables? If so, for what forms, and for which representations of the special orthogonal group of the form?*

I hazard the guess that it is a one-dimensional representation of the special orthogonal group which is relevant. I could make further guesses but I prefer to wait for your response, for I believe you are able to answer the question.

Deinen jüngsten Brief habe ich gestern bekommen. Es würde mich freuen, dein Manuskript lesen zu dürfen.*

Mit herzlichem GrüÙe

Dein
Bob

*Roger Howe had just spent a year in Bonn.

Compiled on April 18, 2025.