Bonn
December 5, 1970

Dear Lang,

I of course will be at Yale next year. Speaking only for myself I should say that I would be pleased to see you at Yale not just for a year but for longer.

Since you are also looking into Shimura let me communicate some of my rough thoughts. You may have some suggestions. These thoughts are based on casual observations and calculations in my head and don't pretend yet to be serious mathematics. I want to understand the relations between the zeta functions of Shimura varieties and the Euler products I introduced in Washington.

For simplicity take the projective theory of Shimura (Annals, 1970, §2.14) and for generality assume that it works for all bounded domains. Thus we have the following: X—bounded symmetric domain; $G_{\mathbf{R}}$—connected group of automorphisms; $\Gamma \subset G_{\mathbf{R}}$—arithmetic subgroup; $V = X/\Gamma$ (assume for simplicity that Γ has no fixed points); $\widetilde{V}$—some compactification of V as a non-singular variety. If one uses (with no right at the moment when V is not compact) the theorem of Matsushima (Jour. of Diff. Geom. vol. 1) to get at the cohomology of $\widetilde{V}$? $\widetilde{V}$ mod $(\widetilde{V} - V)$? (the matter is not clear, cf. also Shimura §2.16) in the middle dimension one sees (see remark at end) that one gets a one-dimensional subspace of the cohomology group every time that a member of the discrete series corresponding to the parameter ρ (where ρ is, with respect to some order on the roots, one-half the sum of the positive roots) occurs in $L^2(\Gamma\backslash G)$. Thus one considers the discrete series with the lowest possible weights. Besides this there may be some trivial stuff coming for example from the constant function but probably (at the moment on the basis of Serre's conjecture about the Γ-factors of Hasse-Weil L-functions or on the basis of a generalized Ramanujan conjecture—but the matter can probably be handled directly) nothing else. The trivial stuff contributes only ordinary zeta-functions (or at worst something that can be handled by induction) to the Hasse-Weil L-function in the middle dimension (or at least whatever part of it Matsushima's theorem represents.) It is the non-trivial part of the Matsushima part that we want to consider.

The number of different discrete series of the required type is the index d of the Weyl group of $G_{\mathbf{R}}$ (or of its maximal compact subgroup—they are the same) in that of $G_{\mathbf{C}}$. These d representations should, in the sense of my Washington lecture, correspond to the same representation of the Weil(!) group in the dual group, should therefore have the same local L-functions, and should not be arithmetically distinguishable. Therefore they should occur with the same multiplicity r in $L^2(\Gamma\backslash G)$. To make matters less theological, this fact should also come out of the Selberg trace formula since the representations have the same formal degree (assume as before that Γ has no fixed points.) The space under consideration has therefore the dimension rd. Now let the Hecke operators act. On the basis of the above principle (although now it would not be just a matter of the formal degree) it should break up into r spaces of dimension d on each of which the Hecke operators act as scalars. To each subspace correspond d different representations of the adèle group but that should have no effect on the global L-function because it has no effect on the local ones. Thus we have only r different (or differentiated) L-functions. If we want to multiply them together to get the relevant part of the Hasse-Weil L-function they have to be of degree d (or maybe a multiple of d because the variety may not be defined over the ground field, i.e. the field over which Γ is considered as defined—such questions are too subtle(!) at this stage.)

According to my definitions (Washington) the possible degrees of these L-functions (degree means degree of a typical denominator in the Euler product) are equal to the possible degrees of the complex representations of what I called the dual group. Here comes the first striking fact. In all cases the connected component of the dual group (by considering only the connected component I am ignoring roughly speaking the twisting necessary to define the group) has a representation of degree d. We have the following picture:

G	(assumed centreless)	$\widehat{G}_0$	dual group (connected component) simply connected because G is centreless
	α root	$\widehat{\alpha}$	dual root $\widehat{\alpha}(\lambda) = 2\frac{(\alpha,\lambda)}{(\alpha,\alpha)}$
L	lattice generated by roots	$\widehat{L} = \mathrm{Hom}(M, \mathbf{Z})$	lattice generated by roots of $\widehat{G}_0$
$\cap$	= weights occurring in representations of G		
M	lattice of weights of Lie algebra	$\widehat{M} = \mathrm{Hom}(L, \mathbf{Z})$	lattice of weights of $\widehat{G}_0$

Fix an order on the roots of G with respect to which every non-compact positive root is totally positive (cf. Harish-Chandra, Amer. Jour., 1956). Claim: there exists $\widehat{\mu}_0$ in $\widehat{M}$ such that $\widehat{\mu}_0(\alpha) = 0$ if α is compact and $\widehat{\mu}_0(\alpha) = 1$ if α is non-compact and positive. The representation σ_0 of $\widehat{G}_0$ with highest weight $\widehat{\mu}_0$ has degree d. This claim could easily be proved directly but I give all the examples for the simple groups.

