The Institute for Advanced Study
Princeton, New Jersey
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Dear Tony,

You may not be willing to have any truck with the point of view exposed in my notes on the K-Z theory, but I would like to correct the second lemma, for the record at least.

The proper statement is more complicated. One weakens the assumption, and supposes only that G is reductive and that the centre of ${}^LG^0$ is connected. Then—

(a) *$S^0\backslash S$ is abelian and every element in it is of order two.*

(b) *There is an exact sequence*

$$1 \longrightarrow A \longrightarrow S^0\backslash S \longrightarrow R \longrightarrow 1\ .$$

In the notation of my notes

$$A = {}^LT \cap S/{}^LT \cap S^0$$

and

$$R = \mathrm{Norm}_S\, {}^LT/{}^LT \cap S.$$

is isomorphic to your R-group.

In the notes I assumed without proof that

$${}^LT \cap S = {}^LT \cap S^0,$$

but this is usually false, even if G is semi-simple and simply-connected.

I am happier with the new statement, for it conforms to a principle suggested by the results of Labesse-Langlands, who were themselves influenced in their exposition by the K-Z theory:

$S^0\backslash S$ is in some sense dual to the L-indistinguishable class associated to φ.

This principle has not been tested in too many cases—over non-archimedean fields, only for groups closely related to SL(2) and for the unramified principal series. Over the complexes or reals, it may be possible to deduce it from your results. The phrase "in some sense" has to be included, as one sees already in Labesse-Langlands. If the group is not quasi-split, each π in the L-indistinguishable class should define a function on $S^0\backslash S$. For quasi-split groups, one may come closer to a real duality. But real groups are anomalous because $H^1(\mathbf{R}, G)$ can be non-trivial even when G is simply-connected, and one must therefore expect the L-indistinguishable class to be smaller than the full dual.

Philosophically speaking, the group $A \subseteq S^0\backslash S$ appears because two different discrete series representations of M may be L-indistinguishable and yield upon induction L-indistinguishable representations of G. An L-indistinguishable family of discrete series is parametrized by an orbit of the complex Weyl group and thus by

$$\Omega_{\mathbf{C}}(T, M)/\Omega_{\mathbf{R}}(T, M)$$

if, as in my notes, T is a fundamental CSG of M.

The formalism exposed in my *Stable conjugacy* leads naturally to an imbedding of this set in $A^\vee$, the dual of A. Let $\mathfrak{D} = \mathfrak{D}(T) = \mathfrak{D}(T/\mathbf{R})$ and $\xi = \xi(T) = \xi(T/\mathbf{R})$ be the two sets of that paper. Then $\mathfrak{D} \subseteq \xi$ and ξ is a group. In Diana's thesis, or at least in some notes of hers, it is shown that

$$\mathfrak{D} \cong \Omega_{\mathbf{C}}(T, M)/\Omega_{\mathbf{R}}(T, M).$$

All we need do is show that A is isomorphic to the dual $\xi^\vee$ of ξ. For non-archimedean fields $\mathfrak{D} = \xi$, but for the reals $\mathfrak{D}$ may be properly smaller than ξ. This is the source of the anomaly mentioned earlier.

Since I don't want you to balk immediately, I forego introducing any cohomology for now, and suppose that $M = G$ is semi-simple and simply-connected. Let $X^*(T)$ be the lattice of weights of T and identify $X_*(T) = \operatorname{Hom}\big(X^*(T), \mathbf{Z}\big)$ with the lattice of weights of ${}^LT$[1]. If $\mu \in X_*(T)$ and $t \in \mathbf{C}^\times$ then $t^\mu \in T(\mathbf{C})$ is defined by

$$\lambda(t^\mu) = t^{\langle \lambda, \mu \rangle} \qquad \lambda \in X^*(T).$$

If $\lambda \in X^*(T)$ and then $t^\lambda \in {}^LT(\mathbf{C})$[2]. The definitions show:

(a) $$\xi = \big\{ (-1)^\mu \bigm| \mu \in X_*(T) \big\}$$

(b) $$A = \Big\{ (-1)^\lambda \Bigm| \lambda \in X^*(T) \Big\}.$$

The pairing is

$$(-1)^\mu \times (-1)^\lambda \longmapsto (-1)^{\langle \lambda, \mu \rangle}.$$

In general, one applies Tate-Nakayama duality, which over $\mathbf{R}$ can be reduced to elementary and easy considerations, to show that

$$\xi \sim \big\{ \mu \in X_*(T_{\mathrm{sc}}) \bigm| \mu + \varphi(\sigma)\mu = 0 \big\} \Big/ \big\{ \mu = \nu - \varphi(\sigma)\nu \in X_*(T_{\mathrm{sc}}), \nu \in X_*(T) \big\}.$$

Here $X_*(T_{\mathrm{sc}})$ is the $\mathbf{Z}$-span of the coroots. Thus

$$\xi \subseteq X_*(T) \Big/ \big\{ \mu = \nu - \varphi(\sigma)\nu,\ \nu \in X_*(T) \big\}.$$

Every character κ of ξ extends to a character of the larger group and thus to an element t of

$$\operatorname{Hom}\big(X_*(T), \mathbf{C}^\times\big) = {}^LT(\mathbf{C}).$$

The condition

$$\nu(t) = \varphi(\sigma)\nu(t)$$

is equivalent to

$$t \in {}^LT \cap S.$$

If we assume that the centre of ${}^LG^0$ is connected, we easily show that the image of t in ${}^LT \cap S / {}^LT \cap S^0$ depends only on κ and obtain in this way an isomorphism

$$\xi^\vee \longrightarrow {}^LT \cap S / {}^LT \cap S^0 = A.$$

The second part of the newly formulated lemma is, I believe, already proved in my notes. To show that $S^0 \backslash S$ is abelian, I have only to verify that if $\alpha^\vee$ is one of the strongly orthogonal roots figuring in Gregg's argument, $w = w_{\alpha^\vee}$ an associated element of the normalizer of ${}^LT$, and $t \in {}^LT \cap S$ then

$$wtw^{-1}t^{-1} \in S^0.$$

[1] Editorial comment: In the original typed document, $\widehat{T}$ has been added handwritten.

[2] Editorial comment: Here, too, in the original typed document, $\widehat{T}$ has been added handwritten.

It is easily seen that

$$wtw^{-1} = t\alpha^\vee(t)^{-\alpha}.$$

Since $\alpha \in {}^L\mathfrak{t}_+$,

$$\alpha^\vee(t)^\alpha \in {}^LT \cap S^0.$$

I look forward to seeing you in Oregon.

Yours,
R. Langlands

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