(i) $G_{\mathbf{R}}$: real symplectic group in $2n$ variables. $\mathfrak{g}$: Lie algebra of $G_{\mathbf{C}}$. Schematically,

$$\mathrm{Cartan}(\mathfrak{g}) = x = \begin{pmatrix} x_1 & & & & & & \\ & x_2 & & & & & \\ & & \ddots & & & 0 & \\ & & & x_n & & & \\ & & & & -x_1 & & \\ & & & & & -x_2 & \\ & 0 & & & & & \ddots \\ & & & & & & & -x_n \end{pmatrix}$$

$$\text{pos. comp. roots:}\quad x \to x_i - x_j, \quad i < j,$$
$$\text{pos. non-comp. roots:}\quad x \to x_i + x_j, \quad i \leqslant j,$$
$$\mu \in M: \quad x \to \sum m_i x_i, \quad d = \frac{2^n n!}{n!} = 2^n.$$

$\widehat{G}_0 = B_n$: simply-connected form of orthogonal group in $2n+1$ variables (dual group always a complex group).

$$\text{Cartan}(\widehat{\mathfrak{g}}) = \begin{pmatrix} y_1 & & & & & & & \\ & y_2 & & & & & & \\ & & \ddots & & & & & \\ & & & y_n & & & & \\ & & & & -y_1 & & & \\ & & & & & -y_2 & & \\ & & & & & & \ddots & \\ & & & & & & & -y_n & \\ & & & & & & & & 1 \end{pmatrix}$$

$$\alpha: \quad x \to x_i \pm x_j; \qquad \widehat{\alpha}: \quad y \to y_i \pm y_j$$
$$\alpha: \quad x \to \pm 2x_i; \qquad \widehat{\alpha}: y \to \pm y_i$$
$$\widehat{\mu}: \quad y \to \sum \widehat{m}_i y_i \Rightarrow \widehat{\mu}(\mu) = \sum \widehat{m}_i m_i$$

Thus $\widehat{\mu}_0 : y \to \sum \frac{y_i}{2}$ and σ_0 is the spinor representation.

(ii) $G_{\mathbf{R}}$: special unitary group of form

$$\begin{pmatrix} I_a & \\ & -I_b \end{pmatrix}, \qquad a + b = n,$$

modulo centre.

$$\text{Cartan}(\mathfrak{g}) = \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix},$$

$$\text{pos. comp. roots:}\quad x \to x_i - x_j, \quad i < j, j \leqslant a \text{ or } i > a,$$
$$\text{pos. non-comp. roots:}\quad x \to x_i - x_j, \quad i \leqslant a, \quad j > a,$$
$$\mu \in M: \quad x \to \sum m_i x_i, \quad \sum m_i = 0, \quad d = \frac{n!}{a!\,b!}.$$

$\widehat{G}_0 = \mathrm{SL}(n)$ $\quad \alpha: \quad x \to x_i - x_j, \quad \widehat{\alpha}: \quad x \to x_i - x_j, \quad \widehat{\mu}: \quad x \to \sum \widehat{m}_i x_i$ with $\sum \widehat{m}_i = 0$. Then $\widehat{\mu}(\mu) = \sum \widehat{m}_i m_i$. In particular

$$\widehat{\mu}_0 = \Big(\underbrace{\frac{s}{n}, \dots, \frac{s}{n}}_{r}, \underbrace{-\frac{r}{n}, \dots, -\frac{r}{n}}_{s}\Big)$$

σ_0 representation on rth exterior powers. Changing the order so that positive roots become negative we get the representation on the sth symmetric powers. The possibility of two σ_0's corresponds to the fact that $G_{\mathbf{R}}$ is of outer type, so that $\widehat{G}$ is not connected and both σ_0's have to occur in the restriction to $\widehat{G}_0$ of a representation of $\widehat{G}$.

(iii) Shimura (for example) in Anal. Fam. (Annals, 1963) represents the third main type as the unitary group of a quaternionic form,

$$G_{\mathbf{R}} = \left\{ V = \begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix} \;\middle|\; A, B \in M_n(\mathbf{C}),\ {}^t\overline{V}\begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix} V = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix} \right\} \Big/ \text{mod centre}$$

The last equation is

$${}^tV\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} V = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

Thus $G_{\mathbf{C}}$ is an orthogonal group (modulo centre) D_n. Schematically

$$\mathrm{Cartan}(\mathfrak{g}) = x = \begin{pmatrix} x_1 & & & & & & \\ & x_2 & & & & & \\ & & \ddots & & & & 0 \\ & & & x_n & & & \\ & & & & -x_1 & & \\ & & & & & -x_2 & \\ & 0 & & & & & \ddots \\ & & & & & & -x_n \end{pmatrix}$$

$$\begin{aligned} &\text{pos. comp.} \quad x \to x_i - x_j, \quad i < j, \\ &\text{pos. non-comp.} \quad x \to x_i + x_j, \quad i < j, \\ &\mu \in M : \quad x \to \sum m_i x_i, \quad d = \frac{2^{n-1} n!}{n!} = 2^{n-1}. \end{aligned}$$

$\widehat{G}_0$ is also an orthogonal group.

$$\mathrm{Cartan}(\widehat{\mathfrak{g}}) = y = \begin{pmatrix} y_1 & & & & & & \\ & y_2 & & & & & \\ & & \ddots & & & & 0 \\ & & & y_n & & & \\ & & & & -y_1 & & \\ & & & & & -y_2 & \\ & 0 & & & & & \ddots \\ & & & & & & -y_n \end{pmatrix}$$

$$\begin{aligned} &\alpha : x \to x_i \pm x_j; \quad \widehat{\alpha} : \quad y \to y_i \pm y_j \\ &\widehat{\mu} : y \to \sum \widehat{m}_i y_i, \\ &\widehat{\mu}(\mu) = \sum \widehat{m}_i m_i \end{aligned}$$

Thus $\widehat{\mu}_0 : y \to \sum \frac{y_i}{2}$ and σ_0 is a spinor representation. In this case there are two spinor representations. However, when n is even the group seems to be of inner type, so only one of them seems to play a role. (Note: this violates no symmetry law.)

(iv) The fourth main type is the real orthogonal group of

$$\begin{pmatrix} I_2 & 0 \\ 0 & -I_n \end{pmatrix}$$

modulo its centre.

(a) n is odd.

$$\text{Cartan}(\mathfrak{g}): \qquad x = \begin{pmatrix} 0 & ix_1 & & & & & & \\ -ix_1 & 0 & & & & & & \\ & & 0 & ix_2 & & & & \\ & & -ix_2 & 0 & \ddots & & & \\ & & & & & 0 & ix_k & \\ & & & & & -ix_k & 0 & \\ & & & & & & & 1 \end{pmatrix}$$

$$k = \frac{n+1}{2}, \qquad \mu : x \to \sum m_i x_i,$$

pos. comp. $x \to x_i \pm x_j$, $2 \leqslant i < j$; $x \to x_i$, $2 \leqslant i$,

pos. non-comp. $x \to x_1 \pm x_j$, x_1, $2 \leqslant j$,

$$d = \frac{2^k k!}{2^{k-1}(k-1)!} = 2k.$$

$\widehat{G}_0 : C_k$—symplectic group in $2k$ variables.

$$\text{Cartan}(\widehat{\mathfrak{g}}) = \begin{pmatrix} y_1 & & & & & & \\ & y_2 & & & & & \\ & & \ddots & & & 0 & \\ & & & y_n & & & \\ & & & & -y_1 & & \\ & & & & & -y_2 & \\ & 0 & & & & & \ddots \\ & & & & & & & -y_n \end{pmatrix}$$

$$\begin{aligned} \alpha :&\quad x \to x_i \pm x_j; &\quad \widehat{\alpha} :&\quad y \to y_i \pm y_j. \\ \alpha :&\quad x \to x_i; &\quad \widehat{\alpha} :&\quad y \to 2y_i. \\ \widehat{\mu} :&\quad y \to \sum \widehat{m}_i y_i, &\quad \widehat{\mu}(\mu) &= \sum \widehat{m}_i m_i. \end{aligned}$$

Thus $\widehat{\mu}_0 = (1, 0, \ldots, 0)$ and σ_0 is the standard representation of C_k.

(b) n even.

$$\text{Cartan}(\mathfrak{g}) = \begin{pmatrix} 0 & ix_1 & & & & & \\ -ix_1 & 0 & & & & & \\ & & 0 & ix_2 & & & \\ & & -ix_2 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & ix_k \\ & & & & & -ix_k & 0 \end{pmatrix}$$

$$k = \frac{n+2}{2}, \qquad \mu : x \to \sum m_i x_i,$$
pos. comp. $\quad x \to x_i \pm x_j, \quad 2 \leqslant i < j$

pos. non-comp. $\quad x \to x_1 \pm x_j, \quad 2 \leqslant j,$

$$d = \frac{2^{k-1} k!}{2^{k-2}(k-1)!} = 2k.$$

$\widehat{G}_0$ is D_k: orthogonal group in $2k$ variables (or covering group thereof).

$$\text{Cartan}(\widehat{\mathfrak{g}}) : y = \begin{pmatrix} y_1 & & & & & & & \\ & y_2 & & & & & 0 & \\ & & \ddots & & & & & \\ & & & y_n & & & & \\ & & & & -y_1 & & & \\ & & & & & -y_2 & & \\ & 0 & & & & & \ddots & \\ & & & & & & & -y_n \end{pmatrix}$$

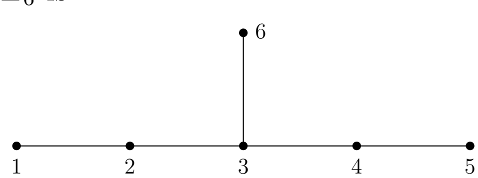
$$\alpha : \quad x_i \pm x_j; \qquad \widehat{\alpha} : y_i \pm y_j.$$
$$\widehat{\mu} : \quad y \to \sum \widehat{m}_i y_i, \quad \widehat{\mu}(\mu) = \sum \widehat{m}_i m_i.$$

Thus $\widehat{\mu}_0 = (1, 0, \dots, 0)$ and σ_0 is the standard representation of D_k.

(v) There is one $G_{\mathbf{R}}$ such that $G_{\mathbf{C}}$ is E_6. It is type $E_{6(-14)}$ in Tits (Springer). Its maximal compact subgroup is $D_5 \times$ Circle. Thus (Sem. Chevalley),

$$d = \frac{72 \cdot 6!}{2^4 \cdot 5!} = 27.$$

The Dynkin diagram of E_6 is

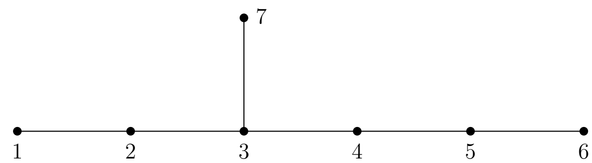


One obtains that of D_5 by removing root #5. $\widehat{G}_0$ is also E_6. After the table in Tits every positive noncompact root has its fifth coefficient equal to 1. Thus $\widehat{\mu}_0$ is the fundamental weight corresponding to root #5. Its dimension is 27 as required.

(vi) There is one $G_{\mathbf{R}}$ such that $G_{\mathbf{C}}$ is E_7. It is type $E_{7(-15)}$ in Tits. Maximal compact is $E_6 \times$ Circle. Thus (Sem. Chev.)

$$d = \frac{56 \cdot 72 \cdot 6!}{72 \cdot 6!} = 56.$$

the Dynkin diagram is



One gets that of E_6 by removing root #1. Since every positive non-compact root has, according to table in Tits, its first coefficient 1, μ_0 is the fundamental weight corresponding to this root ($\widehat{G}_0$ is also E_7.) By Tits degree $\sigma_0 = 56$ as required.

After this, assume for simplicity of discussion that the operation of the Hecke operators really separates the space into subspaces of dimension d (that is, there is no multiplicity—this will presumably almost never be the case but I want to avoid going into pointless detail—the assumption probably does no harm.) Suppose moreover that the whole subspace has a correspondent in the ℓ-adic cohomology. If the Hecke operators are defined over the ground field the Galois group presumably operates on these d-dimensional subspaces. I mention the following question which is perhaps not so important in itself but whose solution may throw some light on a question to follow: Is the image of the Galois group contained in a group whose connected component is a form of the dual group (i.e. its image under σ_0)? To prove this one has presumably to show the existence of certain subvarieties on products of several copies of V (i.e. the existence of certain elements in tensor products of the space dual to the cohomology which are invariant under the Galois group.) For example in case (ii) above with $a = 1$, $b = 2$ the image of σ_0 is just SL(3). This means that in $V \times V \times V$ there should be 3-dimensional algebraic cycles X with the property that the effect on the associated homology class (6-dim) of applying a Hecke operator to the first, second, or third coordinate is independent of the coordinate chosen. At the moment I don't understand the problem at all. I don't know how difficult it is.

Now I can come to the serious question. It is probably simpler at the moment to assume that the group is not obtained by an outer twisting. Then at the non-archimedean places the Hecke algebra over $\mathbf{C}$ is isomorphic to $R_{\mathbf{C}}(\widehat{G}_0)$, the representation ring of $\widehat{G}_0$ over $\mathbf{C}$. Let T_j, $1 \leqslant j \leqslant d$, be the element of the Hecke algebra corresponding to the jth exterior power of σ_0. Let $\lambda_i(T_j)$, $1 \leqslant i \leqslant r$, be the eigenvalue of T_j on the ith of our d-dimensional subspaces and let P_i be the restriction of the Frobenius to this subspace. In order to prove the relation between the Hasse-Weil function and the functions of my Washington lecture mentioned above one has to show that the eigenvalues of P_i are precisely (i.e. with the right multiplicity) the roots of

$$X^d + \sum_{j=1}^{d} (-1)^j \lambda_i(T_j) q^{\frac{t}{2}j} X^{d-j}$$

if q is the number of elements in the residue field and t is the dimension of V. It is probably not so different to show that every eigenvalue of P_i is a root of this equation. It amounts to showing that, for each j, $T'_j = q^{\frac{t}{2}j} T_j$ is in the Hecke algebra over $\mathbf{Z}$ and that

$$P^d + \sum_{j=1}^{d} (-1)^j \widetilde{T}'_j P^{d-j} = 0. \tag{1}$$

This is a relation between correspondences. P is the Frobenius and $\widetilde{T}'_j$ is the reduction modulo the valuation. I will give you an idea in a moment of what one already finds in Shimura. The question however of whether one gets all eigenvalues with the right multiplicity is, so far as I can see, the central difficulty of the affair. It is probably one which Shimura has met met in a context I'll mention later and to which he has given some thought. I would like very much to know how he feels about it.

Take the symplectic $\mathrm{Sp}(n)$ over $\mathbf{Q}$ modulo its centre. I describe the isomorphism above as it was obtained by Satake. Consider rational matrices A all of whose elementary divisors are powers of p and which satisfy

$$ {}^tA \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} A = p^k \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

Then $p^{-k/2}A$ defines a rational element in $\mathrm{Sp}(n)/\text{centre}$. If A is such that

$$p^{-k/2}A = \begin{pmatrix} p^{-\alpha_1} & * & * & * & * & * \\ & \ddots & & & & \\ 0 & & * & * & * & * \\ 0 & 0 & p^{-\alpha_n} & * & * & * \\ 0 & 0 & 0 & p^{\alpha_1} & 0 & 0 \\ & & & & \ddots & \\ 0 & 0 & 0 & * & & 0 \\ 0 & 0 & 0 & * & * & p^{\alpha_n} \end{pmatrix} \tag{2}$$

let $\widehat{\mu}(p^{-k/2}A) = (\alpha_1, \dots, \alpha_n)$ be a weight of the dual group as in case (i) above. Let $\rho(p^{-k/2}A)$ be the integer $n\alpha_1 + (n-1)\alpha_2 + \cdots + \alpha_n$.

Given the double coset $\mathrm{Sp}(n,\mathbf{Z})p^{-k/2}A\,\mathrm{Sp}(n,\mathbf{Z})$ it can be written as a disjoint union

$$\bigcup_i \mathrm{Sp}(n,\mathbf{Z})p^{-k/2}A_i$$

where $p^{-k/2}A$ has the form 1. The representation ring of $\widehat{G}_0$ is the set of elements in the group ring of $\widehat{M}$ which are invariant under permutations and sign changes of the coordinates. The map

$$\mathrm{Sp}(n,\mathbf{Z})p^{-k/2}A\,\mathrm{Sp}(n,\mathbf{Z}) \to \sum p^{\rho(p^{-k/2}A_i)}\widehat{\mu}(p^{-k/2}A_i)$$

is the isomorphism of Satake. If σ is a representation with highest weight

$$\widehat{\mu}_\sigma = (\widehat{m}_1, \dots, \widehat{m}_n)$$

set $\delta_\sigma = n\widehat{m}_1 + \cdots + \widehat{m}_n$ and let χ_σ be the character of σ. Satake's reasoning shows that $\delta_\sigma\chi_\sigma$ lies in the image of the integral Hecke algebra. If $\sigma = \sigma_0$ is the spinor representation then

$$\delta_\sigma = \frac{n(n+1)}{4} = \frac{t}{2}.$$

So $q^{t/2}T_1$ is in the integral Hecke algebra (i.e. algebra of double cosets over $\mathbf{Z}$). So is $q^{\frac{t}{2}j}T_j$.

In order to consider the congruence (1) one has to refer to Shimura (PNAS, vol. 49 and Jour. Math. Soc. of Japan, 1958). Unfortunately here it is not convenient to look at the projective theory. We have to replace $G = \mathrm{Sp}(n)/\text{centre}$ by

$$\widetilde{G} = \left\{ B \in \mathrm{GL}(2n, \mathbf{Q}) \;\middle|\; {}^tB \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} B = \lambda(B) \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \right\}.$$

We have to look at double cosets $\mathrm{Sp}(n, \mathbf{Z}) A \,\mathrm{Sp}(n, \mathbf{Z})$ where A is as above. (Now of course pA is not going to have the same effect as A. To get the projective theory from the non-projective one just ignores the factor p—one restricts to that subfield of the function field on which the Hecke algebras corresponding to scalar matrices operate trivially—this means basically that one makes the field of constants smaller—I am ignoring this but it is always there.) If σ is a representation with highest weight $\widehat{\mu}_\sigma = (\widehat{m}_1, \dots, \widehat{m}_n)$ so that $\widehat{m}_1 \geqslant \cdots \geqslant \widehat{m}_n$ and $\widehat{\mu}_\sigma$ lies in the positive Weyl chamber set $\epsilon(\widehat{\mu}_\sigma) = \epsilon_\sigma = \widehat{m}_1$. We have a map of double cosets of $\widetilde{G}$ into those of G. By Satake's argument the image of

$$\mathrm{Sp}(n, \mathbf{Z}) p^{\widehat{m}_1} \begin{pmatrix} p^{-\widehat{m}_1} & & & & & \\ & \ddots & & & & \\ & & p^{-\widehat{m}_n} & & & \\ & & & p^{\widehat{m}_1} & & \\ & & & & \ddots & \\ & & & & & p^{\widehat{m}_n} \end{pmatrix} \mathrm{Sp}(n, \mathbf{Z})$$

in the representation ring of $\widehat{G}_0 \subseteq$ group ring of $\widehat{M}$ is of the form

$$p^{\delta_\sigma}(\widehat{m}_1, \dots, \widehat{m}_n) + \text{smaller terms} \quad \text{(w.r.t. ordering of weights.)}$$

If $\widehat{\mu}_1$ and $\widehat{\mu}_2$ lie in $\widehat{M}$ and $\widehat{\mu}_2 = \widehat{\mu}_1 - \sum_{\widehat{\alpha} > 0} a_{\widehat{\alpha}} \widehat{\alpha}$, $a_{\widehat{\alpha}} \geqslant 0$ in $\mathbf{Z}$, then $\epsilon(\widehat{\mu}_1) \geqslant \epsilon(\widehat{\mu}_2)$. It follows that $p^{\delta_\sigma} \chi_\sigma$ is the image of a Hecke operator (over $\mathbf{Z}$) of $\widetilde{G}$ which is expressed in terms of cosets $\mathrm{Sp}(n, \mathbf{Z}) A \,\mathrm{Sp}(n, \mathbf{Z})$ with A integral and with $\lambda(A) = p^{2\epsilon_\sigma}$. If σ is the jth exterior power of σ_0 then $j \geqslant 2\epsilon_\sigma$ and $j - 2\epsilon_\sigma$ is even; so $T'_j = p^{\frac{t}{2} j} T_j$ can be represented in terms of such cosets with A integral and $\lambda(A) = p^j$. We do this. Let T^*_j be the corresponding element in the Hecke algebra of $\widetilde{G}$.

Note that this Hecke algebra is going to be isomorphic to the subring of the group ring of

$$\widehat{M}^* = \left\{ (\widehat{\mu}, z) \in \widehat{M} \oplus \mathbf{Z} \;\middle|\; z + \widehat{m}_1 \in \mathbf{Z} \text{ for one and hence for all } i \right\}$$

formed by elements invariant under permutations and sign changes of the coordinates of $\widehat{\mu}$. This isomorphism is constructed as before except that now

$$\widehat{\mu}^* \begin{pmatrix} p^{\beta_1} & * & * & * & * & * \\ 0 & \ddots & * & * & * & * \\ 0 & 0 & p^{\beta_n} & * & * & * \\ 0 & 0 & 0 & p^{\alpha_1} & 0 & 0 \\ 0 & 0 & 0 & * & \ddots & 0 \\ 0 & 0 & 0 & * & * & p^{\alpha_n} \end{pmatrix} = \left(\frac{\alpha_1 - \beta_1}{2}, \dots, \frac{\alpha_n - \beta_n}{2}; \frac{\alpha_1 + \beta_1}{2} \right)$$

Here $\alpha_i + \beta_i$ does not depend on i. We can also consider the group

$$H = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \middle| \, a,\, b \in n \times n \text{ matrices} \quad {}^t ba = \text{a scalar} \right\}$$

as a group over $\mathbf{Q}$. Its Hecke algebra is the set of those elements in the group ring of $\widehat{M}^*$ which are invariant under permutations of the coordinates of $\widehat{\mu}$ (isomorphism obtained as before[1]). Thus we obtain an injection of the Hecke algebra of $\widetilde{G}$ into that of H. In terms of double cosets

$$\mathrm{Sp}(n, \mathbf{Z}) A \,\mathrm{Sp}(n, \mathbf{Z}) = \bigcup_i \mathrm{Sp}(n, \mathbf{Z}) A_i, \qquad A_i = \begin{pmatrix} a_i & * \\ 0 & b_i \end{pmatrix}$$

is mapped to

$$\bigcup_i p^{\phi_i} H_{\mathbf{Z}} \begin{pmatrix} a_i & 0 \\ 0 & b_i \end{pmatrix} \qquad a_i = \begin{pmatrix} p^{-\alpha_1} & & * \\ & \ddots & \\ 0 & & p^{-\alpha_n} \end{pmatrix}, \quad \phi_i = -\frac{n+1}{2}\Big(\sum \alpha_i\Big),$$

which is a union of double cosets counted with multiplicity. Note the image of the double coset

$$H_{\mathbf{Z}} \begin{pmatrix} pI & 0 \\ 0 & I \end{pmatrix} H_{\mathbf{Z}}$$

in the group ring is $\widehat{\mu} = \left(-\frac{1}{2}, \ldots, -\frac{1}{2}, 1\right)$.

In the group ring of $\widehat{M}^*$ we can factor[2]

$$X^d + \sum_{j=1}^{d} (-1)^j \widehat{T}_j^* X^{d-j}$$

as

$$\prod \left(X - p^{t/2} \left(\pm\frac{1}{2}, \ldots, \pm\frac{1}{2}, 1 \right) \right).$$

This is clear from the definitions of T_j, T_j', T_j^*. The product is taken over all possible choices of signs. The map

$$(\widehat{\mu}, z) = (\widehat{m}_1, \ldots, \widehat{m}_n, z) \to p^{+\sum_i (\widehat{m}_i + z)\frac{n+1}{2}} (\widehat{\mu}, z)$$

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[1] If

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} p^{\beta_1} & * & * & 0 & 0 & 0 \\ 0 & \ddots & * & 0 & 0 & 0 \\ 0 & 0 & p^{\beta_n} & 0 & 0 & 0 \\ 0 & 0 & 0 & p^{\alpha_1} & 0 & 0 \\ 0 & 0 & 0 & * & \ddots & 0 \\ 0 & 0 & 0 & * & * & p^{\alpha_n} \end{pmatrix}$$

then

$$\rho \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = +\left\{ \frac{n-1}{2}\alpha_1 + \frac{n-3}{2}\alpha_2 + \cdots - \frac{n-1}{2}\alpha_n \right\}.$$

[2] Added 2009. The hat on T is uncertain and may not have been intended in the letter. I have nevertheless let it stand.

extends to a homomorphism ψ of the group ring of $\widehat{M}^*$ to itself which maps the image of the Hecke algebra of H to itself.

$$X^d + \sum_{j=1}^{d} (-1)^j \psi(\widehat{T}_j^*) X^{d-j} = \prod \left(X - p^{\frac{t}{2} + \sum \pm \frac{n+1}{4}} \left(\pm\frac{1}{2}, \dots, \pm\frac{1}{2}, 1 \right) \right).$$

One root of the polynomial is $\left(-\frac{1}{2}, \dots, -\frac{1}{2}, 1\right)$. So the equation has a root in the Hecke algebra of $H_{\mathbf{Z}}$.

Now look at Shimura (PNAS) I don't really understand that paper yet but let me go ahead anyhow. A coset $\mathrm{Sp}(n, \mathbf{Z})A$, $A \in \widetilde{G}$ and integral, determines a kernel in a generic abelian variety of dimension n (with principal polarization), therefore a quotient, and a new point on the moduli space. Putting these points together appropriately one gets for every double coset a correspondence. Now consider reduction modulo p. We can always arrange that the double coset is represented by

$$\bigcup_i \mathrm{Sp}(n, \mathbf{Z}) A_i,$$

where

$$A_i = \begin{pmatrix} a_i & b_i \\ 0 & d_i \end{pmatrix},$$

and that part of the division points of order a pth power killed by reduction consists of points whose last n coordinates are zero. The point in Shimura seems to be that the effect of A_i on the generic point in the moduli space does not depend on b_i, only on A_i and d_i. Further, along the same lines one can actually get the Hecke algebra of H to act on the reduced variety. Thus the effect of a non-reduced Hecke operator after reduction is obtained by mapping it into the Hecke algebra of H by throwing away b_i. This is our old injection minus the factor p^{d_i}. That is, it is the injection followed by ψ. Since the Frobenius corresponds to the element

$$H_{\mathbf{Z}} \begin{pmatrix} pI & 0 \\ 0 & I \end{pmatrix} H_{\mathbf{Z}}$$

the previous considerations imply that equation (1) is satisfied.

I have kind of ignored groups of outer type but let me make one remark since it connects with a problem about which Shimura has probably thought a great deal. Take a totally real number field F whose imbeddings in $\mathbf{R}$ are $\tau_1, \dots, \tau_g$ and take a quaternion algebra over[3] F which splits exactly at $\tau_1, \dots, \tau_r$ ($r \leqslant g$). Let G be the multiplicative group of M modulo its centre considered as a group over $\mathbf{Q}$. In this case $d = 2^r$. Let K be a Galois extension of F and let $U = \mathfrak{G}(K/\mathbf{Q})$ and $V = \mathfrak{G}(K/F)$. Let $X = V \backslash U$. G operates on X. For each $x \in X$ let $S_x = \mathrm{SL}(2, \mathbf{C})$. Then U operates on $\prod_{x \in X} S_x$ according to

$$g : \prod_x s_x \to \prod_x s_{xg}.$$

We suppose $F \subseteq K \subseteq \mathbf{C}$ and identify each x with a τ_i. The representation

$$s_{x_1} \times \cdots \times s_{x_g} \to s_{x_1} \oplus \cdots \oplus s_{x_r}$$

[3]Added 2009. An error in the text, where M appears rather than F, is corrected here

is a representation of degree 2^r. Let V' be the set of $u \in U$ which map $\{x_1, \dots, x_r\}$ (to itself).[4] I forgot to mention that $\widehat{G}$ may be taken to be the split extension of U by $\prod_{x \in X} S_x = \widehat{G}_0$. It is clear that the given representation of $\widehat{G}_0$ may be extended to a representation of $\widehat{G}_0 \times V' \subseteq \widehat{G}$ of the same degree by letting

$$v' : y_{x_1} \otimes \cdots \otimes y_{x_r} \to y_{x_1 v'} \otimes \cdots \otimes y_{x_r v'}, \qquad v' \in V'.$$

Inducing this representation up to $\widehat{G}$ we get a representation of degree $[U : V']2^r$. I believe that Shimura has conjectured and partially proved that it is the L-functions corresponding to this representation which play a role in the case under consideration. I would like to know if he meets the problems previously mentioned. Note that the fact that the corresponding variety is defined over an abelian extension of the fixed fields of V' looks more natural in this context. It may be better sometimes to reduce the field of definition by considering non-absolutely irreducible varieties.

Finally let me show in a formal way(!) for non-compact V how Matsushima's result gives information about the contribution of the discrete series to the cohomology. According to his formula

$$\dim H^i = \sum_{T \in D_0} N(T) \left(\sum_{i=1}^{s_p} M(T_K, \tau_i^p) \right).$$

Here D_0 is the set of unitary representations of G on which the Casimir operator vanishes, $M(T_K, \tau_i^p)$ is the multiplicity with which the irreducible representation τ_i^p of K occurs in the restriction of T to K, $\tau_1^p, \dots, \tau_{s_p}^p$ are the irreducible constituents, with multiplicity, of the representations of K on the pth exterior power of the tangent space at $K \subseteq K\backslash G = X$. The weights of a Cartan subgroup on this representation are just the set of non-compact roots each counted with multiplicity 1. The weights of K which occur in the pth exterior power for $p = \frac{1}{2}(\text{Real dim } X)$ (in particular) are sums of p non-compact roots in which repetitions are not allowed—of course a root and its negative may occur.

Pick a member T_Λ of the discrete series with parameter Λ. (In the compact case Λ would be (for example) the highest weight plus one-half the sum of the positive roots of T_Λ.) Choose an order on the roots so that Λ lies in the positive Weyl chamber. Fix this order. Since the Casimir operator is to vanish at T_Λ we must have $\langle \Lambda, \Lambda \rangle = \langle \rho, \rho \rangle$ (ρ is $1/2$ the sum of the positive roots for the given order, ρ_+ and ρ_0 are to be $1/2$ of the positive non-compact and positive compact roots respectively.) Since Λ must be integral and must lie in the interior of the chamber the equality is possible only if $\Lambda = \rho$. Let σ be a representation of K with lowest weight μ. The multiplicity of σ in the restriction of T_ρ to K is[5] (at least according to Blattner's conjecture—cf. Schmid, PNAS—for the first time I feel it important to prove the conjecture)

$$(-1)^s \sum_{w \,\in\, \text{Weyl group of } K} \operatorname{sgn} w Q\bigl(\rho + \rho_+ - w(\mu - \rho_k)\bigr)$$

if $s = 1/2\{\dim K - \operatorname{rank} K\}$ and $Q(\omega)$, ω any weight, is the number of ways $-\omega$ can be written as a sum of positive non-compact roots (sum can be empty). If $Q\bigl(\rho + \rho_+ - w(\mu - \rho_k)\bigr)$ is

[4]Added 2009. Some phrase has clearly been omitted in a hastily written sentence. The two words in parentheses are added as they seem to be what is missing.

[5]There is a comment beside the following formula, added at some later date, "Doesn't seem to be quite correct". At the moment, I do not know to what it refers.

not 0,

$$w\mu = \rho_k + w_k + 2\rho_+ + \sum \text{positive non-compact roots.}$$

However $\rho_k + w\rho_k$ is a sum of positive compact roots. If μ occurs in the exterior power in the middle dimension, so does $w\mu$; then $2\rho_+ - w\mu$ is a sum of positive non-compact roots. Since a non-trivial sum of positive roots can never be 0 the above equation is only possible if $\rho_k = -w\rho_k$ so that w is the unique element of the Weyl group which takes positive roots to negative roots and if $w\mu = 2\rho_+$ and if $\sum$ positive non-compact roots is empty. For the given w, $\operatorname{sgn} w = (-1)^s$. Thus T_ρ restricted to K and the representation of K on the exterior power of the tangent space in middle dimension have just the representation with highest weight $2\rho_+$ in common and that occurs with multiplicity 1. This gives the fact alluded to at the beginning.

In any case that is the list of my troubles with Shimura's work. Maybe we can talk about these things next year.

All the best
Yours

Bob L.

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